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A comparison between two competing sixth convergence order algorithms under the same set of conditions

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ABSTRACT. There is a plethora of algorithms of the same convergence order for generating a sequence approximating a solution of an equation involving Banach space operators. But the set of convergence criteria is not the same in general. This makes the comparison between them hard and only numerically. Moreover, the convergence is established using Taylor series and by assuming the existence of high order derivatives not even appearing on these algorithms. Furthermore, no computable error estimates, uniqueness for the solution results or a ball of convergence is given. We address all these problems by using only the first derivative that actually appears on these algorithms and under the same set of convergence conditions. Our technique is so general that it can be used to extend the applicability of other algorithms along the same lines.

1. INTRODUCTION

Let B_1, B_2 denote Banach spaces, $\Omega \subset B_1$ be open and convex and $F : \Omega \longrightarrow B_2$ stand for a continuously differentiable operator (according to Fréchet). A multitude of applications from computational sciences reduces to locating a locally unique solution x^* of equation

$$F(x) = 0 \tag{1.1}$$

using mathematical modeling [1–9, 17, 19, 20, 22, 24]. But the solution x^* can be found in closed or analytic form only in special cases. That explains why most researchers and practitioners utilize algorithms generating sequences approximating x^* under certain conditions on the initial data.

Recently, due also to the development of new and faster computers there is a surge in the development of high convergence order algorithms. The optimality of these algorithms escapes researchers. That is why we have for example many algorithms of the same convergence order (say six) and requiring the same computational effort. As an example, consider the sixth convergence order algorithms defined , respectively for $x_0 \in \Omega$ and all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - \frac{1}{2}T_n^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - 2T_n^{-1}F(z_n), \end{aligned} \tag{1.2}$$

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n + \frac{1}{3}F'(x_n)^{-1}F'(x_n) - \frac{2}{3}T_n^{-1}F(x_n) \\ x_{n+1} &= z_n - \frac{1}{3}F'(x_n)^{-1}F(z_n) - \frac{4}{3}T_n^{-1}F(z_n), \end{aligned} \tag{1.3}$$

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where $T_n = 3F'(y_n) - F'(x_n)$. These algorithms were studied by Cordero et all [11], and Esmaeily et all [14], respectively in the special case, when $B_1 = B_2 = \mathbb{R}^i$ (i a natural number). The convergence order six was shown by assuming the existence of high order derivatives (until seven) which do not appear on these algorithms. Moreover, no computable estimations on $\|x_n - x^*\|$ or uniqueness of the solution x^* or ball of convergence results were given. Notice that these algorithms use two function evaluations, two first derivative evaluations and two inverses evaluations at each step. Moreover, the comparison between them can only be done numerically. To address all these concerns, we provide a unifying convergence analysis using only the first derivative which actually appears on these algorithms and the same set of convergence criteria making the comparison between them easier to carry out. The convergence order six is determined using COC or ACOC [26] that do not require the usage of higher than one derivatives (to be precised in Remark 2.2) but only the usage of the iterates. Moreover, error estimations and uniqueness results are given using $\omega-$continuity conditions.

The assumptions on the seventh order derivative limit the application of these methods, for example: Let $B_1 = B_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(s) = s^3 \log s^2 + s^5 - s^4$$

Then, we have $x^* = 1$, and $f'(s) = 3s^2 \log s^2 + 5s^4 - 4s^3 + 2s^2$, $f''(s) = 6x \log s^2 + 20s^3 - 12s^2 + 10s$, $f'''(s) = 6 \log s^2 + 60s^2 - 24s + 22$. Obviously $f'''(s)$ is not bounded on Ω. So, the convergence of algorithms (1.2) and (1.3) are not guaranteed by the analysis in [11,14].

It becomes clear from the proof, that follows that our technique can be used to extend the applicability of other algorithms along the same lines [1–26]. We just selected these two algorithms because of their usefulness and popularity.

The technique is given in Section 2, the numerical experiments in Section 3 and the conclusions in Section 4.

2. Ball convergence

We first provide the ball convergence of algorithm (1.2) based on some real functions and constants. Let S stand for $[0, \infty)$.

