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Riemann intergability versus continuity for vector-valued functions

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ABSTRACT. The interplay between Riemann integrability and continuity is an interesting topic of modern analysis. In this paper, Riemann integrability of vector-valued continuous functions, property of Lebesgue and weak property of Lebesgue are surveyed and discussed. We also prove that $\ell_1(\mathbb{N}, X)$ has the property of Lebesgue.

1. INTRODUCTION

The study of Riemann integrability for functions taking values in an arbitrary Banach space started in 1927 by Graves [8]. Most of the results concerning the Riemann integrability of real-valued functions remain valid in the vector-valued setting. However, a Riemann integrable function may not be continuous almost everywhere (a.e., for short) in the case of the function taking values in an arbitrary Banach space. There are (infinite dimensional) Banach spaces X where an X valued Riemann integrable function is not continuous a.e.. A Banach space X is said to have the property of Lebesgue if every Riemann integrable function $f : [0, 1] \longrightarrow X$ is continuous a.e. Alexiewicz and Orlicz [1] came up with an example of a Riemann integrable function which is not continuous a.e. and subsequently this property of a Banach space has been widely studied. All finite dimensional Banach spaces have this property, but there are quite a few infinite dimensional spaces which carry the property of Lebesgue. Rejouani [14] proved that ℓ_1 has the property of Lebesgue. This was independently proved by da Rocha [4] who also proved that the Tsirelson space possesses this property.

2. **Continuity implies Riemann integrability**

In this section we will demand the (Riemann) integrability of the function $f : [0, 1] \longrightarrow X$ considering the weaker forms of continuity. We in turn get a characterization of Banach spaces in terms of (Riemann) integrability. We start with the definition of the Riemann integral.

Definition 2.1. [11] Given a Banach space X, a function $f : [a, b] \longrightarrow X$ is said to be Riemann integrable if there exists $\xi \in X$ such that the following holds:
$\forall\, \epsilon > 0\ \exists \delta = \delta(\epsilon)$ such that for each (tagged) partition $\mathcal{P} = \{[x_{(i-1)}, x_i], t_i : 1 \leq i \leq n\}$ of $[a, b]$ where $a = x_0 < x_1 < \cdots < x_n = b$ and $t_i \in [x_{(i-1)}, x_i], 1 \leq i \leq n$, with

$$|\mathcal{P}| = \max_{1\leq i\leq n} (x_i - x_{i-1}) < \delta,$$

we have

$$\|S(f, \mathcal{P}) - \xi\| < \epsilon,$$

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where $\varsigma = \{t_i; 1 \leq i \leq n\}$ and $S(f, \mathcal{P})$ is the Riemann sum of f coresponding to the partition $\mathcal{P}$:

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

The unique vector ξ will be denoted by

$$\int_a^b f(t)dt$$

and is called the Riemann integral of f over $[a, b]$.

As we go on dealing with Riemann integral for functions taking values in a Banach space, several criteria for the equivalence of (Riemann) integral come in place. The following theorem presents some Cauchy criteria for the existence of the Riemann integral. The result is quite useful in proving the properties of the Riemann integral.

Theorem 2.1. [7] *Let $f : [a, b] \longrightarrow X$. The following are equivalent:*

(1) f is Riemann integrable on [a,b].
(2) For each $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \epsilon$, for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of [a,b] with norms less than δ.
(3) For each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of [a,b], such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \epsilon$, for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of [a,b] that refine $\mathcal{P}_\epsilon$.
(4) For each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of [a,b], such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \epsilon$, for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ that have the same points as $\mathcal{P}_\epsilon$.

Definition 2.2. [7] Let $f : [a, b] \longrightarrow X$.

(a) The function f is D_δ- integrable on $[a, b]$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\omega(f, \mathcal{P}) < \epsilon$, whenever $\mathcal{P}$ is a partition of $[a, b]$ with $|\mathcal{P}| < \delta$.

(b) The function f is $D_\triangle$- integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of $[a, b]$ such that $\omega(f, \mathcal{P}) < \epsilon$, whenever $\mathcal{P}$ is a partition of $[a, b]$ that refines $\mathcal{P}_\epsilon$.

The function $f : [a, b] \longrightarrow X$ is said to be Darboux integrable if f is D_δ- integrable or $D_\triangle$-integrable on $[a, b]$.

Definition 2.3. Let τ be a locally convex linear topology on X that is weaker than the norm topology on X. Then X is said to have Schur property with respect to τ if whenever $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ with respect to τ, it follows that $\|x_n - x\| \to 0$.

