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Some Cesáro-Type quasinormal convergences

ABSTRACT. In this paper we introduce the concepts of quasinormal strong Cesàro convergence, quasinormal statistical convergence, lacunary strong quasinormal convergence and lacunary quasinormal statistical convergence of sequences of functions and give some inclusion relations.

1. INTRODUCTION

Let us start with basic definitions from the literature. A sequence $x = (x_k)$ is said to be Cesàro summable to the number ℓ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = \ell,$$

in this case we write $(C, 1) - \lim x_n = \ell$, strongly Cesàro summable to the number ℓ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0,$$

in this case we write $[C, 1] - \lim x_n = \ell$.

The natural density of a set A of positive integers is defined by

$$\delta(A) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in A\}|,$$

where $|k \le n : k \in A|$ denotes the number of elements of A not exceeding n.

A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \quad |x_k - \ell| \ge \epsilon\}| = 0$$

in this case we write $st - \lim x_n = \ell$.

Statistical convergence of sequences of numbers was introduced by Fast [12]. Schoenberg [23] established some basic properties of statistical convergence and also studied the concept as a summability method. The notions of pointwise statistical convergence, pointwise and uniformly statistical convergence order α, λ- statistical convergence of order α and pointwise lacunary statistical convergence of order α of function sequences are introduced and studied in [15, 7, 10, 11] respectively.

A sequence of functions $(f_k(x))$ is said to be statistically convergent to the function $f(x)$ if for every $\epsilon > 0$ and for each x

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |f_k(x) - f(x)| \ge \epsilon\}| = 0.$$

In this case we write $st - \lim f_k(x) = f(x)$. Statistical convergence of function sequences is a natural generalization of ordinary convergence of function sequences. If $\lim f_k(x) = f(x)$, then $st - \lim f_k(x) = f(x)$. The converse does not hold in general.

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Quasinormal convergence was introduced by Bukovská in[2] and Császár and Laczkovich in [8] independently. Authors of [8] called it as *equal convergence*. Also quasinormal convergence (equal convergence) was studied in [9] . For more detail see [20], [3] and [4].

Let f_n, f be real valued functions defined on a nonempty set E. The sequence $(f_n(x))$ is said to be quasinormal convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_n \to 0$ such that for every $x \in E$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \epsilon_n$ for $n \geq n_0$.

The n_0 may generally depend on x. If a sequence (f_n) converges uniformly to f on E, then it converges also quasinormally. On the other hand the quasinormal convergence implies the pointwise one. If all ϵ_n are equal zero, then the corresponding convergence is said to be discrete [2].

2. Quasinormal Cesàro convergence

Let f_n, f be real valued functions defined on a nonempty set E. The sequence $(f_n(x))$ is said to be the Cesàro summable to the function $f(x)$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_k(x) = f(x).$$

Let f_n, f be real valued functions defined on a nonempty set E. The sequence $(f_n(x))$ is said to be the strongly Cesàro summable to the function $f(x)$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |f_k(x) - f(x)| = 0.$$

Definition 2.1. Let f_n, f be real valued functions defined on a nonempty set E. The sequence $(f_n(x))$ is said to be the quasinormal Cesàro convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_n \to 0$ such that for every $x \in E$, there exists $n_0 = n_0(x)$ with

$$\frac{1}{n}\sum_{k=1}^{n} f_k(x) - f(x) < \epsilon_n$$

for $n \geq n_0$.

Definition 2.2. Let f_n, f be real valued functions defined on a nonempty set X. The sequence $(f_n(x))$ is said to be the strongly quasinormal Cesàro convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_n \to 0$ such that for every $x \in E$,there exists $n_0 = n_0(x)$ with

$$\frac{1}{n}\sum_{k=1}^{n} |f_k(x) - f(x)| < \epsilon_n$$

for $n \geq n_0$.

3. Quasinormal statistical convergence

Definition 3.3. Let f_n, f be real valued functions defined on a nonempty set E. The sequence $(f_n(x))$ is said to be the quasinormal statistical convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_k \to 0$ such that for every $x \in E$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |f_k(x) - f(x)| \geq \epsilon_k\}| = 0.$$

The space of all quasinormal statistical convergent function sequences will be denoted QNS.

Theorem 3.1. *Let f_n, f be real valued functions defined on a nonempty set E.*

i) If $(f_n(x))$ is strongly Cesàro convergent to f then $(f_n(x))$ quasinormal statistically convergent to f.

ii) If $(f_n(x))$ is bounded and quasinormal statistically convergent to f then $(f_n(x))$ strongly Cesàro convergent to f.