Suppose that there exists a continuous and nondecreasing function $\omega_0 : S \longrightarrow S$ such that equation

$$\omega_0(s) - 1 = 0 \tag{2.4}$$

has a minimal solution $\rho_0 \in (0, \infty)$. Let S_0 stand for $[0, \rho_0)$. Consider continuous and nondecreasing functions $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$. Define functions g_1 and φ_1 on S_0 as

$$g_1(s) = \frac{\int_0^1 \omega((1-\theta)s)ds + \frac{1}{3}\int_0^1 \omega_1(\theta s)ds}{1 - \omega_0(s)} \text{ and } \varphi_1(s) = g_1(s) - 1.$$

Suppose

$$\varphi_1(s) = 0 \tag{2.5}$$

has a minimal solution $r_1 \in (0, \rho_0)$.

Suppose equation

$$p(s) - 1 = 0 \tag{2.6}$$

has a minimal solution $\rho_p \in (0, \rho_0)$, where $p(s) = \frac{1}{2}(3\omega_0(g_1(s)s) + \omega_0(s))$. Set $\rho_1 = \min\{\rho_0, \rho_p\}$ and $S_1 = [0, \rho_1)$. Define functions g_0, g_2 and φ_2 on S_1 as

$$g_0(s) = \frac{\int_0^1 \omega((1-\theta)s)d\theta}{1 - \omega_0(s)},$$

$$g_2(s) = g_0(s) + \frac{3}{2}\frac{(\omega_0(g_1(s)s) + \omega_0(s)) \int_0^1 \omega_1(\theta s)d\theta}{(1-p(s))(1-\omega_0(s))} \text{ and } \varphi_2(s) = g_2(s) - 1.$$

Suppose that equation

$$\varphi_2(s) = 0 \tag{2.7}$$

has a minimal solution $r_2 \in (0, \rho_1)$.

Suppose that equation

$$\omega_0(g_2(s)s) - 1 = 0 \tag{2.8}$$

has a minimal solution $\rho_2 \in (0, \rho_1)$. Set $\rho_3 = \min\{\rho_1, \rho_2\}$ and $S_2 = [0, \rho_3)$. Define functions g_3 and φ_3 on S_2 as

$$g_3(s) = \left[g_0(g_2(s)s) + \frac{3\omega_0(g_1(s)s) + \omega_0(s) + 2\omega_0(g_2(s)s)) \int_0^1 \omega_1(\theta g_2(s)s)d\theta}{2(1-\omega_0(g_2(s)s))(1-p(s))} \right] g_2(s)$$

and

$$\varphi_3(s) = g_3(s) - 1.$$

Suppose that equation

$$\varphi_3(s) = 0 \tag{2.9}$$

has a minimal solution $r_3 \in (0, \rho_3)$. Define R_1 bt

$$R_1 = \min\{r_j\}, \ j = 1, 2, 3. \tag{2.10}$$

We shall show that R_1 is a radius of convergence for algorithm (1.2). It follows from (2.10) that for each $s \in [0, R_1)$

$$0 \le \omega_0(s) < 1 \tag{2.11}$$

$$0 \le \omega_0(g_1(s)s) < 1 \tag{2.12}$$

$$0 \le \omega_0(g_2(s)s) < 1 \tag{2.13}$$

$$0 \le p(s) < 1 \tag{2.14}$$

and

$$0 \le g_j(s) < 1 \tag{2.15}$$

hold. The notations $U(u, \lambda), \bar{U}(u, \lambda)$ are used for the open and closed balls in B_1 with center $u \in B_1$ and of radius $\lambda > 0$. The following conditions (A) are used:

(a1) There exists a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.

(a2) There exists a continuous and nondecreasing function $\omega_0 : S \longrightarrow S$ such that for all $x \in \Omega$, $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le \omega_0(\|x - x^*\|)$. Set $\Omega_0 = \Omega \cap U(x^*, \rho_0)$, where ρ_0 given in (2.4) exists.

(a3) There exists continuous and nondecreasing functions $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$ such that for each $x, y \in \Omega_0$

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \le \omega(\|y - x\|) \text{ and } \|F'(x^*)^{-1}F'(x)\| \le \omega_1(\|x - x^*\|).$$

(a4) $\bar{U}(x^*, R) \subset \Omega$.

(a5) There exists $R^* \ge R$ such that $\int_0^1 \omega(\theta R^*)d\theta < 1$.