Theorem 2.2. [18] *Let X be a Banach space and τ a locally convex linear space topology on X which is weaker than the norm topology on X. Then the following are equivalent:*

(1) X has Schur property with respect to τ.
(2) If $f : [0, 1] \longrightarrow (X, \tau)$ is τ -continuous, then f is Darboux integrable.
(3) If $f : [0, 1] \longrightarrow (X, \tau)$ is τ -continuous, then f is Riemann integrable.

Definition 2.4. Let $f : [0, 1] \longrightarrow X$. For each $t \in (0, 1)$, $\omega(f, t) = \lim_{\delta \to 0^+} \omega(f, [t-\delta, t+\delta])$ is called the oscillation of f at t.

Note that f is continuous at t if and only if $\omega(f, t) = 0$.

Kadets [10] gave an example of a weak*-continuous function $f : [0, 1] \longrightarrow X$ on an infinite dimensional Banach space X which is not Riemann integrable.

We have the following characterisation of Riemann integrable weak*-continuous functions.

Theorem 2.3. [10] *A Banach space X is finite dimensional if and only if each weak*-continuous function $f : [0, 1] \longrightarrow X^*$ is Rieman integrable.*

Let us recall here that a Fréchet space is a complete, metrizable locally convex space. A locally convex Hausdorff space X in which bounded subsets are relatively compact is Montel if X is also Barreled [12].

Also a Banach space X with the property that separably valued bounded linear operators on X are weakly compact is called a Grothendieck space [5].
By making use of the 'fat' Cantor set and a result of Bonet and Lindstrom [2], Sofi [16] gave a generalization of Kadets's result for functions taking values in an infinite dimensional locally convex space [12] and proved the following result.

Theorem 2.4. [16] *A Fréchet space X is Montel if and only if each weak*-continuous function $f : [0,1] \longrightarrow X^*$ is Rieman integrable.*

Lone [11] proved a similar result and gave charecterization of a class of Banach spaces in terms of scalarly Riemann integrability of a weak*-continuous function.

He gave example of a weak*-continuous function taking values in the dual of a Banach space X which does not come out to be weakly Riemann integrable if X is not a Grothendieck space [5].

Example 2.1. [11] (A weak*-continuous function $f : [0,1] \longrightarrow X^*$ which is not weakly (scalarly) Riemann integrable).

Let us start with the construction of the so-called 'fat' Cantor set in $[0,1]$ which is constructed as follows. Let $B_1 = [0,1]$ and let $d_1^{(1)}$ be the midpoint of $[0,1]$. We remove from the interval $[0,1]$ successively a collection of subintervals as follows: With $d_1^{(1)}$ as centre remove a central subinterval $A_1^{(1)}$ of B_1 of length $\frac{1}{3}$ and denote

$$A_1^{(1)} = (a_1^{(1)}, b_1^{(1)}).$$

This will give us two disjoint subintervals say, $B_2^{(1)} = [0, a_1^{(1)}]$ and $B_2^{(2)} = [b_1^{(1)}, 1]$ each of length $\frac{1}{3}$. With centers $d_2^{(1)}$ and $d_2^{(2)}$ of $B_2^{(1)}$ and $B_2^{(2)}$ respectively remove from $B_2^{(1)}$ and $B_2^{(2)}$ the central subintervals $A_2^{(1)}$ and $A_2^{(2)}$ each of length $\frac{1}{2}\frac{1}{3^2}$. Proceeding likewise and letting d denote the length of a set, we get sequences $A_k^{(i)}$ and $B_k^{(i)}$ such that $A_k^{(i)} \subset B_k^{(i)}$, $d(A_k^{(i)}) = \frac{1}{2^{k-1}}\frac{1}{3^k}$ $d(B_k^{(i)}) = \frac{1}{2^{k-1}}(1 - \sum_{i=1}^{k-1}\frac{1}{3^i}), k = 1, 2, \cdots, 2^{k-1}$. Putting

$$G = \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{2^{k-1}} A_k^{(i)} \text{ and taking } H = [0,1]\backslash G$$

It is easy to see that H is a perfect, nowhere dense set of $[0,1]$ with a positive measure. Let us write $A_k^{(i)} = (a_k^{(i)}, b_k^{(i)}), k = 1, 2, \cdots, i = 1, 2, \cdots, 2^{k-1}$.
For each $k \geq 1$ and $i = 1, 2, \cdots, 2^{k-1}$, define

$$\phi_k^{(i)} : [0,1] \to \mathbb{R}$$

so that it vanishes outside $A_k^{(i)} = (a_k^{(i)}, b_k^{(i)})$ and is piecewise linear on $A_k^{(i)}$.

