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Energy and skew-energy of a modified graph

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ABSTRACT. Graph energies draw the greater attention of the scientific community due to their direct applicability in molecular chemistry. In this paper, we establish the energy of a graph obtained by the means of some graph operations. The energy of the product of graph $K_n \times G$, where K_n is a complete graph and G is a simple undirected graph and energy of the corresponding digraph are estimated. Further, the duplication graph DG is considered and proved that the energy $\mathcal{E}(DG) = 2\mathcal{E}(G)$ and $\mathcal{E}(DG^\sigma) = 2\mathcal{E}(G^\sigma)$.

1. INTRODUCTION

A graph G is an ordered pair (V, E), where $\mid V(G) \mid = m$. Let $A(G)$ be its an adjacency matrix of order m. The characteristic polynomial of a graph G is denoted by $Ch(G;\lambda) = det(\lambda I - G)$, where λ is an eigenvalue of a graph G. The spectrum of G denoted by $Spec(G)$ and is the set of all eigenvalues of G with their multiplicity. In 1978, I. Gutman introduced the concept of energy of a graph [4], the energy of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{m} \mid \lambda_i \mid$. In graph theory, one consistently tries to generate new graphs from a given graph. In this section, we consider some newly generated graphs from a given graph and we check out energy for them.
In 2010, A. Dilek Gungor et al., [2] investigated the Harary energy and Harary Estrada index of a graph and some bounds for the Kirchhoff index of a connected (molecular) graph are reported in [3]. Kinkar Ch. Das et al., [6, 7] investigated some bounds for Laplacian energy and Laplacian-energy-like invariant. The concept of skew energy was introduced by Adiga et al., [1] in 2010 and we also also refere [8,10] for skew energy. The definition of a digraph is utilized to compute the skew energy of some graphs.

The following definition will be needed in our results in the coming sections.

Definition 1.1 ([1]). Let G^σ be a directed graph of order m with the vertex set $V(G^\sigma)$ and the arc set $\Gamma(G^\sigma) \subset V(G^\sigma) \times V(G^\sigma)$. The skew adjacent matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(G^\sigma)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(G^\sigma)$ and $s_{ij} = 0$ otherwise.

Definition 1.2 ([5]). The Cartesian product (denoted by $G_1 \times G_2$) of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has the vertex-set $V(G_1) \times V(G_2)$. For any $u, v \in V(G_1)$ and $x, y \in V(G_2)$, (u, x) is adjacent to (v, y) if either $u = v$ and $xy \in E(G_2)$ or $uv \in E(G_1)$ and $x = y$ and $\mid E(G_1 \times G_2) \mid = \mid V_1 \mid . \mid E_2 \mid + \mid V_2 \mid . \mid E_1 \mid$.

This article is organized as follows. Section 1 consists of the introduction and basic definitions required for the development of the main results. In section 2 the energy of the graph $K_n \times G$ and the energy of the corresponding digraph G^σ is evaluated. In section 3 the energy of a duplication graph DG and digraph DG^σ are computed.

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2. Energy of the graph $K_n \times G$

In this section, the energy of $K_n \times G$ is calculated, where K_n is a complete graph and G is a simple undirected graph and the energy of corresponding digraph G^σ is also calculated.

Theorem 2.1. *Let $G = (V, E)$ be a simple graph with $\mid V \mid = m$. Let λ_j, $j = 1, 2, \ldots, m$ be eigenvalues of G. Then*

$$\mathcal{E}(K_n \times G) = \sum_{j=1}^{m} (n-1) \mid \lambda_j - 1 \mid + \mid \lambda_j + n - 1 \mid .$$

Proof. Let $V(K_n) = \{u_1, \ldots, u_n\}$ and $V(G) = \{v_1, \ldots, v_m\}$ then adjacency matrix of G is given by

$$A(G) = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & 0 & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & 0 & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & 0 \end{pmatrix}.$$