Set $\Omega_1 = \Omega \cap \bar{U}(x^*, R^*)$. Here R stands for R_1, R_2, if algorithms (1.2) and (1.3) are used, respectively.

Next, we are in a position to present the ball convergence result for algorithm (1.2) based on the aforementioned terminology are the conditions (A).

Theorem 2.1. *Suppose the conditions (A) hold. Then, sequence $\{x_n\}$ starting from $x_0 \in U(x^*, R_1) - \{x^*\}$ is well defined in $U(x^*, R_1)$, remains in $U(x^*, R_1)$ for each $n = 0, 1, 2, \ldots$ and $\lim_{n\longrightarrow\infty} x_n = x^*$. Moreover, the following assertions hold for $e_n = \|x_n - x^*\|$ and each $n = 0, 1, 2, \ldots$.*

$$\|y_n - x^*\| \leq g_1(e_n)e_n \leq e_n < R_1, \tag{2.16}$$

$$\|z_n - x^*\| \leq g_2(e_n)e_n \leq e_n, \tag{2.17}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(e_n)e_n \leq e_n, \tag{2.18}$$

where R_1 is defined in (2.10) and the functions g_j are given previously. Moreover, the limit point $x^* \in \bar{U}(x^*, R_1)$ is the only solution of equation $F(x) = 0$ in Ω_1 given in (a5).

Proof. Let $v \in U(x^*, R_1)$. Using (a1), (a2), (2.10) and (2.11), we obtain in turn

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq \omega_0(\|v - x^*\|) \leq \omega_0(R_1) < 1, \tag{2.19}$$

so by a lemma on invertible operators due to Banach [19,20], $F'(v)$ is invertible and

$$\|F'(v)^{-1}F'(x^*)\| \leq \frac{1}{1 - \omega_0(\|v - x^*\|)}. \tag{2.20}$$

If we set $v = x_0$, then y_0 is well defined by the first substep of algorithm (1.2), and we can write

$$y_0 - x^* = (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + \frac{1}{3}F'(x_0)^{-1}F(x_0). \tag{2.21}$$

Then, by (2.15) (for $j = 1$), (2.10), (a3), (2.20) (for $v = x_0$) and (2.21) we have in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| + \frac{1}{3}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta \right\| \\
&\leq \frac{[\int_0^1 \omega((1-\theta)e_0)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta e_0)d\theta]e_0}{1 - \omega_0(e_0)} \leq g_1(e_0)e_0 \leq e_0 < R_1,
\end{aligned} \tag{2.22}$$

showing $y_0 \in U(x^*, R_1)$ and (2.16) for $n = 0$. By (2.12), (2.14), (2.22) and (a3), we get in turn that

$$\begin{aligned}
\|(2F'(x^*))^{-1}(3F'(y_0) - F'(x_0) - 2F'(x^*))\| &\leq \frac{1}{2}[3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\
&\quad + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|] \\
&\leq \frac{1}{3}(3\omega_0(\|y_0 - x^*\|) + \omega_0(e_0)) \leq p(e_0) < 1,
\end{aligned} \tag{2.23}$$

so

$$\|T_0^{-1}F'(x^*)\| \leq \frac{1}{2(1 - p(e_0))}, \tag{2.24}$$

z_0 is well defined by the second substep of algorithm (1.2), and we can write

$$\begin{aligned}
z_0 - x^* &= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) \\
&\quad +[I - \frac{1}{2}T_0^{-1}(3F'(y_0) + F'(x_0))]F'(x_0)^{-1}F(x_0) \\
&= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) \\
&\quad +T_0^{-1}[3F'(y_0) - F'(x_0) - \frac{1}{2}(3F'(y_0) + F'(x_0))]F'(x_0)^{-1}F(x_0) \\
&= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) \\
&\quad +3T_0^{-1}[F'(y_0) - F'(x^*) + (F'(x^*) - F'(x_0))]F'(x_0)^{-1}F(x_0).
\end{aligned} \tag{2.25}$$