$$\phi_k^{(i)}(t) = \begin{cases} \frac{2}{b_k^{(i)} - a_k^{(i)}}\left(t - a_k^{(i)}\right), & t \in [a_k^{(i)}, a_k^{(i)} + \frac{b_k^{(i)} - a_k^{(i)}}{2}] \\ \frac{2}{b_k^{(i)} - a_k^{(i)}}\left(b_k^{(i)} - t\right), & t \in [a_k^{(i)} + \frac{b_k^{(i)} - a_k^{(i)}}{2}, b_k^{(i)}] \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that $\phi_k^{(i)}(t)$ is continuous and so is $h_k : [0,1] \longrightarrow \mathbb{R}$ where

$$h_k(t) = \sum_{i=1}^{2^{k-1}} \phi_k^{(i)}(t).$$

Let us define $f : [0, 1] \longrightarrow X^*$ by

$$f(t) = \sum_{k=1}^{\infty} h_k(t)x_k^*,\ t \in [0, 1].$$

We show that f is a well-defined weak*-continuous function.
To this end, note that since $f(t) = 0$ for $t \in H$, it follows that the series defining f is actually a finite sum in X^* and for $t \notin H$ there exists $k \geq 1$ such that $t \in (a_{k_0}^{(i)}, b_{k_0}^{(i)})$ for some k_0 where $1 \leq k_0 \leq 2^{k-1}$ and so, f reduces to $h_{k_0}(t)x_k^*$. Therefore in either case the above series in X^* actually appears as a finite sum. Now as h_k is continuous for all $k \geq 1$, to prove that f is weak*-continuous it suffices to show that the series defining f is weak*-uniformly convergent in X^*.
To this end, fix $\epsilon > 0$ and $x \in X$. We can choose K_0 such that $|x_k^*(x)| < \epsilon$ for all $k \geq K_0$. By the definition of $h_k(t)$, it follows that

$$\sum_{k=K_0+1}^{\infty} h_k(t)x_k^* = h_{k_0}(t)x_{k_0}^* \ \text{ for } \ t \in \bigcup_{k=K_0+1}^{\infty} \bigcup_{i=1}^{2^{k-1}} A_k^{(i)}.$$

In either case, we get $\langle f(t), x\rangle = \langle h_{k_0}(t)x_{k_0}^*, x\rangle = |h_{k_0}(t)||\langle x_{k_0}^*, x\rangle| < \epsilon$, using the fact that $|h_k(t)| \leq 1$, for all $t \in [0, 1],\ k \geq 1$.
Now by making use of Theorem 2.1 it is easy to prove that f is not Riemann integrable.

3. **Riemann integrability implies continuity**

While discussing the theme of integrability and its interply with the topological structure of a Banach space some natural questions which are enunciated below can be asked:

(1) Find necessary and sufficient conditions for a vector-valued Riemann integrable function to be continuous a.e.
(2) Make charecterizations of Banach spaces in terms of Riemann integrability of a function.
(3) Find necessary and sufficient condition for vector-valued Riemann integrable function to be weakly continuous a.e.

Gordon [7] introduced a property possessed by some Banach spaces which he called as ”property of Lebesgue”. A Banach space is said to have the property of Lebesgue (LP for short) if every Riemann integrable function $f : [a, b] \longrightarrow X$ is continuous a.e. Wang [17] introduced a similar notion in Banach spaces, known as ”weak property of Lebesgue”. A Banach space is said to have weak property of Lebesgue (WLP for short) if every Riemann integrable function $f : [a, b] \longrightarrow X$ is weakly continuous a.e. In what follows, X will always denote a Banach space unless otherwise mentioned, μ Lebesgue measure and

$$\ell_1(\mathbb{N}, X) = \left\{ \widehat{x} : \widehat{x} = (x_n), \sum_{n=1}^{\infty} \|x_n\| < \infty \right\}.$$

If $\mathcal{P} = \{t_i : 0 \leq i \leq n\}$ is a partition of $[a, b]$, then

$$\omega(f, \mathcal{P}) = \sum_{i=1}^{n} \omega\left(f, [t_{i-1}, t_i]\right)(t_i, t_{i-1})$$

where $\omega(f, [t_{i-1}, t_i]) = \sup\{\|f(v) - f(u)\| : u, v \in [t_{i-1}, t_i]\}$ is the oscillation of f on $[t_{i-1}, t_i]$. Let us recall that by a partition of $[a, b]$ we mean a finite set of points $\{x_i : 0 \leq i \leq n\}$. A tagged partition is defined as the set of pairs $\{[x_{i-1}, x_i], t_i\}_{i=1}^n$ where $x_0 < x_1 < \cdots < x_n$ and t_i's are tags of the partition.

