CREAT. MATH. INFORM. **30** (2021), No. 1, 29 - 40
Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Rotational $\lambda-$hypersurfaces in Euclidean spaces

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ABSTRACT. Self-similar flows arise as special solution of the mean curvature flow that preserves the shape of the evolving submanifold. In addition, $\lambda-$hypersurfaces are the generalization of self-similar hypersurfaces. In the present article we consider $\lambda-$hypersurfaces in Euclidean spaces which are the generalization of self-shrinkers. We obtained some results related with rotational hypersurfaces in Euclidean $4-$space $\mathbb{R}^4$ to become self-shrinkers. Furthermore, we classify the general rotational $\lambda-$hypersurfaces with constant mean curvature. As an application, we give some examples of self-shrinkers and rotational $\lambda-$hypersurfaces in $\mathbb{R}^4$.

1. INTRODUCTION

Let $x : M \to \mathbb{R}^{n+d}$ be an isometric immersion, where M is an $n-$dimensional differentiable manifold. Under this isometric immersion, M is called the $n-$dimensional hypersurface immersed in $\mathbb{R}^{n+1}$. One of the most important geometric objects of M is the x position vector. This vector is also a vector of the Euclidean space $\mathbb{R}^{n+1}$ defined in the form $x = \overrightarrow{op}$, also known as the location vector or radius vector, and the position of the point $p \in M$ to the reference point $o \in \mathbb{R}^{n+1}$. One of the important invariants of M is the mean curvature vector field $\overrightarrow{H}$. In physics, the mean curvature vector field is the torsional field applied on the hypersurface. It is used for surface tension, surface stress or surface free energy in materials science [6]. The well known formula of E. Beltrami is between the position vector field x and the mean curvature vector vector field $\overrightarrow{H}$ of M provides a simple relationship in the form $\Delta x = -n\overrightarrow{H}$. Here Δ denotes the Laplacian of M with respect to the induced metric. From this equation, the necessary and sufficient condition for M to be minimal (i.e., $\overrightarrow{H} = 0$) is $\Delta x = 0$. In other words, M is harmonic [5]. The position vector field x of M has a natural decomposition given by

$$x = x^T + x^N, \tag{1.1}$$

where $x^T$and x^N are the tangential and normal components of x respectively [5]. In 2017, B.Y. Chen presented a study of various topics in differential geometry associated with the location vector fields of Euclidian submanifolds [6].

The mean curvature flow is the gradient flow of the functional area of the n-dimensional hypersurface M. From the perspective of the analysis, this flow is produced by a nonlinear parabolic equation. Although the classified results of the analysis show the short-term presence of the average curvature flow, understanding the long-term behavior is a difficult problem that requires checking for possible singularities that may occur throughout the flow. The mean curvature vector field $\overrightarrow{H}$ is one of the most important invariants of the hypersurface M. In physics, the average curvature vector field is the torsion field applied to the hypersurface originated from the ambient space. The mean curvature flow is the gradient flow of the area functional on the space of the hypersurface M. The self-similar

Received: 14.07.2020. In revised form: 02.12.2020. Accepted: 09.12.2020
2010 *Mathematics Subject Classification.* 53C40, 53C42.
Key words and phrases. *Rotational hypersurface, mean curvature flow, $\lambda-$hypersurface.*
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flows arise as special solution of the mean curvature flow that preserve the shape of the evolving hypersurface [20]. The most important mean curvature flow is self-similar flow which is obtained when the evolution becomes a homothety. Such self-similar hypersurface M with curvature vector field $\overrightarrow{H}$ satisfying the following non-linear elliptic system $\overrightarrow{H}+\lambda x^{N}=0$,where x^{N} is the normal component of x and λ is a real valued function. If λ is any strictly positive constant, then the hypersurface shrinks infinite time to a single point, under the action of the mean curvature flow, its shape remaining unchanged. If λ is strictly negative, then the hypersurface will expand its shape again remain the same; in this case the hypersurface is necessarily non compact. The case of vanishing λ is the well-known case of a minimal hypersurface, which of course is stationary under the action of the flow [20].

In [11] Chang and Wei introduced a $\lambda-$hypersurfaces of weighted volume-preserving mean curvature flow in Euclidean space giving a natural generalization of self-shrinkers in the hypersurface case. According to [11], a hypersurface $M\subset\mathbb{R}^{n+1}$ is called a $\lambda-$ hypersurface if its mean curvature H satisfies $H+\langle x,N\rangle=\lambda$ for some real function λ, where N is the unit normal of the hypersurface. Recently, Li and Chang made a generalization of both self-shrinkers and $\lambda-$hypersurfaces, by introducing the concepts of $\xi-$submanifolds [24].