Hence, by (2.10), (2.15) (for $j = 2$), and (2.22)-(2.25), we obtain in turn

$$\begin{aligned}
\|z_0 - x^*\| &\leq [g(e_0) + \frac{3(\omega_0(\|y_0 - x^*\|) + \omega_0(e_0)) \int_0^1 \omega_1(\theta e_0)d\theta}{2(1 - p(e_0))(1 - \omega_0(e_0))}]e_0 \\
&\leq g_2(e_0)e_0 \leq e_0,
\end{aligned} \tag{2.26}$$

showing $z_0 \in U(x^*, R_1)$, (2.17) holds for $n = 0$, and (2.20) is verified for $v = z_0, x_1$ is well defined by the third substep of algorithm (1.2), and we can write

$$\begin{aligned}
x_1 - x^* &= (z_0 - x^* - F'(z_0)^{-1}F(z_0)) + (F'(z_0)^{-1} - 2T_0^{-1})F(z_0) \\
&= (z_0 - x^* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}(T_0 - 2F'(z_0))T_0^{-1}F(z_0) \\
&= (z_0 - x^* - F'(z_0)^{-1}F(z_0)) \\
&\quad +F'(z_0)^{-1}(2(F'(y_0) - F'(z_0)) + (F'(y_0) - F'(x_0))]T_0^{-1}F(z_0).
\end{aligned} \tag{2.27}$$

Hence, we get by (2.10), (2.15) (for $j = 3$), (2.22), (2.26) and (2.27) that

$$\begin{aligned}
e_1 &\leq [g_0(\|z_0 - x^*\|) + \frac{(2(\omega_0(\|y_0 - x^*\|) + \omega_0(\|z_0 - x^*\|)) + \omega_0(\|y_0 - x^*\|}{2(1 - \omega_0(\|z_0 - x^*\|))(1 - p(e_0))} \\
&\quad \left. + \frac{\omega_0(\|y_0 - x^*\| + \omega_0(e_0)) \int_0^1 \omega_1(\theta\|z_0 - x^*\|)d\theta}{2(1 - \omega_0(\|z_0 - x^*\|))(1 - p(e_0))}\right] \|z_0 - x^*\| \\
&\leq g_3(e_0)e_0 \leq e_0,
\end{aligned} \tag{2.28}$$

which completes the induction for estimations (2.16)-(2.18) for $n = 0$. Suppose these estimations hold for all $m = 0, 1, 2, \ldots, n$. Then, by simply switching x_0, y_0, z_0, x_1 by x_m, y_m, z_m, x_{m+1} in the preceding calculations, we complete the induction for estimations (2.16)-(2.18). Then, using the estimation

$$e_{m+1} \leq ce_m < R_1, \tag{2.29}$$

where $c = g_3(e_0) \in [0, 1)$, we conclude that $\lim_{m\longrightarrow\infty} x_m = x^*$, and $x_{m+1} \in U(x^*, R_1)$. Consider $q \in \Omega_1$ such that $F(q) = 0$. Then, in view of (a2) and (a5)

$$\|F'(x^*)^{-1}(M - F'(x^*))\| \leq \int_0^1 \omega_0(\theta\|x^* - q\|)d\theta < 1, \tag{2.30}$$

so $x^* = q$ follows from the invertability of M and the identity $0 = F(x^*) - F(q) = M(x^* - q)$. □

Remark 2.1. 1. By (a2), and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \leq 1+w_0(\|x-x^*\|)\end{aligned}$$

second condition in (a3) can be dropped, and w_1 be defined as $w_1(t)=1+w_0(t)$. Notice that, if $w_1(t)<1+w_0(t)$, then R_1 can be larger (see Example 3.1).

2. The results obtained here can be used for operators G satisfying autonomous differential equations [2–9] of the form $F'(x)=T(F(x))$, where T is a continuous operator. Then, since $F'(x^*)=T(F(x^*))=T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x)=e^x-1$. Then, we can choose: $T(x)=x+1$.
3. The local results obtained here can be used for projection algorithms such as the Arnoldi's algorithm , the generalized minimum residual algorithm (GMRES), the generalized conjugate algorithm (GCR) for combined Newton/finite projection algorithms and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2–9, 17].
4. Let $w_0(t)=L_0t$, and $w(t)=Lt$. The parameter $r_A=\frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's algorithm [2]