Definition 3.5. [7] Given a Banach space X, a function $f : [a, b] \longrightarrow X$ is said to be R_δ-integrable if there exists $x \in X$ for which the following is true: for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|S(f, \mathcal{P}) - x\| < \epsilon$$

whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ with $|\mathcal{P}| < \delta$.

Definition 3.6. [7] Given a Banach space X, a function $f : [a, b] \longrightarrow X$ is said to be R_Δ-integrable if there exists $x \in X$ for which the following is true: for each $\epsilon > 0$ there exists a partition $\mathcal{P}_\epsilon$ of $[a, b]$ such that

$$\|S(f, \mathcal{P}) - x\| < \epsilon$$

whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that refines $\mathcal{P}_\epsilon$.

A preliminary investigation indicates that the function f must be bounded in either of the definitions given above, and also the vector $x \in X$ has to be unique. It is clear that a R_δ-integrable function is R_Δ-integrable to make the two definitions equivalent, we present the following theorem.

Theorem 3.5. [7] *A function $f : [a, b] \longrightarrow X$ is R_Δ-integrable on $[a, b]$ if and only if it is R_δ-integrable on $[a, b]$.*

Definition 3.7. A function $f : [a, b] \longrightarrow X$ is Riemann integrable on $[a, b]$ if f is either R_δ-integrable or R_Δ-integrable.

It is a well-known fact that a real-valued Riemann integrable function is continuous almost everywhere (a.e.). This phenomenon does not carry over to vector version that is for a function taking values in an arbitrary Banach space.

As mentioned before, Alexiewicz and Orlicz [1] were the first ones to present an example of a Riemann integrable function which is not continuous a.e.

Definition 3.8. Let $f : [a, b] \longrightarrow X$.
(a) The function f is said to be scalarly measurable if $x^* f$ is measurable for each $x^* \in X^*$.
(b) The function f is said to be of weak bounded variation on $[a, b]$ if $x^* f$ is of bounded variation on $[a, b]$ for each $x^* \in X^*$.
(c) The function f is said to be of outside bounded variation on $[a, b]$ if

$$\sup_{[x_{i-1}, x_i] \in [a,b]} \left\{ \left\| \sum_{i}^{n} \Big(f(x_i) - f(x_{i-1}) \Big) \right\| \right\}$$

is finite.

Theorem 3.6. *If $f : [a, b] \longrightarrow X$ is of outside bounded variation on $[a, b]$ then f is Riemann intgerable.*

Proof. To show that f is Riemann integrable it suffices to show that: for every $\epsilon > 0$ there esists a partition $\mathcal{P}_\epsilon$ of $[a, b]$ such that

$$\|S(f, \mathcal{P}_1) - S(f, \mathcal{P}_2)\| < \epsilon \tag{3.1}$$

for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$ that have the same points as $\mathcal{P}_\epsilon$. Let $\epsilon > 0$ and M outside variation of f on $[a, b]$. Choose n such that $\dfrac{b-a}{n} < \dfrac{\epsilon}{M}$. Let $\mathcal{P}_\epsilon = \{x_i : 0 \le i \le n\}$ be the partition of $[a, b]$ for which $x_i = x_0 + \dfrac{i}{n}(b - a)$.

Now take $\mathcal{P}_1 = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ and $\mathcal{P}_2 = \{[x_{i-1}, x_i], s_i\}_{i=1}^n$ two tagged partitions of $[a, b]$. Then we have

$$\begin{aligned} \|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)\| &= \left\| \sum_{i=1}^{n} \Big(f(s_i) - f(t_i) \Big) \Delta x_i \right\| \\ &= \frac{b-a}{n} \left\| \sum_{i=1}^{n} \Big(f(s_i) - f(t_i) \Big) \right\| \\ &\leq \frac{b-a}{n} M \\ &< \epsilon \end{aligned}$$

which proves 3.1. Hence f is Riemann integrable. □

As mentioned earlier the relation between Riemann integrability and continuity does not go smoothly for vector-valued functions as in the scaler case. Here we will present some (counter) examples which will illustrate the pathological charecteristic of the vector-valued Riemann integral and makes the relation between continuity and Riemann integrability very interesting. The following examples are a modified version of results given by Rejouni [14].