This paper is organized as follows: In section 2, we give some basic concepts of the second fundamental form and curvatures of the hypersurface in $\mathbb{R}^{n+1}$. In section 3, we give some well known results of self-similar hypersurfaces and $\lambda-$hypersurfaces in Euclidean spaces $\mathbb{R}^{n+1}$. Further, we give some well known examples satisfying the self-shrinking condition. In section 4 we consider rotational hypersurfaces in $\mathbb{R}^{n+1}$. We obtained some results related with these type of hypersurfaces to become self-shrinkers. Furthermore, we classify the rotational $\lambda-$hypersurfaces with constant mean curvature. As an application, we give some examples of self-shrinkers and rotational $\lambda-$hypersurfaces in $\mathbb{R}^{4}$.

2. Preliminaries

Let M be an $n-$dimensional smooth hypersurface in $\mathbb{R}^{n+1}$ given with the isometric immersion (position vector), $x(s,u_1,...,u_n):(s,u_1,...,u_{n-1})\in U\subset\mathbb{R}^{n+1}$. The tangent space to M at an arbitrary point $p=x(s,u_1,...,u_{n-1})$ of M^n span $\left\{x_s,x_{u_1}...,x_{u_{n-1}}\right\}$. In the chart $(s,u_1,...,u_{n-1})$ the coefficients of the first fundamental form of M are given by

$$g_{ij}=\left\langle x_{u_i},x_{u_j}\right\rangle,\ 0\leq i,j\leq n-1 \tag{2.2}$$

where $\langle,\rangle$ is the Euclidean inner product. Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent and normal to M, respectively. Given any local orthonormal vector fields X_1, $X_2,...,X_n$ tangent to M, consider the second fundamental map $h:\chi(M)\times\chi(M)\to\chi^{\perp}(M)$;

$$h(X_i,X_j)=\widetilde{\nabla}_{X_i}X_j-\nabla_{X_i}X_j\quad 1\leq i,j\leq n. \tag{2.3}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{R}^{n+1}$, respectively. This map is well-defined, symmetric and bilinear [5].

For the normal vector field N of M, recall the shape operator $A:\chi^{\perp}(M)\times\chi(M)\to\chi(M)$;

$$A_NX_j=-\widetilde{\nabla}_{X_j}N,\quad 1\leq i,j\leq n,\quad X_j\in\chi(M^n). \tag{2.4}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_NX_j,X_i\rangle=\langle h(X_i,X_j),N\rangle=h_{ij},\quad 1\leq i,j\leq n; \tag{2.5}$$

where h_{ij} are the coefficients of the second fundamental form. The equations (2.3) and (2.4) are called *Gaussian formula* and *Weingarten formula* respectively. In addition,

$$h(X_i, X_j) = h_{ij}N,\ 1 \leq i, j \leq n, \tag{2.6}$$

and

$$A_N X_i = -\sum_{j=1}^{n} h_{ij} X_j, 1 \leq i \leq n, \tag{2.7}$$

hold. The mean curvature vector $\overrightarrow{H}$ and the square length $\|h\|^2$ of the second fundamental form h are defined respectively by

$$\overrightarrow{H} = \frac{1}{n}\sum_{k=1}^{n} h(X_k, X_k). \tag{2.8}$$

$$\|h\|^2 = \sum_{j=1}^{n} (h_{ij})^2 \tag{2.9}$$

The norm of the mean curvature vector $H = \left\|\overrightarrow{H}\right\|$ is called the *mean curvature* of M^n [5].

3. Material and methods

Let the $n-$dimensional hypersurface M be given by isometric immersion $x : M \to \mathbb{R}^{n+1}$. A family of differentiable immersions is defined as

$$x(p,t) : M \to \mathbb{R}^{n+1}, x(p,0) = x(p).$$

In this case, the mean curvature vector at the point $x(p,t)$ of the hypersurface $M_t = x(M,t)$ becomes $\overrightarrow{H}(t) = \overrightarrow{H}(p,t)$. If the following equality holds, this family is called the*average mean curvature flow* ([3]);

$$\left(\frac{\partial}{\partial t} x_t(p)\right)^{\perp} = H(p,t),\ x_0 = x \tag{3.10}$$

where $\upsilon^{\perp}$ denotes the projection of υ into the normal space of $x_t(M)$. The mean curvature flow is also considered in [31], [26] and [30]. For higher dimensional case see [13] and [32]. In addition, the mean curvature flow of entire graphs was analyzed in [16]. See, also [25] for lecture notes on mean curvature flow.