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n) \text{ for each } n=0,1,2,\cdots \tag{2.31}$$

under the conditions (a1)-(a3) (w_1 is not used). It follows that the convergence radius R of algorithm (1.2) cannot be larger than the convergence radius r_A of the second order Newton's algorithm (2.31). As already noted in [3] r_A is at least as large as the convergence ball given by Rheinboldt [20] $r_{TR}=\frac{2}{3L_1}$, where L_1 is the Lipschitz constant on Ω, $L_0\leq L_1$ and $L\leq L_1$. In particular, for $L_0<L_1$ or $L<L_1$, we have that

$$r_{TR}<r_A \text{ and } \frac{r_{TR}}{r_A}\to\frac{1}{3} \quad as \quad \frac{L_0}{L_1}\to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_{TR} was given by Traub [24].
5. It is worth noticing that solver (1.2) is not changing, when we use the conditions (A) of Theorem 2.1 instead of the stronger conditions used in [11, 14]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi=\ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right)/\ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1=\ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)/\ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the existence of seventh Fréchet derivatives of operator F.

Concerning the convergence of algorithm (1.3), we define analogously functions

$$\bar{\varphi}_1(s)=g_0(s)-1,$$

$$\bar{g}_2(s)=g_0(s)+\frac{(\omega_0(g_0(s)s)+\omega_0(s))\int_0^1\omega_1(\theta s)d\theta}{2(1-p(s))}$$

$$\bar{\varphi}_2(s)=\bar{g}_2(s)--1,$$

$$\begin{aligned}\bar{g}_3(s) &= [g_0(\bar{g}_2(s)) + \left(\frac{\omega_0(\bar{g}_2(s)s)+\omega_0(s)}{(1-\omega_0(\bar{g}_2(s)s))(1-\omega_0(s))}\right.\\ &+ \left.\frac{\omega_0(g_0(s)s)+\omega_0(s)}{(1-\omega_0(s))(1-p(s))}\right)\int_0^1 \omega_1(\theta\bar{g}_2(s))d\theta\Big]\,\bar{g}_2(s)\end{aligned}$$

and

$$\bar{\varphi}_3(s) = \bar{g}_3(s) - 1.$$

Moreover $\bar{r}_j$ are the minimal positive solutions (if they exist) of equations $\bar{\varphi}_j(s) = 0$, respectively, and

$$R = R_2 = \min\{\bar{r}_j\}. \tag{2.32}$$

These functions are realized because of the estimations

$$\|y_n - x^*\| = \|x_n - x^* - F'(x_n)^{-1}F(x_n)\| \le \bar{g}_0(e_n)e_n \le e_n < R_2,$$

$$\begin{aligned}\|z_n - x^*\| &= \|y_n - x^* + \frac{1}{3}(F'(x_n)^{-1} - 2A_n^{-1})F(x_n)\|\\ &\le \|y_n - x^*\| + \frac{1}{3}\|F'(x_n)^{-1}F'(x^*)\|\|F'(x^*)^{-1}(T_n - 2F'(x_n))T_n^{-1}F(x_n)\|\\ &\le \|y_n - x^*\| + \|F'(x_n)^{-1}F'(x^*)\|\\ &\quad\times[\|F'(x^*)^{-1}(F'(x_n)-F'(x^*))\| + \|F'(x^*)^{-1}(F'(y_n)-F'(x^*))\|]\\ &\quad\times\|T_n^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_n)\|\\ &\le [g_0(e_n) + \frac{(\omega_0(\|y_n-x^*\|)+\omega_0(e_n))\int_0^1\omega_1(\theta e_n)d\theta}{2(1-p(e_n))}]e_n \le \bar{g}_2(e_n)e_n \le e_n,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|z_n - x^* + [F'(z_n)^{-1} - \frac{1}{3}F'(x_n)^{-1} - \frac{4}{3}T_n^{-1}]F(z_n)\|\\ &\le [g_0(\|z_n - x^*\|) + \left(\frac{\omega_0(\|z_n-x^*\|)+\omega_0(e_n)}{(1-\omega_0(\|z_n-x^*\|))(1-\omega_0(e_n))}\right.\\ &\quad+\left.\frac{\omega_0(\|y_n-x^*\|)+\omega_0(e_n)}{(1-\omega_0(e_n))(1-p(e_n))}\right)\int_0^1\omega_1(\theta\|z_n-x^*\|)d\theta]\|z_n-x^*\|\\ &\le \bar{g}_3(e_n)e_n \le e_n,\end{aligned}$$