Example 3.2. Let $f : [0, 1] \longrightarrow c_0$ be defined by

$$f(t) = \begin{cases} 0, & \text{if t is irrational} \\ e_n, & \text{if } t = r_n \end{cases}$$

where (r_n) are the rationals in $[0, 1]$. Now looking at

$$\sup_{[x_{i-1}, x_i] \in [a,b]} \left\{ \left\| \sum_{i}^{n} \Big(f(x_i) - f(x_{i-1}) \Big) \right\| \right\}$$

it is easy to see that the supremum for the function f if finite. Therefore, by Theorem 3.6 f is Riemann integrable. But by just looking at the definition of the function it is clear that f is not continuous a.e on $[a, b]$.

Example 3.3. Let $f : [0, 1] \longrightarrow \ell_2$ be defined by

$$f(t) = \begin{cases} 0, & \text{if t is irrational} \\ e_n, & \text{if } t = r_n \end{cases}$$

Claim: f is Riemann integrable on $[0, 1]$.
Let $\epsilon > 0$ and $\delta = \epsilon^2$. Let $\mathcal{P} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^n$ be a tagged partition of $[0, 1]$ with $|\mathcal{P}| < \delta$. Then

$$\begin{aligned} \|S(f,\mathcal{P})\| &= \left\| \sum_{i=1}^{n} f(t_i) \Delta x_i \right\| \\ &= \left\| \sum_{i=1}^{n} e_i \Delta x_i \right\| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^{n} \|e_i \Delta x_i\| \\ &= \left\{ \sum_{i=1}^{n} (\Delta x_i)^2 \right\}^{\frac{1}{2}} \\ &\leq |\mathcal{P}| \left\{ \sum_{i=1}^{n} (\Delta x_i)^2 \right\}^{\frac{1}{2}} < \epsilon \end{aligned}$$

which proves our claim. But f is not of outside bounded variation on $[0,1]$. On this account let n be a positive integer and for each i, let s_i be an irrational number in the interval $\left(\frac{1}{i-1}, \frac{1}{i}\right)$. Then

$$\left\| \sum_{i=1}^{n} \left(f\left(\frac{1}{i}\right) - f(s_i) \right) \right\| = \left(\sum_{i=1}^{n} 1 \right)^{\frac{1}{2}} = \sqrt{n}$$

which does not exist finitely. Hence f is not of outside bounded variation.

The examples presented above give us enough motivation to present the following concept.

Definition 3.9. A Banach space X is said to have the Lebesgue property or property of Lebesgue (LP for short) if every Riemann integrable function $f : [a,b] \longrightarrow X$ is continuous a.e. on $[a,b]$.

All finite dimensional Banach spaces have this property. But there are quite a few infinite dimensional Banach spaces having property of Lebesgue. The next theorem is instrumental in determining if a Banach space has property of Lebesgue.

Theorem 3.7. [7] *Let X be a Banach space and Y a subspace of X.*
(a) If X has the property of Lebesgue then Y has it.
(b) If Y does not have the property of Lebesgue then X doesn't have it.

Now with help of the preceeding theorem (Theorem 3.7) and Example 3.2 we will present examples of some spaces which do not possess the property of Lebesgue.

(1) $c_0, c, \ell_\infty, C[a,b], \ell_\infty[a,b], L_\infty[a,b]$. By Example 3.2, c_0 doesn't have property of Lebesgue and we know that c_0 embeds in $c, \ell_\infty, C[a,b], \ell_\infty[a,b], L_\infty[a,b]$. Therefore, by Theorem 3.7 these spaces too doesn't have LP.
(2) Example 3.3 shows that $\ell_p, 1 < p < \infty$ does not have the property of Lebesgue. Also ℓ_2 embeds in $L_1[a,b]$ so, by above theorem $L_1[a,b]$ does not have the property of Lebesgue.

Definition 3.10. [15] A Banach space X is said to be uniformly convex if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x and y in X with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq 1$, we have $\|x + y\| \leq 2(1 - \delta)$.

Theorem 3.8. *An infinite dimensional uniformly convex Banach space does not have the property of Lebesgue.*

Proof. We know that infinite dimensional Hilbert spaces are uniformly convex, also $L_p[a,b]$ for $1 < p < \infty$ is uniformly convex. These spaces does not possess the property of Lebesgue by preceeding theorem. □

Now let us turn to positive side of the property, upto now we have given examples of spaces which does not have the property of Lebesgue. There are spaces with this property (indeed quite a few infinite dimensional) as mentioned earlier, all finite dimensional Banach spaces have this property. For the case of infinite dimensional Banach spaces Nemirovski, Ochan and Rejouani [13] proved the following