Definition 3.1. An immersed hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is called self-similar solution of (3.10) if the curvature vector field $\overrightarrow{H}$ of M satisfies the following non-linear elliptic system:

$$\overrightarrow{H} + \lambda x^N = 0 \tag{3.11}$$

where x^N is the normal component of x and λ is a real valued function. It is called self-shrinker if $\lambda = 1$, and self-expander if $\lambda = -1$. The case of vanishing λ is the well-known case of a minimal surface, which of course is stationary under the action of the flow [20].

A classification and analysis of low index mean curvature flow and self-shrinker are considered in [22]. See also [27] for complete self-shrinkers. Recently self-similar solution of surfaces has been considered in [17]. A survey of closed self-shrinkers with symmetry can be seen in [15]. See also [19] for some results on self-shrinkers and singularity model of the mean curvature flow.

Let us denote $H = \left\| \overrightarrow{H} \right\|$ and N the mean curvature and unit normal vector of the hypersurface $M \subset \mathbb{R}^{n+1}$ respectively. So, multiplying both sides of the equation (3.11) by N we obtain

$$H + \lambda \left\langle x^N, x \right\rangle = 0. \tag{3.12}$$

A similar definition is given below in the case of $\lambda = 1$.

Definition 3.2. Let M be an $n-$dimensional smooth hypersurface in $\mathbb{R}^{n+1}$ given with the isometric immersion $x : M \to \mathbb{R}^{n+1}$. If the equality

$$H + \left\langle x^N, x \right\rangle = 0 \tag{3.13}$$

holds, M is called a self-shrinker hypersurface [11]. Here $\langle , \rangle$ is the standard inner product of $\mathbb{R}^{n+1}$.

Self-shrinking hypersurfaces play an important role in studies on average curvature flux. These define all possible breaks at a given singular point of curvature flux [12]. If $M = \Gamma \subset \mathbb{R}^2$ is a curve then the all solutions of (3.11) have been classified by Abresch and Langer in [1]. Except the straight lines passing the origin, the curvature κ is positive for all of them. In higher dimension any self-shrinker curve $\gamma \subset \mathbb{R}^n$ lies in a flat linear two-space $\mathbb{E}^2 \subset \mathbb{R}^n$and coincides with one of the Abresch$-$Langer curves Γ in $\mathbb{E}^2$, because then (3.10) becomes an ODE of order 2. Abresch and Langer proved that all the differentiable and closed self-shrinking curves in $\mathbb{E}^2$ were circles.

Definition 3.3. A family of differentiable immersions is defined as $x(.,t) : M \to \mathbb{R}^{n+1}$, $x(.,0) = x(.)$. The average curvature flux defined in the form

$$\frac{\partial x(t)}{\partial t} = -\alpha(t)N(t) + \vec{H}(t), x(t) = x(.,t) \tag{3.14}$$

is called the average curvature flux that maintains the weighted volume. Here

$$\alpha(t) = \frac{\int\limits_M H(t) \langle N(t), N \rangle e^{-\frac{\|x\|^2}{2}} d\mu}{\int\limits_M \langle N(t), N \rangle e^{-\frac{\|x\|^2}{2}} d\mu}, \tag{3.15}$$

$\vec{H}(t) = \vec{H}(.,t)$ and $N(t)$ are the mean curvature vector the normal vector of $M_t = x(M,t)$ respectively and N is the unit normal vector of M (see, [11] and [9]).

