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An area formula for the triangle of residual centroids and its generalizations

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ABSTRACT. In this paper we consider an inscribed triangle XYZ to a triangle ABC and we establish a relation between the area of these two triangles and the area of the triangle determined by the centroids of the residual triangles AZY, BXZ and CYX. Moreover we generalize this relation to the case of a general barycenter instead of centroid and also to quadrilaterals.

1. INTRODUCTION

In this paper we consider the points X, Y, Z on the sides BC, CA and AB of an arbitrary triangle ABC and we denote by G_A, G_B, G_C the centroids of the residual triangles AZY, BXZ and CYX respectively. We establish a relation (see Theorem 2) between the area of the triangles $G_AG_BG_C$, XYZ and ABC. This is motivated by some earlier results published by M. Dalcín in [3] and E. Eckart in [5] and [6] concerning triangle centers of residual triangles. More exactly in [3] the author asserted the following property without proof:

Theorem 1.1 (Proposition 8). *An inscribed triangle and its triangle of residual orthocenters have equal areas.*

This is a consequence of a general property of hexagons having three pairs of parallel sides:

Theorem 1.2. *If the pairs of opposite sides of the hexagon $ABCDEF$ are parallel, then the triangles ACE and BDF have equal area.*

This property was a contest problem in 1958 at the famous József Kürschák Mathematical Competition and the solution can be found in [2].

In this paper we establish a relation between the area of the initial triangle ABC, the inscribed triangle XYZ and the triangle determined by the centroids of the residual triangles AZY, BXZ and CYX. Moreover we give a possible generalization for quadrilaterals.

2. MAIN RESULTS

Theorem 2.3. *The area of the triangle of residual centroids satisfies the following relation:*

$$9 \cdot Area(G_AG_BG_C) = 2 \cdot Area(ABC) + Area(XYZ). \tag{2.1}$$

Proof. Let $x, y, z \in \mathbb{R}$ such that the normalized barycentric coordinates of the points X, Y, Z are

$$X = (0, 1-x, x), Y = (y, 0, 1-y), Z = (1-z, z, 0).$$

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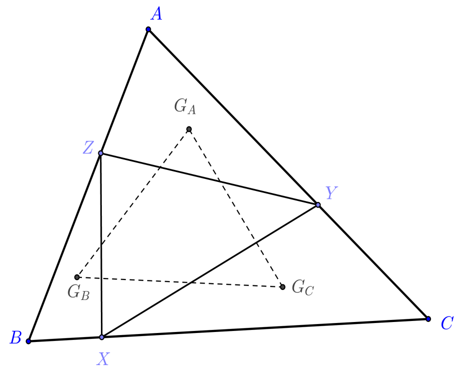


FIGURE 1. The centroids of the residual triangles

With these notations we have

$$\begin{aligned}Area(XYZ) =& Area(ABC)\cdot\begin{vmatrix} 0 & 1-x & x\\ y & 0 & 1-y\\ 1-z & z & 0\end{vmatrix}\\ =& Area(ABC)\cdot(xyz+(1-x)(1-y)(1-z)). \end{aligned}\tag{2.2}$$

The barycentric coordinates of the centroid G_A, G_B and G_C can be expressed as follows:

$$G_A=\frac{1}{3}(2+y-z,z,1-y), G_B=\frac{1}{3}(1-z,2+z-x,x), G_C=\frac{1}{3}(y,1-x,2+x-y)$$

so the area of the triangle $G_AG_BG_C$ is

$$\begin{aligned}Area(G_AG_BG_C) =& \frac{1}{27}Area(ABC)\cdot\begin{vmatrix} 2+y-z & z & 1-y\\ 1-z & 2+z-x & x\\ y & 1-x & 2+x-y\end{vmatrix}\\ =& \frac{1}{9}Area(ABC)\cdot\begin{vmatrix} 1 & z & 1-y\\ 1 & 2+z-x & x\\ 1 & 1-x & 2+x-y\end{vmatrix}\\ =& \frac{1}{9}Area(ABC)\cdot(2+xyz+(1-x)(1-y)(1-z)). \end{aligned}\tag{2.3}$$

From (2.3) and (2.2) we obtain (2.1), so the proof is complete. □

Remark 2.1. The area is considered as oriented one, so the relation remains true even if, the points X, Y, Z are outside the segments BC, CA, AB.