Proof. i) Let $(f_n(x))$ be strongly Cesàro convergent to f. For $\epsilon_k \to 0$ and every $x \in E$, we get

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n}|f_k(x)-f(x)| \\
&= \left(\frac{1}{n}\sum_{\substack{k=1\\|f_k(x)-f(x)|\geq\epsilon_k}}^{n}|f_k(x)-f(x)| + \frac{1}{n}\sum_{\substack{k=1\\|f_k(x)-f(x)|<\epsilon_k}}^{n}|f_k(x)-f(x)|\right) \\
&\geq \frac{1}{n}\sum_{\substack{k=1\\|f_k(x)-f(x)|\geq\epsilon_k}}^{n}|f_k(x)-f(x)| \\
&\geq \frac{1}{n}|\{1\leq k\leq n : |f_k(x)-f(x)|\geq\epsilon_k\}|\epsilon
\end{aligned}$$

where $\epsilon = \min\{\epsilon_k : k = 1, 2, ..., n\}$. Hence, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{1\leq k\leq n : |f_k(x)-f(x)|\geq\epsilon_k\}| = 0$$

that is, $(f_n(x))$ quasinormal statistically convergent to f.

ii) Now suppose that $(f_n(x))$ is bounded and strongly quasinormal statistically convergent to, since $(f_n(x))$ is bounded, say $|f_k(x) - f(x)| \leq K$ for all k and every $x \in E$. For $\epsilon_k \to 0$ and every $x \in E$ we get

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}|f_k(x)-f(x)| \\
&= \frac{1}{n}\left(\sum_{\substack{k=1\\|f_k(x)-f(x)|\geq\epsilon_k}}^{n}|f_k(x)-f(x)| + \sum_{\substack{k=1\\|f_k(x)-f(x)|<\epsilon_k}}^{n}|f_k(x)-f(x)|\right) \\
&\leq \frac{1}{n}\left(K\sum_{\substack{k=1\\|f_k(x)-f(x)|\geq\epsilon_k}}^{n}1 + \sum_{\substack{k=1\\|f_k(x)-f(x)|<\epsilon_k}}^{n}|f_k(x)-f(x)|\right) \\
&\leq K\frac{1}{n}|\{1\leq k\leq n : \quad |f_k(x)-f(x)|\geq\epsilon_k\}| + \frac{1}{n}\sum_{k=1}^{n}\epsilon_k.
\end{aligned}$$

Since $\epsilon_k \to 0$, we know that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\epsilon_k = 0$. Hence, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|f_k(x)-f(x)| = 0,$$

that is $(f_n(x))$ strongly Cesàro convergent to f. □

4. Lacunary quasinormal convergence

A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

For the concepts of lacunary strong summability, lacunary statistical convergence and lacunary statistical summability of number sequences see [14], [18] and [19].

The concepts of strong lacunary convergence and lacunary statistical convergence of function sequences are defined and studied in [16, 17].

Let f_n and f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the lacunary summable to the function $f(x)$ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} f_k(x) = f(x).$$

Let f_n, f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the strongly lacunary summable to the function $f(x)$ if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |f_k(x) - f(x)| = 0.$$

Let f_n, f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the lacunary statistically convergent to $f(x)$ if for every $\epsilon > 0$ and for every $x \in E$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \epsilon\}| = 0.$$

Definition 4.4. Let f_n and f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the lacunary quasinormal convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_r \to 0$ such that for every $x \in E$, there exists $r_0 = r_0(x)$ with

$$\frac{1}{h_r} \sum_{k\in I_r} f_k(x) - f(x) < \epsilon_r$$

for $r \geq r_0$.

Definition 4.5. Let f_n and f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the strongly lacunary quasinormal convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_r \to 0$ such that for every $x \in E$,there exists $r_0 = r_0(x)$ with

$$\frac{1}{h_r} \sum_{k\in I_r} |f_k(x) - f(x)| < \epsilon_r$$

for $r \geq r_0$.

5. Lacunary quasinormal statistical convergence

Definition 5.6. Let f_n and f be real valued functions defined on a nonempty set E and θ be any lacunary sequence. The sequence $(f_n(x))$ is said to be the lacunary quasinormal statistical convergent to $f(x)$ if there is a sequence of positive reals $\epsilon_k \to 0$ such that for every $x \in E$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |f_k(x) - f(x)| \geq \varepsilon_k\}| = 0.$$

The space of all lacunary quasinormal statistical convergent function sequences will be denoted QNS_θ.

Theorem 5.2. *Let f_n and f be real valued functions defined on a nonempty set E and θ be any lacunary sequence.*

i) If $(f_n(x))$ is lacunary strongly convergent to f then $(f_n(x))$ lacunary quasinormal statistically convergent to f.
ii) If $(f_n(x))$ is bounded and quasinormal statistically convergent to f then $(f_n(x))$ lacunary strongly convergent to f.

The proof of theorem similar to that of the Theorem 1, so we omit it.

By using the similar techniques to that in the Lemma 2.1 and Lemma 2.2 of [14], we can prove following theorem.

Theorem 5.3. *Let θ be any lacunary sequence. Then*

i) $QNS \subseteq QNS_\theta$ if and only if $\liminf_r \frac{k_r}{k_{r-1}} > 1$.
ii) $QNS_\theta \subseteq QNS$ if and only if $\limsup_r \frac{k_r}{k_{r-1}} < \infty$.
iii) $QNS = QNS_\theta$ if and only if

$$1 < \liminf_r \frac{k_r}{k_{r-1}} \leq \limsup_r \frac{k_r}{k_{r-1}} < \infty.$$

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