The average curvature flux given with (3.14) preserves the weighted volume given by its equation

$$V(t) = \int\limits_M \langle x(t), N \rangle e^{-\frac{\|x\|^2}{2}} d\mu, \tag{3.16}$$

where $d\mu$ is the outer volume element defined by (see, [12]);

$$d\mu = \sqrt{\det(g_{ij})}, g_{ij} = \left\langle \frac{\partial x}{\partial x_i}, \frac{\partial x}{\partial x_j} \right\rangle. \tag{3.17}$$

However, the weighted area functional is defined by

$$A(t) = \int\limits_M e^{-\frac{\|x\|^2}{2}} d\mu_t. \tag{3.18}$$

Here, the function $d\mu_t$ is the area element of M, which is reduced by the metric with the help of $x(.,t)$. So, $V(t)$ is constant for each t, being a family (variation) of differentiable immersions defined as $x(.,t) : M \to \mathbb{R}^{n+1}$, $x(.,0) = x(.)$. The $x(.,t)$ family is said to be a family of $x(.)$ that preserves the weighted volume. The necessary and sufficient condition for each family $x(.)$ that preserves the weighted volume to be a critical point of the weighted area functional $A(t)$ is the equality

$$H + < x, N >= \lambda, \tag{3.19}$$

holds identically. Here H is the mean curvature of M and λ is a real constant. If the equation (3.19) yields, M is called a (*proper*)$\lambda-$*hypersurface* with $\lambda \neq 0$. For the case $\lambda = 0$, M is self-shrinking [11]. A rigidity results of $\lambda-$hypersurfaces have given in [29] and [10]. In literature one can find the following well-known examples (see, [11]);

Example 3.1. The n-dimensional sphere $S^n(r)$ with radius r is a compact λ-hypersurface of $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$.

Example 3.2. The hypercylinder $S^k(r) \times \mathbb{R}^{n-k}$,$1 \leq k \leq n-1$ of dimension n is a complete and non-compact $\lambda-$hypersurface of $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$

Example 3.3. $\mathbb{R}^n$ is a $n-$dimensional complete, and non-compact $\lambda-$hypersurface of $\mathbb{R}^{n+1}$ with $\lambda = 0$.

Proposition 3.1. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface given with constant mean curvature. If M is a $\lambda-$hypersurface then it is isometric to a part of a sphere $S^n(\sqrt{n})$ with radius $r = \sqrt{n}$.*

A generalization of the $\lambda-$hypersurface is given in the following definition.

Definition 3.4. Let M be a hypersurface given with the isometric immersion $x : M \to \mathbb{R}^{n+1}$. If the mean curvature H and the the unit normal vector N of M satisfy the equality

$$H + w < x, \vec{N} >= \lambda, \tag{3.20}$$

then M is called a $\lambda-$hypersurface corresponding to the weight function w [24].

In the case of $w = 0$, hypersurfaces with constant curvature are obtained. In case of $w = c$ and $\lambda = 0$, M hypersurfaces is self-shrinker. In the case of $w = -1/2$ and $\lambda = 0$, it was concluded in [21] that the compact self-shrinking hypersurfaces with non-negative mean curvature consist of $x(M) = S^n(\sqrt{n})$, for $n \geq 2$. In the same study, the following result has been proved.

Theorem 3.1. [21] *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface given with the mean curvature $H > 0$. If M satisfies the condition*

$$H - \frac{1}{2} < x, \vec{N} >= 0 \tag{3.21}$$

then M is equal to one of the following;
(1) S^n,
(2) $S^{n-m} \times \mathbb{R}^m$,
(3) $\Gamma \times \mathbb{R}^{n-1}$,
where, Γ is a Abresch-Langer curve.

In [23] Kim and Pyo gave the following definition.

Definition 3.5. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with the isometric immersion $x : M \to \mathbb{R}^{n+1}$. If the mean curvature vector $\overrightarrow{H}$ and the the unit normal vector N of M satisfy the equality

$$\left\langle \overrightarrow{H}, x \right\rangle = -\frac{1}{c}, \tag{3.22}$$

then M is called a homothetic soliton. Here c is a nonzero constant. Homothetic soliton is a self-similar solution of mean curvature flux.

The following results are due to [8];

Definition 3.6. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with the isometric immersion $x : M \to \mathbb{R}^{n+1}$. Given any any smooth vector field $Z \in T(M)$ if the divergent of Z vanishes identically, i.e. $div(Z) = 0$, then Z is called an incompressible vector field [8].

Theorem 3.2. [8] *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with the isometric immersion $x : M \to \mathbb{R}^{n+1}$ then the tangential component x^T of the position vector x is an incompressible vector field if and only if*

$$\left\langle \overrightarrow{H}, x \right\rangle = -1 \tag{3.23}$$

holds identically.

As a consequence of Definition 3.6 with Theorem 3.2 one can get the following result.