Remark 2.2. Using Theorem 1.1. we obtain the following equivalent formulation:

$$9\cdot Area(G_AG_BG_C)=2\cdot Area(ABC)+Area(H_AH_BH_C),\tag{2.4}$$

where H_A, H_B and H_C are the orthocenters of the residual triangles AZY, BXZ and CYX respectively.

Theorem 2.4. *Consider the arbitrary points X, Y, Z on the sides BC, CA and AB of an arbitrary triangle ABC. The points A_1, A_2 and A_3 have normalized barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_3, \lambda_1, \lambda_2)$ and $(\lambda_2, \lambda_3, \lambda_1)$ respectively with respect to the triangle AZY. In a similar way we consider the points B_1, B_2 and B_3 in the triangle BXZ and C_1, C_2, C_3 in the triangle CYX.*

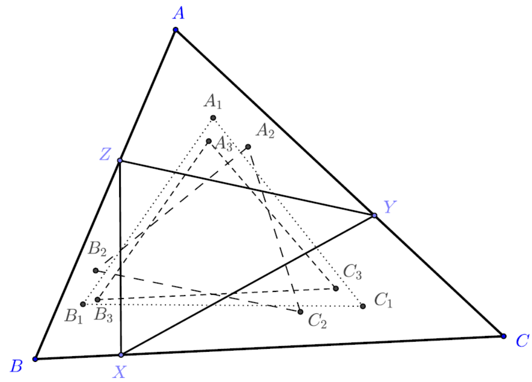


FIGURE 2. Replacing the centroid with 3 points

With these notations we have

$$Area(A_1B_1C_1) + Area(A_2B_2C_2) + Area(A_3B_3C_3) = c_1 \cdot Area(ABC) + c_2 \cdot Area(XYZ), \tag{2.5}$$

where

$$c_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1 \cdot \lambda_2 + \lambda_2 \cdot \lambda_3 + \lambda_3 \cdot \lambda_1,$$
$$c_2 = 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_1 \cdot \lambda_2 + \lambda_2 \cdot \lambda_3 + \lambda_3 \cdot \lambda_1).$$

Remark 2.3. If $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, we have $A_1 = A_2 = A_3 = G_A$ the centroid of AYZ, $B_1 = B_2 = B_3 = G_B$ the centroid of BZX and $C_1 = C_2 = C_3 = G_C$ the centroid of CXY. Moreover in this case $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so we obtain (2.1).

Proof of theorem 2.3. Using the same notations as in proof of Theorem 2.3 we obtain for the coordinates of the points A_1, B_1 and C_1 the following expressions:

$$A_1 = (\lambda_1 + \lambda_2 \cdot (1-z) + \lambda_3 \cdot y, \lambda_2 \cdot z, \lambda_3 \cdot (1-y)),$$
$$B_1 = (\lambda_1 \cdot (1-z), \lambda_1 + \lambda_2 \cdot (1-x) + \lambda_3 \cdot z, \lambda_2 \cdot x),$$
$$C_1 = (\lambda_1 \cdot y, \lambda_2 \cdot (1-x), \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot x).$$

So for the ratio $R_1 = \frac{Area(A_1B_1C_1)}{Area(ABC)}$ we obtain (using $\lambda_1 + \lambda_2 + \lambda_3 = 1$):

$$\begin{aligned}
R_1 &= \begin{vmatrix} \lambda_1 + \lambda_2 \cdot (1-z) + \lambda_3 \cdot y & \lambda_2 \cdot z & \lambda_3 \cdot (1-y) \\ \lambda_1 \cdot (1-z) & \lambda_1 + \lambda_2 \cdot (1-x) + \lambda_3 \cdot z & \lambda_2 \cdot x \\ \lambda_1 \cdot y & \lambda_2 \cdot (1-x) & \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot x \end{vmatrix} \\
&= \begin{vmatrix} 1 & \lambda_2 \cdot z & \lambda_3 \cdot (1-y) \\ 1 & \lambda_1 + \lambda_2 \cdot (1-x) + \lambda_3 \cdot z & \lambda_2 \cdot x \\ 1 & \lambda_2 \cdot (1-x) & \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot x \end{vmatrix} \\
&= \begin{vmatrix} \lambda_1 + \lambda_2 \cdot (1-x-z) + \lambda_3 \cdot z & \lambda_2 \cdot x - \lambda_3 \cdot (1-y) \\ \lambda_2 \cdot (1-x-z) & \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot (x+y-1) \end{vmatrix} \\
&= (\lambda_1 + \lambda_3 \cdot z) \begin{vmatrix} 1 & \lambda_2 \cdot x - \lambda_3 \cdot (1-y) \\ 0 & \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot (x+y-1) \end{vmatrix} + \\
&\quad \lambda_2(1-x-z) \begin{vmatrix} 1 & \lambda_2 \cdot x - \lambda_3 \cdot (1-y) \\ 1 & \lambda_1 + \lambda_2 \cdot (1-y) + \lambda_3 \cdot (x+y-1) \end{vmatrix}.
\end{aligned} \tag{2.6}$$