Now, $V(K_n \times G) = w_i =$

$$\{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_m),$$

$$(u_2, v_1), (u_2, v_2), \ldots, (u_2, v_m),$$

$$\vdots$$

$$(u_n, v_1), (u_n, v_2), \ldots, (u_n, v_m)\}.$$

Then $A(K_n \times G)$ can be written as a block matrix as given below,

$$A(K_n \times G) = \begin{pmatrix} A(G) & I_m & I_m & \cdots & I_m & I_m \\ I_m & A(G) & I_m & \cdots & I_m & I_m \\ I_m & I_m & A(G) & \cdots & I_m & I_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ I_m & I_m & I_m & \cdots & I_m & A(G) \end{pmatrix}.$$

Where I_m is the identity matrix of order m.
By the laws of matrix algebra, $\lambda_j - 1$ ($n-1$ times) and $\lambda_j + n - 1$ are eigenvalues of matrix $A(K_n \times G)$,where λ_j, $j = 1, 2, \ldots, m$ are the eigenvalues of G. Hence the energy of $A(K_n \times G)$ is given by

$$\mathcal{E}(K_n \times G) = \sum_{j=1}^{m} (n-1) \mid \lambda_j - 1 \mid + \mid \lambda_j + n - 1 \mid .$$

□

Illustration 1 : Consider the cycle C_3 and the cartesian product graph $K_2 \times C_3$ with $V(K_2 \times C_3) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. and $A(K_2 \times C_3)$ be its adjacency matrix.

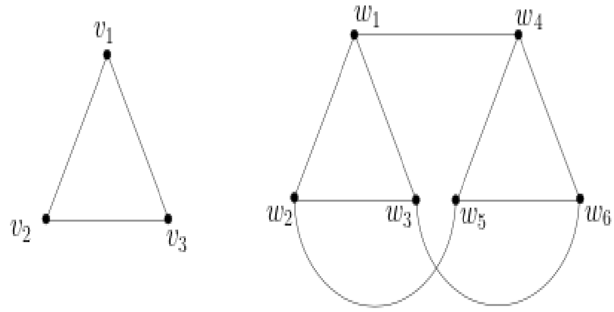

FIGURE 1. The graph of C_3 and $K_2 \times C_3$

$$A(K_2 \times C_3) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

$$Spec(K_2 \times C_3) = \begin{pmatrix} -2 & 0 & 1 & 3 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{E}(K_2 \times C_3) = 8$$

$$A(C_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } A(K_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues of C_3 are -1, -1, 2 and eigenvalues of K_2 are 1, -1. Therefore by using Theorem 2.1, the eigenvalues of $K_2 \times C_3$ are 0, 0, -2, -2, 3, 1 and the energy is $\mathcal{E}(K_2 \times C_3) = 8$. Hence the result is verified. In Figure 1, the graph of $K_2 \times C_3$ is shown.

Theorem 2.2. *Let G^σ be the oriented graph on m vertices and K_n^σ be the oriented complete graph on n vertices with the orientations of all the arcs go from low labels to high labels. Let $i\lambda_j$, $j = 1, 2, \ldots, m$ and $i = \sqrt{-1}$ be the eigenvalues of G^σ. Then*

$$\mathcal{E}(K_n^\sigma \times G^\sigma) = \sum_{j=1}^{m} \mid (\lambda_j - cot(2k+1)\frac{\pi}{2n}) \mid, \quad k = 0, 1, \ldots, n-1.$$

Proof. Let $V(K_n^\sigma) = \{u_1, \ldots, u_n\}$ and $V(G^\sigma) = \{v_1, \ldots, v_m\}$ then adjacency matrix of G^σ (by using the definition of digraph) is given by

$$A(G^\sigma) = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ -a_{21} & 0 & a_{23} & \cdots & a_{2m} \\ -a_{31} & -a_{32} & 0 & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & 0 \end{pmatrix}.$$

Then $A(K_n^\sigma \times G^\sigma)$ can be written as a block matrix as given below

$$A(K_n^\sigma \times G^\sigma) = \begin{pmatrix} A(G^\sigma) & I_m & I_m & \cdots & I_m & I_m \\ -I_m & A(G^\sigma) & I_m & \cdots & I_m & I_m \\ -I_m & -I_m & A(G^\sigma) & \cdots & I_m & I_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -I_m & -I_m & -I_m & \cdots & -I_m & A(G^\sigma) \end{pmatrix}.$$