Corollary 3.1. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with the isometric immersion $x : M \to \mathbb{R}^{n+1}$. If the tangential component x^T of the position vector x is an incompressible vector field then the hypersurface M is a homothetic soliton with $c = 1$.*

4. Results

Rotational hypersurfaces are one of the important issues of modern differential geometry. These hypersurfaces are widely used in $\mathbb{R}^3$ especially in computer aided geometric design and surface modeling. In the present section, we consider rotational $\lambda-$hypersurfaces in Euclidean spaces.

4.1. **Rotational $\lambda-$surfaces in $\mathbb{R}^3$.**

A *rotational surface M* in $\mathbb{R}^3$ is defined by the parametrization

$$x(u, v) = (f(u), g(u)\cos v, g(u)\sin v), \tag{4.24}$$

where $u \in J, 0 \leq v < 2\pi$,and $\alpha(u) = (f(u), g(u))$ is the meridian curve of the rotation [18]. The orthonormal frame field of M is given by

$$\begin{aligned} e_1 &= \frac{1}{\varphi(u)}\frac{\partial}{\partial u} \\ e_2 &= \frac{1}{g(u)}\frac{\partial}{\partial v} \\ e_3 &= \frac{1}{\varphi(u)}(g'(u), -f'(u)\cos v, -f'(u)\sin v) \end{aligned} \tag{4.25}$$

where

$$\varphi(u) = \sqrt{(f'(u))^2 + (g'(u))^2}, \tag{4.26}$$

is the smooth function on M [4]. With respect to this frame we can obtain the second fundamental maps;

$$\begin{aligned} h(e_1, e_1) &= \frac{\kappa}{\varphi^3}e_3 \\ h(e_1, e_2) &= 0 \\ h(e_2, e_2) &= \frac{f'}{g\varphi}e_3 \end{aligned} \tag{4.27}$$

where

$$\kappa = f''g' - f'g'' \tag{4.28}$$

is the smooth function on M.

Consequently, by the use of (2.8) with (4.27) the mean curvature vector $\overrightarrow{H}$ of M becomes

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2}\{h(e_1,e_1)+h(e_2,e_2)\} \\ &= \frac{1}{2\varphi}\left(\frac{\kappa}{\varphi^2}+\frac{f'}{g}\right)e_3. \end{aligned} \tag{4.29}$$

From the orthogonal decomposition (1.1) of the position vector x of M we obtain

$$x^N = x - \rho'(u)e_1 \tag{4.30}$$

where $\rho(u) = \frac{1}{2}\|x\|^2$ is the square norm of the distance function of the position vector x such that

$$\rho'(u) = f(u)f'(u) + g(u)g'(u). \tag{4.31}$$

The gradient of the distance function is given by

$$\text{grad}\,(\|x\|) = \sum_{j=1}^{2}\frac{\langle x, e_j\rangle}{\|x\|}e_j = \frac{\rho'(u)}{\|x\|}e_1 \tag{4.32}$$

Due to [6] we obtain the following results.

Theorem 4.3. *Let M be a general rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). Then $x = x^N$ holds identically if and only if M is a spherical surface of $\mathbb{R}^3$.*

Proof. Assume that M is a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). If $x = x^N$ holds identically, then $\rho'(u) = 0$ holds. So, the equation (4.32) yields the the distance function of M has zero gradient so by Example 4.1. of [7] M is a spherical surface in $\mathbb{R}^3$. Infact, $f_1(u)f_1'(u)+f_2(u)f_2'(u) = 0$, i.e. $f_1^2(u)+f_1^2(u) = r_0^2$ implies that the meridian curve α is an open part of a circle parametrized by

$$f_1(u) = r_0\cos\left(\frac{u}{r_0}\right), f_2(u) = r_0\sin\left(\frac{u}{r_0}\right), \tag{4.33}$$

where r_0 is a positive real number.

The converse is clear. □

Theorem 4.4. *Let M be a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). Then $x = x^T$ holds identically if and only if M is a conic surface with the vertex at the origin.*

Proof. Assume that M is a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). If $x = x^T$ holds identically, then $x = \rho'(u)e_1$ holds identically. So, the equation (4.32) yields that gradient of the distance function has constant length

$$\|\text{grad}\,(\|x\|)\| = \frac{|\rho'(u)|}{\|x\|} = 1. \tag{4.34}$$

So by Proposition 5.2 of [7] M is a conic surface in $\mathbb{R}^3$ with the vertex at the origin. Infact, $x = \rho'(u)e_1$ yields $f'g - fg' = 0$. Consequently the meridian curve α is an open part of a straight line passing through origin.