So we have

$$R_1 = \lambda_1^2 + \lambda_2^2(1-x-y)(1-x-z) + \lambda_3^2(zx + zy - z) +$$

$$+\lambda_1 \cdot \lambda_2(2-x-y-z) + \lambda_2 \cdot \lambda_3(z+x-zy-xz-x^2) + \lambda_3 \cdot \lambda_1(x+y+z-1).$$

By a similar calculation we deduce a relation for $R_2 = \frac{Area(A_2B_2C_2)}{Area(ABC)}$ and for $R_3 = \frac{Area(A_3B_3C_3)}{Area(ABC)}$. Due to the symmetry of the notations by adding the three ratios we obtain

$$R_1 + R_2 + R_3 = c_1 + c_2(1-x-y-z+xy+yz+zx),$$

which completes the proof.

□

Example 2.1. If M_1 is the midpoint of ZY, the midpoint of AM_1 has barycentric coordinates $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, so applying the previous result we deduce

$$Area(A_1B_1C_1) + Area(A_2B_2C_2) + Area(A_3B_3C_3) = \frac{11}{16} \cdot Area(ABC) + \frac{7}{16} \cdot Area(XYZ), \tag{2.7}$$

where A_1, A_2, A_3, B_1, B_2, B_3 and C_1, C_2, C_3 are the midpoints of the corrsponding medians in the triangles AZY, BXZ and CYX respectively.

In what follows we give a more general version for the relation (2.1).

Theorem 2.5. *If G_1, G_2 and G_3 are the centroids of the triangles $A_{11}A_{12}A_{13}$, $A_{21}A_{22}A_{23}$ and $A_{31}A_{32}A_{33}$, then*

$$\begin{aligned} 9Area[G_1G_1G_3] = &Area[A_{11}A_{21}A_{31}] + Area[A_{12}A_{22}A_{32}] + Area[A_{13}A_{23}A_{33}] + \\ &+ Area[A_{11}A_{22}A_{33}] + Area[A_{12}A_{23}A_{31}] + Area[A_{13}A_{21}A_{32}] + \\ &+ Area[A_{11}A_{23}A_{32}] + Area[A_{12}A_{21}A_{33}] + Area[A_{13}A_{22}A_{31}] \end{aligned} \tag{2.8}$$

Proof. For the area of an arbitrary triangle UVT we use it's expression in complex numbers (see [1], page 109, [7], [4]):

$$Area[UVT] = \frac{1}{2} Im(u\overline{v} + v\overline{t} + t\overline{u}),$$

where u, v, t are the affixes of the points U, V, T. Using this expression and the affixes of the centroids, we have

$$\begin{aligned} 9Area[G_1G_2G_3] = \frac{1}{2} Im((a_{11} + a_{12} + a_{13})\overline{(a_{21} + a_{22} + a_{23})} \\ +(a_{21} + a_{22} + a_{23})\overline{(a_{31} + a_{32} + a_{33})} + (a_{31} + a_{32} + a_{33})\overline{(a_{11} + a_{12} + a_{13})}). \end{aligned}$$