By using the result in [5], the eigenvalues of the matrix are given by $i(\lambda_j - cot(2k+1)\frac{\pi}{2n}) \mid$, where $k = 0, 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, m$ and $i = \sqrt{-1}$.

$$\mathcal{E}(K_n^\sigma \times G^\sigma) = \sum_{j=1}^{m} \mid (\lambda_j - cot(2k+1)\frac{\pi}{2n}) \mid .$$

□

Illustration 2: Consider the cycle C_3^σ(all the arcs go from low labels to high labels) and the cartesian product $K_2^\sigma \times C_3^\sigma$.

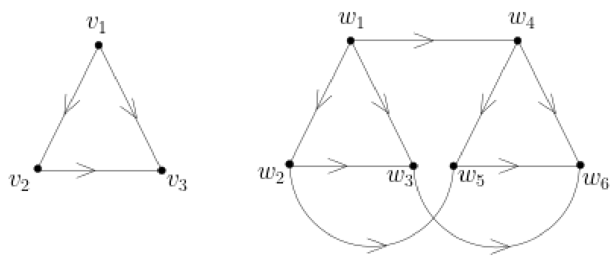

FIGURE 2. The graph of C_3^σ and $K_2^\sigma \times C_3^\sigma$

Adjacency matrix of $K_2^\sigma \times C_3^\sigma$ is given by

$$A(K_2^\sigma \times C_3^\sigma) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

$$Spec(K_2^\sigma \times C_3^\sigma) = \begin{pmatrix} i & -i & 2.7321i & -2.7321i & 0.7321i & -0.7321i \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{E}(K_2^\sigma \times C_3^\sigma) = 8.92884.$$

Now,

$$A(C_3^\sigma) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \text{ and } A(K_2^\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of C_3^σ are $\pm 1.7321i$, 0 and eigenvalues of K_2^σ are $\pm i$ therefore by using Theorem 2.2, the eigenvalues of $K_2^\sigma \times C_3^\sigma$ are i(1.7321-1), i(1.7321+1), i(-1.7321-1), i(-1.7321+1), i, -i. and the energy is $\mathcal{E}(K_2^\sigma \times C_3^\sigma) = 8.92884$. Hence the result is verified. In Figure 2, the graph of $K_2^\sigma \times C_3^\sigma$ is shown.

3. Energy of a duplication graph

In [9] S. K. Vaidya et al., has estimated the energy of a splitting graph and a shadow graph. In this section, we reckon the energy of a duplication graph and its corresponding digraph.

Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then the tensor product (or Kronecker product) of A and B is defined as the matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Proposition 3.1 ([5]). *Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$ and let λ be the eigenvalue of matrix A with corresponding eigenvector x and μ be the eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Definition 3.3. Let G be a graph with $V(G) = \{v_1, \dots, v_m\}$. Take another set $U = \{u_1, \dots, u_m\}$. Make u_j adjacent to all the vertices in $N(v_j)$ in G for each j and remove edges of G only. The resulting graph H is called the duplication graph of G denoted by DG.

Theorem 3.3. *Let $V(G) = \{v_1, \dots, v_m\}$ be the vertex set of a graph G and $A(G)$ be its adjacency matrix. Then $\mathcal{E}(DG) = 2\mathcal{E}(G)$.*

Proof. The adjacency matrix of G is given by

$$A(G) = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & 0 & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & 0 & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & 0 \end{pmatrix}.$$

Let $U =\{u_1, \dots, u_m\}$. Make u_j adjacent to all the vertices in $N(v_j)$ in G for each j and remove edges of G only. We get duplication graph DG with $V(DG) = \{v_1, v_2, \dots, v_m, u_1, \dots, u_m\}$. The adjacency matrix of duplication graph DG is given by

$$A(DG^{\sigma}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & 0 & \cdots & 0 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & 0 & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \\ 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} & 0 & 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & -a_{23} & \cdots & -a_{2n} & 0 & 0 & 0 & \cdots & 0 \\ -a_{31} & -a_{32} & 0 & \cdots & -a_{3n} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$A(DG) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & 0 & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & 0 & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & 0 \end{pmatrix}$$

$$Spec(DG) = \begin{pmatrix} -\lambda_j & \lambda_j \\ m & m \end{pmatrix},$$

where λ_j, $j = 1, 2, ..., m$ are the eigenvalues of G while $\pm$ 1 are the eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\mathcal{E}(DG) = \sum_{j=1}^{m} \mid (\pm 1)\lambda_j \mid = 2\sum_{j=1}^{m} \mid \lambda_j \mid = 2\mathcal{E}(G).$$