where we also used

$$\begin{aligned}&F'(z_n)^{-1} - F'(x_n)^{-1} + \frac{2}{3}F'(x_n)^{-1} - \frac{4}{3}T_n^{-1}\\ &= (F'(z_n)^{-1} - F'(x_n)^{-1}) + \frac{2}{3}(F'(x_n)^{-1} - 2T_n^{-1})\\ &= F'(z_n)^{-1}(F'(x_n) - F'(z_n))F'(x_n)^{-1} + \frac{2}{3}F'(x_n)^{-1}(T_n - 2F'(x_n))T_n^{-1}\\ &= F'(z_n)^{-1}(F'(x_n) - F'(z_n))F'(x_n)^{-1} + 2F'(x_n)^{-1}(F'(y_n) - F'(x_n))T_n^{-1}.\end{aligned}$$

Hence, we arrived at the corresponding convergence result for algorithm (1.3).

Theorem 2.2. *Suppose that conditions of Theorem 2.1 hold. Then, the conclusions hold but with g_i, R_1, replaced by $\bar{g}_i$ and R_2, respectively and $R = R_2$ in conditions (A).*

3. Numerical examples

Example 3.1. Let $B_1 = B_2 = \mathbb{R}$. Define $F(x) = \sin x$. Then, we get that $x^* = 0,\ \omega_0(s) = \omega(s) = s$ and $\omega_1(s) = 1$. Then, we have

Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$	Radius	$\omega_1(s) = 1$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.4444	0.4000	$\bar{r}_1$	0.6667	0.6667
r_2	0.0551	0.0392	$\bar{r}_2$	0.4431	0.4236
r_3	0.0710	0.6737	$\bar{r}_3$	0.5241	0.4839

TABLE 1. Radius for Example 3.1

Example 3.2. Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.33}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad \text{for each } \ \xi \in \Omega.$$

Then, we get that $x_* = 0,\ \omega_0(s) = \omega_1(s) = \frac{15}{2}s, \omega_1(s) = 2$. This way, we have that

Radius	$\omega_1(s) = 2$	$\omega_1(s) = 1 + \omega_0(s)$	Radius	$\omega_1(s) = 2$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0296	0.0533	$\bar{r}_1$	0.0889	0.0889
r_2	0.0044	0.0053	$\bar{r}_2$	0.0509	0.0593
r_3	0.0062	0.0967	$\bar{r}_3$	0.0672	0.0719

TABLE 2. Radius for Example 3.2

Example 3.3. Let $B_1 = B_2 = \mathbb{R}^3, \Omega = U(0,1), x_* = (0,0,0)^T$ and define F on Ω by

$$F(x) = F(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{3.34}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $G'(x_*) = diag(1,1,1)$, we get by conditions (A) $\omega_0(s) = (e-1)s$, $\omega(s) = e^{\frac{1}{e-1}}s$, and $\omega_1(s) = e^{\frac{1}{e-1}}$. Then, we have

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$	Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.1544	0.2299	$\bar{r}_1$	0.3827	0.3827
r_2	0.0210	0.0228	$\bar{r}_2$	0.2219	0.2482
r_3	0.0292	0.0412	$\bar{r}_3$	0.2861	0.2904

TABLE 3. Radius for Example 3.3

Example 3.4. Returning back to the motivational example at the introduction of this study, we have $\omega_0(s) = \omega(s) = 96.662907s$, $\omega_1(s) = 1.0631$. Then, we have

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$	Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0045	0.0041	$\bar{r}_1$	0.0069	0.0069
r_2	0.0005	0.0005	$\bar{r}_2$	0.0048	0.0046
r_3	0.0007	0.0035	$\bar{r}_3$	0.0057	0.0001

TABLE 4. Radius for Example 3.4

4. CONCLUSIONS

We present a new technique for comparing competing algorithms based only on the first derivative that actually appears on them in contrast to earlier ones using higher than one derivatives. Our technique is so general that it can be used to extend the applicability of other algorithms along the same lines. Our technique also provides computable error estimations and uniqueness results based on $\omega-$ continuity conditions on F' not possible in earlier works [1–26].

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