CREAT. MATH. INFORM. Online version at https://creative-mathematics.cunbm.utcluj.ro/
30 (2021), No. 1, 69 - 74 Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Application of subordination for estimating the Hankel determinant for subclass of univalent functions

SH. NAJAFZADEH, H. RAHMATAN and H. HAJI

ABSTRACT. By using subordination structure, a new subclass of convex functions is introduced. The estimate of the third Hankel determinants is also investigated.

1. INTRODUCTION

The coefficient estimate of univalent functions is one of the most important subjects which has been addressed in many recent articles. See [5], [6] and [7].
The well- known Fekete- Szegö inequality and the bounds of second and third Hankel determinants are investigated. See [1], [3], [4], [8] and [9].

Let $\mathcal{A}$ denote the class of univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized univalent in $\mathbb{U}$.

Let f and g be analytic in $\mathbb{U}$. Then f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function ω analytic in $\mathbb{U}$, with $\omega(0) = 0$, $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$.
If g is univalent, the $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
Suppose that $\mathcal{P}$ denote the class of analytic functions p of the type

$$p(z) = 1 + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

such that $Re\, p(z) > 0$. Also

$$SL^{\mathcal{C}} = \left\{ f \in A : \left|(1 + \frac{zf''(z)}{f'(z)})^2 - 1\right| < 1, \quad z \in \mathbb{U} \right\}.$$

Thus the values of $1 + \frac{zf''(z)}{f'(z)}$ where $f \in SL^{\mathcal{C}}$ lies in the region which bounded by the right half of the lemniscate of Bernoulli given by $|\omega^2 - 1| < 1$.

Received: 16.06.2020. In revised form: 15.09.2020. Accepted: 22.09.2020
2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. *Convex functions, subordination, Hankel determinants.*
Corresponding author: H. Rahmatan; h.rahmatan@gmail.com

It is easy to see that $f \in SL^{\mathcal{C}}$ if it satisfies the condition

$$1+\frac{zf''(z)}{f'(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{U}. \tag{1.3}$$

The determinants of the q^{th} Hankel matrix denoted by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

and is called the q^{th} Hankel determinant. In the particular cases
$q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$.
the Hankel determinant simplifies respectively to
$H_2(1) = |a_3 - a_2^2|$ and $H_2(2) = |a_2a_4 - a_3^2|$.
In the case $q = 3$ and $n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$H_2(2)$ and $H_3(1)$ are respectively called second and third Hankel determinant
For $f \in \mathcal{S}, a_1 = 1$ so that,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2).$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

In this paper, we introduce a new subclass of convex functions assciated with diffrential subordination and extremum point.

Lemma 1.1. [7] *If $p \in \mathcal{P}$ be of the form 1.2, Then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

for some x,$|x| \leq 1$, and

$$4p_3 = p_1^3 + 2(4-p_1^2)p_1x - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)z$$

for some z, $|z| \leq 1$.

2. Main results

In this section we obtain some coefficient estimates and bounds of third Hankel determinant.

Lemma 2.2. *If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each k.* [2]

Theorem 2.1. *Let f of the form 1.1 be in the class $SL^{\mathcal{C}}$, Then*

$$|a_3 - a_2^2| \leq \frac{5}{24}.$$

Proof. If $f \in SL^{\mathcal{C}}$, then it follows from 1.3 that

$$1+\frac{zf''(z)}{f'(z)} \prec \phi(z), \tag{2.4}$$

where $\phi(z) = \sqrt{1+z}$. Define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad .$$

It is clear that $p \in \mathcal{P}$. This implies that

$$\omega(z) = \frac{p(z)-1}{p(z)+1}.$$

From 2.4, we have

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)).$$

with

$$\phi(\omega(z)) = \left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}}.$$

Now

$$\left(\frac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}} = 1 + \frac{1}{4}p_1 z + \big(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\big)z^2 + \big(\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\big)z^3 \dots \quad .$$

Similarly,

$$\frac{1+zf''(z)}{f'(z)} = 1 + 2a_2 z + \big(6a_3 - 4a_2^2\big)z^2 + \big(12a_4 - 18a_2a_3 + 8a_2^3\big)z^3 \dots \quad .$$