So

$$9Area[G_1G_2G_3] = \frac{1}{2} Im \sum_{i=1}^{3} (\sum_{j=1}^{3} a_{ij})(\sum_{j=1}^{3} \overline{a_{i+1j}})$$

$$\begin{aligned} &= \frac{1}{2} Im\left((a_{11}\overline{a_{21}} + a_{21}\overline{a_{31}} + a_{31}\overline{a_{11}}) + (a_{12}\overline{a_{22}} + a_{22}\overline{a_{32}} + a_{32}\overline{a_{12}})\right. \\ &\qquad \left.+(a_{13}\overline{a_{23}} + a_{23}\overline{a_{33}} + a_{33}\overline{a_{31}})\right) \\ &+ \frac{1}{2} Im\left((a_{11}\overline{a_{22}} + a_{22}\overline{a_{33}} + a_{33}\overline{a_{11}}) + (a_{12}\overline{a_{23}} + a_{23}\overline{a_{31}} + a_{31}\overline{a_{12}})\right. \\ &\qquad \left.+(a_{13}\overline{a_{21}} + a_{21}\overline{a_{32}} + a_{32}\overline{a_{13}})\right) + \\ &+ \frac{1}{2} Im\left((a_{11}\overline{a_{23}} + a_{23}\overline{a_{32}} + a_{32}\overline{a_{11}}) + (a_{12}\overline{a_{21}} + a_{21}\overline{a_{33}} + a_{33}\overline{a_{12}})\right. \\ &\qquad \left.+(a_{13}\overline{a_{22}} + a_{22}\overline{a_{31}} + a_{31}\overline{a_{13}})\right). \end{aligned}$$

Identifying the terms in the right-hand side as areas we obtain (2.8), so the proof is complete. □

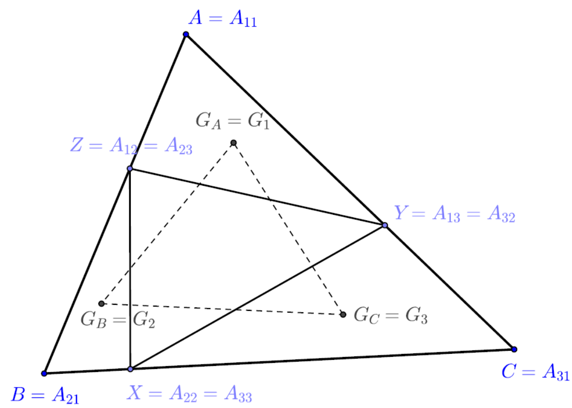


FIGURE 3. Special configuration of three triangles

Remark 2.4. Using the notations of the figure 3 ($X = A_{22} = A_{33}, \dots$) we obtain

$$\begin{aligned} 9Area[G_1G_2G_3] =& Area[ABC] + 2Area[XYZ] \\ &+ Area[AZY] + Area[BXZ] + Area[CYX], \end{aligned} \tag{2.9}$$

so if the points XYZ are on the sides of the triangle ABC, we have

$$Area[ABC] = Area[XYZ] + Area[AZY] + Area[BXZ] + Area[CYX],$$

which compared to (2.9) implies (2.1).

On the other hand the relation (2.9) is valid even in the case when $X \notin BC$, $Y \notin CA$ and $Z \notin AB$, so it implies the following interesting property for a hexagon $AZBXCY$:

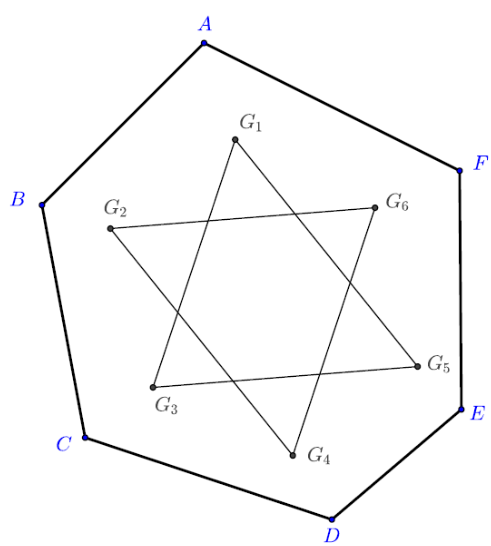


FIGURE 4. An interesting property of the hexagon

Corollary 2.1. *If $AZBXCY$ is an arbitrary hexagon, G_1, G_2 and G_3 are the centroids of the triangles AZY, BXZ and CYX respectively, then relation* (2.9) *holds.*

Applying this corollary twice and interchanging the role of the points ABC and XYZ we obtain the following property:

Corollary 2.2. *If in the hexagon $ABCDEF$ G_1, G_2, G_3, G_4, G_5 and G_6 are the centroids of the triangles ABF, BCA, CDB, DEC, EFD and FAE respectively (see figure 4), then*