Theorem 3.9. [13] *ℓ_1 has the property of Lebesgue.*

The above theorem was independently proved by da Rocha [4]. Although the proofs which were given independently use the same technique. In [4] da Rocha proved that the Tsirelson space has property of Lebesgue. We will present that result after giving the constructuion of the Tsirelson space as given by Casazza and Shura [3].
(i) For A, B finite and nonempty subsets of $\mathbb{N}$, we write $A \leq B$ if

$$\max\{n : n \in A\} \leq \max\{n : n \in B\}$$

and write $A < B$ if the inequality is strict.
(ii) Let $c_{00} = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R}, x_n \neq 0 \text{ for finite } n\}$.
(iii) Let $\{e_n\}$ be the canonical unit vector basis of c_{00}.
(iv) For any $x = \sum_n a_n e_n \in c_{00}$ and a subset $A \subset \mathbb{N}$, define $Ax = \sum_{n \in A} a_n$.
(v) Define inductively a sequence of norms $\{\|.\|\}_{m=0}^{\infty}$ on c_{00} as follows: Fix

$$x = \sum_n a_n e_n \in c_{00}.$$

Let

$$\|x\|_0 = \max_n |a_n|$$

and

$$\|x\|_{m+1} = \max\left\{\|x\|_m, \frac{1}{2}\left[\sum_{j=1}^{k} \|A_j x\|_m\right]\right\},$$

for $m \geq 0$ where the supremum is taken over all collections of finite subsets of $\{A_j\}$ of positive integers $k \leq A_1 < A_2 \cdots < A_k$ and all positive integers k.
(vi) Now it is easy to see that these norms increase with m on c_{00} and that $\|x\|_{\ell_\infty} \leq \|x\|_m \leq \|x\|_{\ell_1}$ for all $x \in c_{00}$ and for all m.

Definition 3.11. [6] Define $\|x\| = \lim_{m \longrightarrow \infty} \|x\|_m$. The Tsirelson space T is $\|.\|$ completion of c_{00}.

Theorem 3.10. *Tsirelson space has property of Lebesgue.*

Definition 3.12. Let $f : [0,1] \longrightarrow X$, for each $t \in (0,1)$ $\omega(f,t) = \lim_{\delta \to 0^+} \omega(f, [t-\delta, t+\delta])$ is said to be the oscillation of f at t.
Note that f is continuous at t if and only if $\omega(f,t) = 0$.

Theorem 3.11. *$\ell_1(\mathbb{N}, X)$ has the property of Lebesgue.*

Proof. Let $f : [0,1] \longrightarrow \ell_1(\mathbb{N}, X)$ be a bounded function which is not continuous a.e. on $[0,1]$. To establish the result it suffices to prove that f is not Riemann integrable. Since f is not continuous a.e. there exist $a > 0$ and $b > 0$ such that $\mu(H) = a$ where

$$H = \{t \in [0,1] : \omega(f,t) \geq b\}.$$

<u>Claim:</u> For each $\delta > 0$ there exists tagged partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[0,1]$ with $|\mathcal{P}_1| < \delta$ and $|\mathcal{P}_1| < \delta$ such that

$$\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| \geq \frac{ab}{4}.$$

Given $\delta > 0$ choose a positive integer N such that $\frac{1}{N} < \delta$ and $\mathcal{P}_N = \left\{\frac{k}{N} : 0 \le k \le N\right\}$.
Let $\{[c_i, d_i] : 1 \le i \le k\}$ be all the intervals of $\mathcal{P}_N$ for which

$$\mu\left(H \bigcap (c_i, d_i)\right) > 0$$

and $\frac{k}{N} \ge a$. For each positive integer j, let K_j be the set of discontinuities of $e_j f$ on $[0, 1]$. Then $\mu(K_j) = 0$ otherwise $e_j f$ and consequently f is not Riemann integrable on $[0, 1]$. So $K = \cup_j K_j$ has measure zero and every $e_j f$ is continuous on $[0, 1] - K$. To establish this we let $n_0 = 0$ and choose $t_1 \in (H - K) \cap (c_1, d_1)$. Since $\omega(f, t_1) \ge b$ there exists a point $s_1 \in (c_1, d_1)$ such that $\|f(t_1) - f(s_1)\| \ge \frac{b}{2}$. Let $\{x^i_j\} = f(t_1) - f(s_1)$ and choose an integer n_1 such that

$$\sum_{j=n_1}^{\infty} \|x^i_j\| < \frac{\epsilon}{2}.$$

Now choose $t_2 \in (H - K) \cap (c_2, d_2)$. Since $\omega(f, t_2) \ge b$ and as $e_j f$ is continuous at t_2 for each $1 \le j \le n_1$ there exists a point $s_2 \in (c_1, d_1)$ such that $\|f(t_2) - f(s_2)\| \ge \frac{b}{2}$ and