Now, equating the coefficients we get

$$2a_2 = \frac{1}{4}p_1,$$

then

$$a_2 = \frac{1}{8}p_1. \tag{2.5}$$

and

$$6a_3 - 4a_2^2 = \frac{1}{4}p_2 - \frac{5}{32}p_1^2,$$

$$a_3 = \frac{1}{24}p_2 - \frac{1}{64}p_1^2. \tag{2.6}$$

and

$$12a_4 - 18a_2a_3 + 8a_2^3 = \frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3$$

$$a_4 = \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3. \tag{2.7}$$

Applying Lemma 2.2 for the coefficients p_1 and p_2, we get:

$$|a_3 - a_2^2| \le \frac{5}{24}.$$

□

Theorem 2.2. *Let f of the form 1.1 be in the class $SL^{\mathcal{C}}$, Then*

$$|a_2a_4 - a_3^2| \le \frac{1}{72}.$$

Proof. From 2.5, 2.6 and 2.7, we obtain

$$\begin{aligned} a_2a_4-a_3^2 &= \frac{1}{384}\left(p_1p_3-\frac{7}{8}p_1^2p_2+\frac{13}{64}p_1^4\right)-\left(\frac{1}{24}p_2-\frac{1}{64}p_1^2\right)^2 \\ &= \frac{1}{384}p_1p_3-\frac{7}{3072}p_1^2p_2+\frac{13}{24576}p_1^4-\frac{1}{576}p_2^2 \\ &\quad +\frac{1}{768}p_1^2p_2-\frac{1}{4096}p_1^4 \\ &= \frac{1}{24576}\left(64p_1p_3+24p_1^2p_2-\frac{128}{3}p_2^2+7p_1^4\right). \end{aligned}$$

Putting the values of p_2 and p_3 from Lemma 1.1, letting $p>0$ and taking $p_1=p\in[0,2]$, we get

$$\begin{aligned} |a_2a_4-a_3^2| &= \frac{1}{24576}\bigg|16p_1(p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2 \\ &\quad +2\big(4-p_1^2)(1-|x|^2)z\big)+12p_1^2\big(p_1^2+x(4-p_1^2)\big) \\ &\quad -\frac{32}{3}\big(p_1^2)+x(4-p_1^2)^2\big)+16p_1^2\big(p_1^2+x(4-p_1^2)\big)-6p_1^4\bigg|. \end{aligned}$$

After a simple calculation, we get

$$\begin{aligned} |a_2a_4-a_3^2| &= \frac{1}{24576}\bigg|\frac{82}{3}p^4+\frac{116}{3}(4-p^2)p^2x \\ &\quad +32(4-p^2)p(1-|x|^2)z-\frac{16}{3}(4-p^2)(p^2+8)x^2\bigg|. \end{aligned}$$

Now, applying the triangle inequality and replacing $|x|$ by ρ, we obtain

$$\begin{aligned} |a_2a_4-a_3^2| &\leq \frac{1}{73728}\Big|82p^4+192p(4-p^2)+116(4-p^2)p^2\rho+16\rho^2(4-p^2)(p^2+8)\Big| \\ &= F(p,\rho). \end{aligned}$$

Differentiating with respect to ρ, we have

$$\frac{\partial F(p,\rho)}{\partial\rho}=\frac{1}{73728}\big(116(4-p^2)p^2+32\rho(4-p^2)(p^2+8)\big).$$

It is clear that $\frac{\partial F(p,\rho)}{\partial\rho}>0$, which shows that $F(p,\rho)$ is an increasing function on the closed interval $[0,1]$. This implies that maximum occurs at $\rho=1$. Therefore $Max\ F(p,\rho)=F(p,1)=G(p)$. Now

$$G(p)=\frac{1}{73728}\big(-148p^4+336p^2+1024\big).$$

Therefore

$$G'(p)=\frac{1}{73728}\big(-592p^3+672p\big)$$

and

$$G''(p)=\frac{1}{73728}\big(-1776p^2+672\big)<0$$

for $p=0$. This shows that maximum of $G(p)$ occurs at $p=0$. Hence, we obtain

$$\begin{aligned} |a_2a_4-a_3^2| &\leq \frac{1024}{73728} \\ &= \frac{1}{72}. \end{aligned}$$