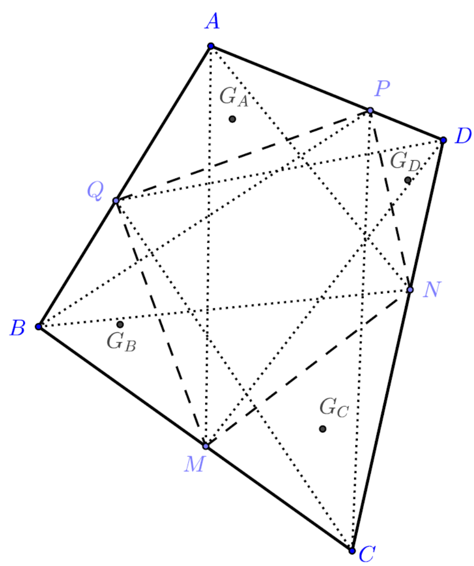
$$Area(G_1G_3G_5) = Area(G_2G_4G_6).$$

This property also reduces to Theorem 1.2. since the opposite sides of the hexagon $G_1G_2G_3G_4G_5G_6$ are parallel to the diagonals AD, BE and CF.

Theorem 2.5. can also be generalized to more than 3 triangles, but the number of triangles in the right-hand side expression is too large (16). For this reason we do not assert this case or the higher dimensional analogous properties, we prove only a property for quadrilaterals that is analogous to Theorem 2.3.

Theorem 2.6. *If $ABCD$ is a quadrilateral, $M \in BC, N \in CD, P \in DA$, $Q \in AB$ and we denote by G_A, G_B, G_C, G_D the centroids of the triangles AQP, BMQ, CNM and DPN, then*

$$9Area[G_AG_BG_CG_D] = 2Area[ABCD]+2Area[MNPQ]+Area[AMDQCPBNA] \quad (2.10)$$



Remark 2.5. For a better understanding of the term $Area[AMDQCPBNA]$ from the previous theorem we illustrate it in Figure 5. In fact it is the sum of the areas A_1, A_2 and A_3 shown in this figure.

Proof. Using complex numbers as in the proof of theorem 2.5. we have

$$9Area(G_AG_BG_CG_D) = \frac{9}{2} Im(g_A \cdot \overline{g_B} + g_B \cdot \overline{g_C} + g_C \cdot \overline{g_D} + g_D \cdot \overline{g_A}) \quad (2.11)$$

$$= \frac{1}{2} Im((p+a+q) \cdot \overline{q+b+m} + (q+b+m) \cdot \overline{m+c+n}) \quad (2.12)$$

$$+ \frac{1}{2} Im((m+c+n) \cdot \overline{n+d+p} + (n+d+p) \cdot \overline{p+a+q}) \quad (2.13)$$