$$\sum_{j=1}^{n_1} \|e_j f(t_2) - e_j f(s_2)\| < \frac{\epsilon}{2^2}.$$

Let $\{x^2_j\} = f(t_2) - f(s_2)$ then

$$\sum_{j=1}^{n_1} \|x^2_j\| < \frac{\epsilon}{2^2}.$$

Now choose an integer $n_2 > n_1$ such that

$$\sum_{j=n_2}^{\infty} \|x^i_j\| < \frac{\epsilon}{2^2}.$$

Continuing this process for k steps we obtain the sets

$$\{t_i : 1 \le i \le k\} \text{ where } t_i \in (H - K) \cap (c_i, d_i) \text{ for each } i$$

and

$$\{s_i : 1 \le i \le k\} \text{ where } s_i \in (c_i, d_i) \text{ for each } i$$

and $\{n_i : 0 \le i \le k\}$ where each n_i is an integer with $0 = n_0 < n_1 < \cdots < n_k$ with the following properties. Let n_i be the smallest integer such that

$$\sum_{j=n_i}^{\infty} \|x^i_j\| < \frac{\epsilon}{2^i},\ \forall\, i \ge 1$$

and n_{i-1} be the largest integer such that

$$\sum_{j=1}^{n_{i-1}} \|x^i_j\| < \frac{\epsilon}{2^i},\ \forall\, i \ge 2$$

where $\{x^i_j\} = f(t_i) - f(s_i)$ with $\|\{x^i_j\}\| \ge \frac{b}{2}$.
Let or each $1 \le i \le k$

$$y^i_j = \sum_{j=n_{i-1}+1}^{n_i - 1} x^i_j e_j.$$

Then

$$\|x_j^i - y_j^i\| = \sum_{j=1}^{n_{i-1}} \|x_j^i\| + \sum_{j=n_i}^{\infty} \|x_j^i\| < 2\frac{\epsilon}{2^i}$$

and

$$\|y_j^i\| = \|x_j^i\| - \|x_j^i - y_j^i\| \geq \frac{1}{2}b - 2\frac{\epsilon}{2^i}.$$

This is true for all $1 \leq i \leq k$. By summing from 1 to k, we get

$$\left\|\sum_{i=1}^{k} x_j^i\right\| \geq \left\|\sum_{i=1}^{k} y_j^i\right\| - \left\|\sum_{i=1}^{k}(y_j^i - x_j^i)\right\| \geq \sum_{i=1}^{k} \|y_j^i\| - \sum_{i=1}^{k} \|y_j^i - x_j^i\|$$

$$\geq \sum_{i=1}^{k}\left(\frac{1}{2}b - \frac{2\epsilon}{2^i}\right) - \sum_{i=1}^{k}\frac{2\epsilon}{2^i} \geq \frac{1}{2}kb - 4\epsilon.$$

Now, to show that f is not Riemann integrable let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two tagged partitions of $[0,1]$ that have the same points as $\mathcal{P}_N$. Let t_i be the tag of $[c_i, d_i]$ for $\mathcal{P}_1$ and s_i for $\mathcal{P}_2$. Then $|\mathcal{P}_1| < \delta$ and $|\mathcal{P}_2| < \delta$, and so we have

$$\|S(f,\mathcal{P}_1) - S(f,\mathcal{P}_2)\| = \left\|\sum_{i=1}^{k}\frac{1}{N}x_j^i\right\| \geq \frac{1}{2}\frac{k}{N}b - \frac{4}{N}\epsilon \geq \frac{1}{2}ab - \frac{1}{4}ab = \frac{ab}{4}$$

which shows that f is not Riemann integrable. □

Example 3.4. Define

$$f : [0,1] \longrightarrow \ell_\infty[0,1]$$

by

$$f(t) = \chi_{[0,t]}.$$

Then it is easy to see that f is of outside bounded variation on $[0,1]$. Therefore, by Theorem 3.6 f is Riemann integrable. But f is not measurable and hence not continuous a.e. on $[0,1]$.

Wang Chonghu [17] introduced the weak property of Lebesgue of Banach spaces and proved that every Banach space with seperable dual has weak property of Lebesgue.

Before we will present this result and reconstruct its proof we would like to give some definitions and some results which come in use of the proof.

Definition 3.13. A Banach space X is said to have weak property of Lebesgue (WLP for short) if every Riemann integrable function $f : [0,1] \longrightarrow X$ is weakly continuous a.e.

If a Banach space X has property of Lebesgue then it has the weak property of Lebesgue. So, every finite dimensional Banach space has the weak property of Lebesgue. Also ℓ_1 and Tsirelson space T^* have the weak property of Lebesgue. The weak property of Lebesgue is topologically invariant as shown in the following.