□

Theorem 2.3. *Let f of the form 1.1 be in the class $SL^{\mathcal{C}}$, Then*

$$|a_2a_3 - a_4| \leq \frac{1}{12}.$$

Proof. Since

$$\begin{aligned} a_2 &= \frac{1}{8}p_1, \\ a_3 &= \frac{1}{24}p_2 - \frac{1}{64}p_1^2, \\ a_4 &= \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3, \end{aligned}$$

therefore, by using Lemma 1.2 and replacing $|x|$ by ρ, we have

$$|a_2a_3 - a_4| \leq \frac{1}{3072}\left|39p^3 + 64\left(4 - p^2\right) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2)\right|.$$

Let

$$F_1(p, \rho) = \frac{1}{3072}\left|39p^3 + 64(4 - p^2) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2)\right|. \tag{2.8}$$

We assume that the upper bound occurs at the interior of the rectangle $[0, 2] \times [0, 1]$. Differentiating 2.8 with respect to ρ, we have

$$\frac{\partial F_1(p, \rho)}{\partial \rho} = \frac{1}{3072}\left(4p(4 - p^2) + 64p\rho(4 - p^2)\right).$$

For $0 < \rho < 1$, and fixed $p \in (0, 2)$, it is clear that $\frac{\partial F_1(p,\rho)}{\partial \rho} < 0$, which shows that $F_1(p, \rho)$ is an decreasing function of p, which contradicts our assumption, therefore $Max\ F_1(p, \rho) = F_1(p, 0) = G_1(p)$. Now

$$G_1(p) = \frac{1}{3072}(-4p^3 + 16p).$$

So

$$G_1'(p) = \frac{1}{3072}(-12p^2 + 16)$$

and

$$G_1''(p) = \frac{1}{3072}(-24p) < 0$$

for $p = 0$. This shows that maximum of $G_1(p)$ occurs at $p = 0$. Hence, we get the required result.

□

Lemma 2.3. *If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $SL^{\mathcal{C}}$, then*

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{12}, \quad |a_4| \leq \frac{1}{24}, \quad |a_5| \leq \frac{1}{40}.$$

These estimations are sharp.

Theorem 2.4. *Let f of the form 1.1 be in the class $SL^{\mathcal{C}}$, Then*

$$|H_3(1)| \leq \frac{49}{8640}.$$

Proof. Since

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2),$$

Now, using the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

Using the fact that $a_1 = 1$ with the results of Theorem 2.1, Theorem 2.2, Theorem 2.3 and Lemma 2.3, we obtain

$$|H_3(1)| \leq \frac{49}{8640}.$$

□

References

[1] Babalola, K. O., *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6** (2010), 1–7

[2] Duren, P. L., *Univalent functions*, Grundlehren der Mathematischen Wissenschaften Band 259, Springer-Verlag, New York, (1983)

[3] Janteng, A., Halim, S. A. and Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., **1** (2007), No. 13, 619–625

[4] Kanas, S., Analouei Adegani, E. and Zireh, A., *An unified approach to second Hankel determinant of bi-subordinate functions*. Mediterr. J. Math., **14** (2017), No. 6, Paper No. 233, 12 pp.

[5] Murugusundramurthi, G. and Magesh, N., *Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant*, Bull. Math. Anal. Appl., **1** (2009), No. 3, 85–89

[6] Mehrok, B. S. and Singh, Ga., *Estimate of second Hankel determinant for certain classes of analytic functions*, Scientia Magna, **3** (2012), 85–94

[7] Raza, M. and Malik, S. N., *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl., **2013**, 2013:412, 8pp.

[8] Singh, Ga., *Hankel determinant for a new subclass of analytic functions*, Scientia Magna, **4** (2012), 61–65

[9] Shanmugam, G., Adolf Stephen, B. and Babalola, K. O., *Third Hankel determinant for α- starlike functions*, Gulf Journal of Mathematics, **2** (2014), 107–113

Department of Mathematics
Faculty of Science
Payame Noor University
P.O.Box 19395-3697, Tehran, Iran
Email address: najafzadeh1234@yahoo.ie
Email address: h.rahmatan@gmail.com
Email address: hhaji88@yahoo.com