The converse is clear. □

However, the inner product of e_3 with the position vector x gives

$$\langle e_3, x\rangle = \frac{f(u)g'(u) - g(u)f'(u)}{\varphi}. \tag{4.35}$$

We obtain the following result;

Theorem 4.5. *Let M be a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24).Then M is a homothetic soliton if and only if*

$$c(\kappa g + f'\varphi^2)\delta + 2g\varphi^4 = 0 \tag{4.36}$$

holds identically. Here

$$\delta = f(u)g'(u) - g(u)f'(u) \tag{4.37}$$

and φ, κ are differentiable functions defined in the equations (4.26) and (4.28), respectively.

Proof. Let $M \subset \mathbb{R}^3$ be a rotational surface given with the parametrization (4.24). If M is a homothetic soliton, then

$$\left\langle \overrightarrow{H}, x \right\rangle = H \left\langle e_3, x \right\rangle = -\frac{1}{c}.$$

holds. Thus, with the help of equations (4.29) and (4.35) we get (4.36).

The proof of the converse statement is obvious. □

We give the following examples;

Example 4.4. Every sphere $S^2(r) \subset \mathbb{R}^3$ is a homothetic soliton with $c = 1$.

Example 4.5. Every cylinder given with the meridian curve $\alpha(u) = (au + b, d)$ is a homothetic soliton with $c = 2$.

As a consequence of the equations (4.25), (4.29), and (3.20) we get the following results;

Theorem 4.6. *Let M be a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). Then M is a $\lambda-$surface corresponding to the weight function w if and only if*

$$w = \frac{-(kg + f'\varphi^2) + 2\lambda g\varphi^3}{2g\delta\varphi^2})$$

holds identically.

Theorem 4.7. *Let M be a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). Then M is a $\lambda-$surface if and only if*

$$kg + f'\varphi^2 + 2g\delta\varphi^2 - 2\lambda g\varphi^3 = 0 \tag{4.38}$$

holds identically. Here φ, κ, δ are smooth functions defined in (4.26),(4.28) and (4.37) respectively.

Thus, in the case of $\lambda = 0$, a result of Theorem 4.6 is given below.

Corollary 4.2. *Let M be a rotational surface in $\mathbb{R}^3$ given with the parametrization (4.24). Then M is a self-shrinker if and only if*

$$kg + f'\varphi^2 + 2g\delta\varphi^2 = 0$$

holds identically.

We give the following examples;

Example 4.6. Let $f(u) = a$, $g(u) = bu + c$. In this case, $\lambda = a$ is obtained. This surface specifies a plane and is self-shrinking for $a = 0$.

Example 4.7. Let $f(u) = au + b$, $g(u) = c$. In this case, $\varphi = a$, $\delta = -ac$, $\kappa = 0$. Thus $\lambda = \frac{1-2c^2}{2c}$ is obtained. This surface specifies a cylinder and is self-shrinking for the value $c = \pm\frac{1}{\sqrt{2}}$.

Example 4.8. Let $f(u) = au+b$, $g(u) = cu+d$. In this case if the equation (4.38) is provided then

$$\lambda = \frac{a + 2(cu+d)(bc-ad)}{2(cu+d)\sqrt{a^2+c^2}}$$

holds. Thus, the following situations are hold;

(1) If $a = 0$, then the resultant surface specifies a plane given in Example 4.6.

(2) If $c = 0$, the resultant surface specifies a cylinder given in Example 4.7.

(3) If $a \neq 0$ and $c \neq 0$, the resultant surface specifies a cone which is not a $\lambda-$surface.

4.2. **Rotational $\lambda-$hypersurfaces in $\mathbb{R}^4$.** Let $M \subset \mathbb{R}^4$ be a rotational hypersurface given with the regular coordinate patch

$$x(s,u,v) = (f(s), g(s)\sin u, g(s)\cos u \sin v, g(s)\cos u\cos v) \tag{4.39}$$

where $\gamma(s) = (f(s), g(s))$ is a regular curve. For the rotational hypersurfaces with constant curvature see for example [14].