Theorem 3.12. [17] *Let X and Y be topologically isomorphic Banach spaces. Then both have the weak property of Lebesgue if either has it.*

Proof. Suppose X has weak property of Lebesgue. Let $f : [a,b] \longrightarrow Y$ be Riemann integrable.

<u>Claim:</u> f is weakly continuous a.e. on $[a,b]$.

To this end let T be the topological isomorphism and take $y^* \in Y^*$. Then there exist $x^* \in X^*$ such that

$$y^*(f(t)) = (T^*)^{-1}x^*(f(t)) = (T^{-1})^*x^*(f(t)) = x^*T^{-1}(f(t)).$$

Since f is Riemann integrable and T^{-1} is linear and continuous. $T^{-1}f : [a,b] \longrightarrow X$ is Riemann integrable. Also X has WLP there exists $E \subset [a,b]$ with $\mu(E) = b-a$ for any $x^* \in X^*, x^*T^{-1}(f(t))$ is continuous at every $t \in E$. Therefore, $y^*f(t)$ is continuous at $t \in E$ for any $y^* \in Y^*$ which proves our claim and the result. □

Theorem 3.13. [17] *If X has WLP and $Y \subset X$ is a subspace of X, then Y has WLP.*

$C[a,b]$ does not have the WLP. Noting that every separable Banach space is topologically isomorphic to a suitable quotient space of ℓ_1 [17] and $C[a,b]$ does not have WLP, it follows that the quotient space $\dfrac{X}{Y}$ (Y is a closed subspace of X) may not have LP or WLP even though X has it.

Now we are going to discuss some properties of Banach spaces with shrinking bases and especially see that this kind of Banach spaces have WLP.

We begin by presenting the following

Definition 3.14. Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. If for every $x^* \in X^*$

$$\limsup_{n\to\infty} \left\{ |x^*(y_n)| : y_n = \sum_{i=n}^{\infty} \alpha_i x_i, \|y_n\| = 1 \right\} = 0,$$

then $\{x_n\}_{n=1}^{\infty}$ is called the shrinking basis of X.

Lemma 3.1. [17] *Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ and A a bounded set of X. Then the sequence of subsets*

$$A_n = \left\{ \sum_{i=n}^{\infty} \alpha_i x_i : x = \sum_{i=1}^{\infty} \alpha_i x_i \in A \right\}, n = 1, 2, \ldots$$

are uniformly bounded.

Theorem 3.14. *Let X be a Banach space with a shrinking basis $\{x_n\}_{n=1}^{\infty}$ and $f : [a,b] \longrightarrow X$ be defined by*

$$f(t) = \sum_{i=1}^{\infty} f_i(t) x_i$$

where f_i is a real-valued function on $[a,b]$. Then if f is bounded, the series $\sum_{i=1}^{\infty} f_i(t)x^*(x_i)$ is uniformly convergent on $[a,b]$.

Proof. Since f is bounded by Lemma 3.1 we can assume that for all $t \in [a,b]$ there exists $M > 0$ such that

$$\left\| \sum_{i=1}^{\infty} f_i(t) x_i \right\| \le M.$$

Then for any $x^* \in X^*$

$$\sup_{t\in[a,b]} \left| \sum_{i=n}^{\infty} f_i(t) x^*(x_i) \right| = \sup_{t\in[a,b]} \left| x^* \left(\sum_{i=n}^{\infty} f_i(t) x_i \right) \right| \le \sup \left\{ M \left| x^* \left(\sum_{i=n}^{\infty} \alpha_i x_i \right) \right| : \left\| \sum_{i=n}^{\infty} \alpha_i x_i \right\| = 1 \right\}.$$

Now $\{x_n\}_{n=1}^{\infty}$ is a shrinking basis of X, $\sup_{t\in[a,b]} |\sum_{i=n}^{\infty} f_i(t)x^*(x_i)| \longrightarrow 0$ (as $n \longrightarrow \infty$) is uniformly convergent on $[a,b]$. □

Theorem 3.15. *Let X be a Banach space with a shrinking basis $\{x_n\}_{n=1}^{\infty}$ and $f : [a,b] \longrightarrow X$ be defined by*

$$f(t) = \sum_{i=1}^{\infty} f_i(t) x_i$$

where f_i is a real-valued function on $[a, b]$. Then f is weakly continuous if and only if f is bounded and f_i is continuous for all $i = 1, 2, \ldots$

Theorem 3.16. *If X is a Banach space with a shrinking basis, then X has WLP.*

Corollary 3.1. *Any Banach space with a separable dual has WLP.*

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