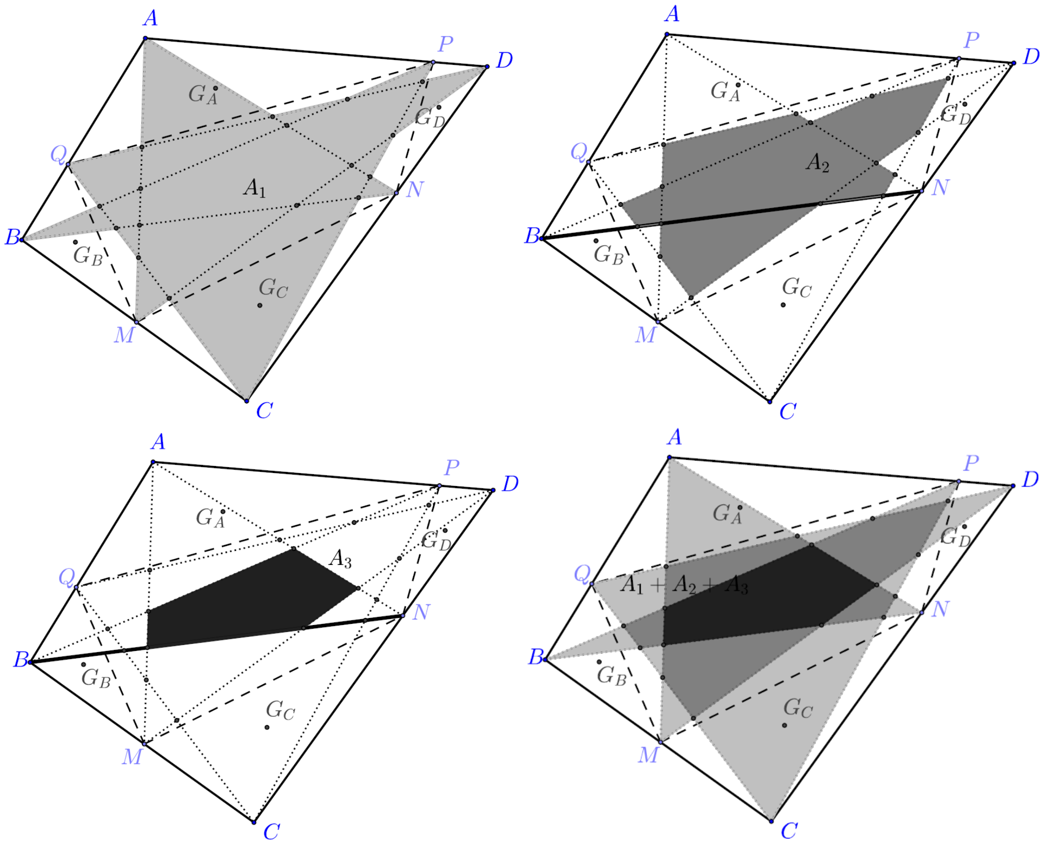


FIGURE 5. $Area(AMDQCPBNA) = A_1 + A_2 + A_4$

Expanding the products in the right-hand side we obtain 36 terms. We can omit the terms $q \cdot \overline{q}$, $m \cdot \overline{m}$, $n \cdot \overline{n}$ and $p \cdot \overline{p}$ since these terms are real numbers, so their imaginary part is 0. The sum of terms containing $p \cdot \overline{q}$, $q \cdot \overline{m}$, $m \cdot \overline{n}$ and $n \cdot \overline{p}$ is $Area(MNPQ)$, while the sum of terms containing $a \cdot \overline{b}$, $b \cdot \overline{c}$, $c \cdot \overline{d}$ and $d \cdot \overline{a}$ is $Area(ABCD)$. We have also the terms $q \cdot \overline{m}$, $m \cdot \overline{n}$, $n \cdot \overline{p}$ and $p \cdot \overline{q}$, whose sum gives the second $Area(MNPQ)$ on the right-hand side.

In order to identify the second $Area(ABCD)$ observe that the terms $a \cdot \overline{q}$, $q \cdot \overline{n}$, $n \cdot \overline{d}$, $d \cdot \overline{p}$ and $p \cdot \overline{d}$ form $Area(AQNDPA)$, while the terms $q \cdot \overline{b}$, $b \cdot \overline{m}$, $m \cdot \overline{c}$, $c \cdot \overline{n}$ and $n \cdot \overline{q}$ form $Area(QBMCNQ)$, so the sum of these two areas is $Area(ABCD)$. The remaining terms are $a \cdot \overline{m}$, $m \cdot \overline{d}$, $d \cdot \overline{q}$, $q \cdot \overline{c}$, $c \cdot \overline{p}$ $p \cdot \overline{b}$, $b \cdot \overline{n}$ and $n \cdot \overline{a}$ whose sum gives $Area(AMDQCPBNA)$, so the proof is complete. □

In the previous proof we used that the points M, N, P, Q are on the sides of the quadrilateral only in identifying the area of $AQNDPA$ with the area of $AQND$, so a more general identity is valid for any octogon $AQBMCNDP$.

Theorem 2.7. *If in the octogon $AQBMCNDP$ G_A, G_B, G_C and G_D are the centroids of the triangles AQP, BMQ, CNM and DPN respectively, then*

$$\begin{aligned} 9Area[G_AG_BG_CG_D] =& Area[ABCD] + Area[AQBMCNDP] \\ &+ 2Area[MNPQ] + Area[AMDQCPBNA] \end{aligned}$$

Applying this theorem twice we obtain the following property:

Corollary 2.3. *If in the octogon* $A_1A_2A_3A_4A_5A_6A_7A_8$ *we denote for all* $i \in \{1, 2, 3, \ldots, 8\}$ *by* G_i *the centroid of the triangle* $A_{i-1}A_iA_{i+1}$ *(the indeces are taken circulary, so* $A_9 = A_1$ *and* $A_0 = A_8$*), then*

$$Area(A_1A_3A_5A_7) - Area(A_2A_4A_6A_8) = 9(Area(G_2G_4G_6G_8) - Area(G_1G_3G_5G_7)).$$

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