The orthonormal frame field of M is given by

$$\begin{aligned} \vec{e}_1 &= \frac{1}{\varphi}\frac{\partial x}{\partial s}, & \vec{e}_2 &= \frac{1}{g}\frac{\partial x}{\partial u}, \\ \vec{e}_3 &= \frac{1}{g\cos u}\frac{\partial x}{\partial v}, & \vec{e}_4 &= \frac{1}{\varphi}(g', -f'\sin u, -f'\cos u\sin v, -f'\cos u\cos v) \end{aligned} \tag{4.40}$$

where φ is differentiable function defined in the equation (4.26). With respect to this frame we can obtain the second fundamental maps;

$$\begin{aligned} h(e_1,e_1) &= \frac{\kappa}{\varphi^3}\vec{e}_4, \quad h(e_2,e_2) = \frac{f'}{\varphi g}\vec{e}_4 \\ h(e_3,e_3) &= \frac{f'}{\varphi g}\vec{e}_4, \quad h(e_1,e_2) = h(e_1,e_3) = h(e_2,e_3) = 0. \end{aligned} \tag{4.41}$$

Consequently, by the use of (2.8) with (4.41) the mean curvature vector $\overrightarrow{H}$ of M becomes

$$\overrightarrow{H} = \left(\frac{\kappa g + 2\varphi^2 f'}{3g\varphi^3}\right)\vec{e}_4, \tag{4.42}$$

where φ, κ are differentiable functions defined in the equations (4.26) and (4.28), respectively

However, the inner product of $\vec{e}_4$ with the position vector x gives

$$\langle x, e_4\rangle = \frac{\delta}{\varphi} \tag{4.43}$$

where δ is the smooth function defined in (4.37).

Consequently, by taking account of (4.39)-(4.43) with (3.22) and (3.19) we obtain the following results;

Theorem 4.8. *Let $M \subset \mathbb{R}^4$ be a rotational hypersurface given with the parametrization (4.39).Then M is a homothetic soliton if and only if*

$$c(\kappa g + 2f'\varphi^2)\delta + 3g\varphi^4 = 0 \tag{4.44}$$

holds identically.

Theorem 4.9. *Let M be a rotational hypersurface in $\mathbb{R}^4$ given with the parametrization (4.39). Then M is a $\lambda-$hypersurface if and only if*

$$kg + 2f'\varphi^2 + 3g\delta\varphi^2 - 3\lambda g\varphi^3 = 0 \tag{4.45}$$

holds identically. Here φ, κ, δ are smooth functions defined in (4.26),(4.28) and (4.37) respectively.

We give the following examples;

Example 4.9. Let $M \subset \mathbb{R}^4$ be a rotational hypersurface given by $f(s) = \cos s$ and $g(s) = \sin s$. In this case, the hypersurface given with the patch

$$x(s,u,v) = (\cos s, \sin s \sin u, \sin s \cos u \sin v, \sin s \cos u \cos v)$$

represents a hypersphere $S^3(1)$ which is self-shrinking i.e., $\lambda = 0$.

Example 4.10. Let $M \subset \mathbb{R}^4$ be a rotational hypersurface given by $f(s) = as + b$ and $g(s) = r_0$, $0 \neq r_0 \in \mathbb{R}$. In this case, the hypersurface given with the patch

$$x(s,u,v) = (as + b, r_0 \sin u, r_0 \cos u \sin v, r_0 \cos u \cos v)$$

represents a circular hypercylinder $S^2(r_0) \times \mathbb{R}$ which is a $\lambda-$hypersurface with $\lambda = \frac{2-3r_0^2}{3r_0}$.

Example 4.11. Let $M \subset \mathbb{R}^4$ be a rotational hypersurface given by $f(s) = b$ and $g(s) = cs + d$, $b, c, d \in \mathbb{R}$. In this case, the hypersurface given with patch

$$x(s,u,v) = (b, (cs + d) \sin u, (cs + d) \cos u \sin v, (cs + d) \cos u \cos v)$$

represents a part of a plane which is a $\lambda-$hypersurface with $\lambda = b$.