□

Illustration 3: Consider the cycle C_4 and its duplication graph DC_4 with vertex set $V(DC_4) = \{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$.

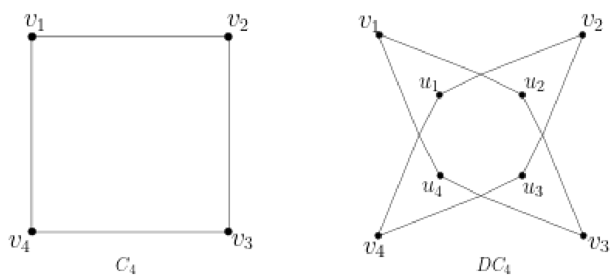

FIGURE 3. The duplication graph of C_4

Adjacency matrix of DC_4 is given by

$$A(DC_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$Spec(DC_4) = \begin{pmatrix} -2 & 0 & 2 \\ 2 & 4 & 2 \end{pmatrix} \text{ and } Spec(C_4) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{E}(DC_4) = 8 \text{ and } \mathcal{E}(C_4) = 4.$$

Hence

$$\mathcal{E}(DC_4) = 2\mathcal{E}(C_4)$$

Corollary 3.1. *Let $V(G^\sigma) = \{v_1, \ldots, v_m\}$ be the vertex set of a graph G^σ with arc set $\Gamma(G^\sigma)$ and $A(G^\sigma)$ be its adjacency matrix. Then $\mathcal{E}(DG^\sigma) = 2\mathcal{E}(G^\sigma)$.*

Illustration 4 : Consider the directed cycle C_4^σ (all the arcs in clockwise direction) and Let $V(DC_4^\sigma) = \{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$ be the vertex set of duplication graph DC_4^σ .

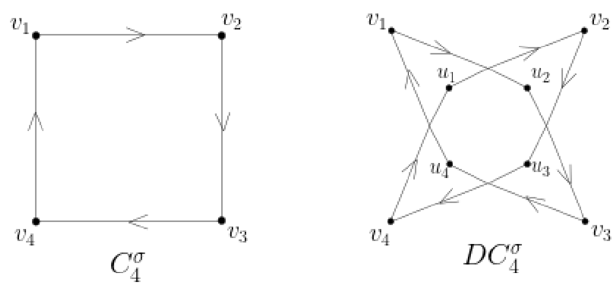

FIGURE 4. Duplication graph of digraph C_4^σ

Adjacency matrix of DC_4^σ is given by

$$A(DC_4^\sigma) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A(DG^\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes C_4^\sigma$$

$$Spec(DC_4^\sigma) = \begin{pmatrix} 0 & -2i & 2i \\ 4 & 2 & 2 \end{pmatrix} \text{ and } Spec(C_4^\sigma) = \begin{pmatrix} 0 & -2i & 2i \\ 2 & 1 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{E}(DC_4^\sigma) = 8 \text{ and } \mathcal{E}(C_4^\sigma) = 4.$$

Hence

$$\mathcal{E}(DC_4^\sigma) = 2\mathcal{E}(C_4^\sigma)$$

4. CONCLUSION

We initiated investigation on energy of a graph obtained via some graph operations. Here we considered duplication graph DG and it has been disclosed that $\mathcal{E}(DG) = 2\mathcal{E}(G)$ and $\mathcal{E}(DG^\sigma) = 2\mathcal{E}(G\sigma)$ and also the energy of product of graph $K_n \times G$ and energy of corresponding digraph $K_n^\sigma \times G^\sigma$ are determined.

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