4.3. **Birotational $\lambda-$hypersurfaces in $\mathbb{R}^4$.** A *birotational hypersurface* M in $\mathbb{R}^4$ is defined by the parametrization (see, [15]);

$$x(s,u,v) = (f(s) \cos u, f(s) \sin u, g(s) \cos v, g(s) \sin v) \tag{4.46}$$

where $\gamma(s) = (f(s), g(s))$ is a regular curve in $\mathbb{R}^2$. The orthonormal frame field of M is given by

$$\begin{aligned} \vec{e}_1 &= \frac{1}{\varphi}\frac{\partial x}{\partial s}, \quad \vec{e}_2 = \frac{1}{f}\frac{\partial x}{\partial u}, \\ \vec{e}_3 &= \frac{1}{g}\frac{\partial x}{\partial v}, \quad \vec{e}_4 = \frac{1}{\varphi}(g' \cos u, g' \sin u, -f' \cos v, -f' \sin v) \end{aligned} \tag{4.47}$$

where φ is differentiable function defined in the equation (4.26). With respect to this frame we can obtain the second fundamental maps;

$$\begin{aligned} h(e_1, e_1) &= \frac{\kappa}{\varphi^3}\vec{e}_4, \quad h(e_2, e_2) = -\frac{g'}{\varphi f}\vec{e}_4 \\ h(e_3, e_3) &= \frac{f'}{\varphi g}\vec{e}_4, \quad h(e_1, e_2) = h(e_1, e_3) = h(e_2, e_3) = 0. \end{aligned} \tag{4.48}$$

Consequently, by the use of (2.8) with (4.48) the mean curvature vector $\overrightarrow{H}$ of M becomes

$$\overrightarrow{H} = \left(\frac{\kappa f g + \varphi^2 \mu}{3 f g \varphi^3} \right) \vec{e}_4, \tag{4.49}$$

where φ, κ are differentiable functions defined in the equations (4.26) and (4.28), respectively and

$$\mu = f f' - g g'. \tag{4.50}$$

In [15] one can see the following example;

Example 4.12. Let the meridian curve be a circle of radius r. In this case, M is a birotational hypersurface with parameterization

$$x(s,u,v) = (r \cos s \cos u, r \cos s \sin u, r \sin s \cos v, r \sin s \sin v).$$

An easy computation gives that the mean curvature of M is $H = -\frac{1}{3r}$.

Further, the inner product of $\vec{e}_4$ with the position vector x gives $\langle x, e_4\rangle = \frac{\delta}{\varphi}$ where δ is the smooth function defined in (4.37).

Consequently, by taking account of (4.46)$-$(4.50) with (3.22) and (3.19) we obtain the following results;

Theorem 4.10. *Let $M \subset \mathbb{R}^4$ be a birotational hypersurface given with the parametrization (4.46).Then M is a homothetic soliton if and only if*

$$c(\kappa fg + \mu\varphi^2)\delta + 3fg\varphi^4 = 0 \tag{4.51}$$

holds identically. Here $\varphi, \kappa, \delta, \mu$ are smooth functions defined in (4.26),(4.28), (4.37) and (4.49) respectively.

Theorem 4.11. *Let $M \subset \mathbb{R}^4$ be a birotational hypersurface in $\mathbb{R}^4$ given with the parametrization (4.46). Then M is a $\lambda-$hypersurface if and only if*

$$kfg + \varphi^2(\mu + 3fg\delta - 3\lambda fg\varphi) = 0 \tag{4.52}$$

holds identically.

Corollary 4.3. *Let $M \subset \mathbb{R}^4$ be a birotational hypersurface in $\mathbb{R}^4$ given with the parametrization (4.46). Then M is a self-shrinker if and only if*

$$kfg + \varphi^2(\mu + 3fg\delta) = 0 \tag{4.53}$$

holds identically.

We give the following example;

Example 4.13. Let $M \subset \mathbb{R}^4$ be a birotational hypersurface whose meridian curve is a line passing through the origin as $f(s) = g(s)$. In this case, the hypersurface given with the patch

$$x(s, u, v) = f(s)(\cos u, \sin u, \cos v, \sin v)$$

represents a minimal surface in $\mathbb{R}^4$. Especially when $\gamma(s) = (s, s)$ this hypersurface given with

$$x(s, u, v) = (s\cos u, s\sin u, s\cos v, s\sin v)$$

is self-shrinking and minimal Clifford cone [15].

5. Conclusion

The mean curvature flow is a kind of gradient flow of a given hypersurface. However, the self-similar flows arise as special solution of the mean curvature flow that preserve the shape of the evolving hypersurface. Self-shrinkers are the special version of self-similar flows. In addition, $\lambda-$hypersurfaces are the generalization of self-shrinkers. In this study, we obtain some results of rotation $\lambda-$hypersurfaces in Euclidean spaces. We have shown that the spherical and cylindrical rotation hypersurfaces are self-shrinkers. We hoped that this study will contribute to the mean curvature flow calculations of high dimensional rotational submanifolds.

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