CREAT. MATH. INFORM. 30 (2021), No. 1, 81 - 88
Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Some 3×3 dimensional nonsingular matrices related to generalized Fibonacci numbers

TUĞBA PETIK, HILAL AKBULUT and HALIM ÖZDEMIR

ABSTRACT. The relation between integer powers of the generalized Fibonacci matrix $Q_g = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$ and generalized Fibonacci numbers is well known, where p and q are nonzero real numbers. Inspired by this relation, a procedure is presented to find some 3×3 dimensional nonsingular matrices whose powers are related to generalized Fibonacci numbers.

1. INTRODUCTION

The Fibonacci sequence $\{F_n\}_{n \geq 0}$ is defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \quad \text{for all integers} \quad n \geq 1,$$

where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence with negative subscripts is determined by the relation $F_{-n} = (-1)^{n+1} F_n$ for all integers $n \geq 1$, see, for instance, [14].

Fibonacci sequences appear in many mathematical problems in number theory and applied sciences, see, for instance, [2,4,14,17,19]. In addition, Fibonacci sequences are very useful for deriving interesting properties in mathematics. For example, there is a well known relation between the Fibonacci numbers and matrices: If $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n \in \mathbb{Z}$, see, for instance, [11,13,14]. Many identities associated with Fibonacci numbers were derived by using the relation between the matrices Q and Q^n. For example, the identity

$$F_{n+1} F_{n-1} - {F_n}^2 = (-1)^n$$

which is known as Cassini identity can be proved by using the determinants of the matrices Q and Q^n.

On the other hand, for all $n \in \mathbb{Z}$, the equalities

$$\alpha^n = \alpha F_n + F_{n-1} \quad \text{and} \quad \beta^n = \beta F_n + F_{n-1}$$

hold, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the polynomial $x^2 - x - 1$, which is the characteristic polynomial of the matrix Q, see, for instance, [11,14]. By using such properties, the matrix Q mentioned above, and matrix methods, many identities related to Fibonacci numbers were derived. For detail information, it can be looked at, for instance, [2,4,11,14,19].

Received: 10.07.2020. In revised form: 02.12.2020. Accepted: 09.12.2020
2010 *Mathematics Subject Classification.* 11B39, 15A18.
Key words and phrases. *Generalized Fibonacci numbers, matrix methods, eigenvalue-eigenvector.*
Corresponding author: Halim Özdemir; hozdemir@sakarya.edu.tr

Fibonacci numbers have been generalized in different ways by many authors. For example, in [5], Horadam defined the generalized Fibonacci numbers by the relation

$$H_n = H_{n-1} + H_{n-2} \quad n \geq 3; \quad H_1 = p, \quad H_2 = p + q,$$

where p, q are arbitrary integers. And then, Horadam gave a more general definition for the sequences of this type: The sequences $\{W_n(a, b; p, q)\}$ are defined by the general recurrence relation

$$W_n = pW_{n-1} - qW_{n-2} \quad n \geq 2; \quad W_0 = a, W_1 = b$$

where a, b, p, q are integers (see, for instance, [6–8]). For some special values of a, b, p, and q, Fibonacci and Lucas sequences are obtained; $F_n = W_n(0, 1; 1, -1)$, $L_n = W_n(2, 1; 1, -1)$.

Recently, Gupta et al. handled generalized Fibonacci sequences by the relation

$$F_k = pF_{k-1} + qF_{k-2}, \quad k \geq 2 \quad \text{with} \quad F_0 = a, \quad F_1 = b,$$

where p, q, a, and b are positive integers [3]. Also, the authors emphasized that many sequences could be determined for different values of p, q, a, and b.

In this work, we use a generalization similar to that given by Gupta et al. taking initial conditions as 0 and 1, and also taking the coefficients of recurrence relation in nonzero real numbers:

Let p and q be arbitrary nonzero real numbers. Consider a generalized Fibonacci sequence as

$$G_{n+1} = pG_n + qG_{n-1} \quad \text{for all integers} \quad n \geq 1 \quad \text{with} \quad G_0 = 0, \quad G_1 = 1. \tag{1.1}$$

This generalization corresponds exactly to the generalization in [18]. Generalized Fibonacci numbers with negative subscripts are determined by the relation $G_{-n} = \dfrac{-G_n}{(-q)^n}$ [18]. If $p = q = 1$ is taken, then the sequence (1.1) turns into the classical Fibonacci sequence.

Now, suppose that $p^2 + 4q > 0$. It is well known that

$$G_n = \frac{\alpha_{p,q}^n - \beta_{p,q}^n}{\alpha_{p,q} - \beta_{p,q}},$$

where $\alpha_{p,q} = \dfrac{p + \sqrt{p^2 + 4q}}{2}$ and $\beta_{p,q} = \dfrac{p - \sqrt{p^2 + 4q}}{2}$ (see, for instance, [18]). This identity is known as Binet's Formula. Notice that $\alpha_{p,q}$ and $\beta_{p,q}$ are the roots of the polynomial $x^2 - px - q$. It is obvious that

$$\alpha_{p,q} + \beta_{p,q} = p,$$
$$\alpha_{p,q} - \beta_{p,q} = \sqrt{p^2 + 4q},$$

and

$$\alpha_{p,q}\beta_{p,q} = -q.$$

From now on, for the sake of simplicity, α and β will be used instead of $\alpha_{p,q}$ and $\beta_{p,q}$, respectively.

There are many identities concerning generalized Fibonacci numbers mentioned here. Many of them can be proved by using Binet's Formula, induction, and matrix methods. For example, Şiar and Keskin proved that $Q_g^n = \begin{pmatrix} G_{n+1} & qG_n \\ G_n & qG_{n-1} \end{pmatrix}$ with $Q_g = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$, and

$$\alpha^n = \alpha G_n + qG_{n-1}, \quad \beta^n = \beta G_n + qG_{n-1} \tag{1.2}$$

for all $n \in \mathbb{Z}$ [18]. For detail information related to generalized Fibonacci numbers, it can be looked at, for instance, [1,3,5–9,12,15,16,18].

In [10], the authors obtained some 3×3 dimensional matrices, whose powers are related to Fibonacci numbers, thanks to a procedure using the diagonalization method of matrices. Basic idea of the procedure is based on the relation between the matrices $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for $n \in \mathbb{Z}$. The procedure established in this work is similar to the one in [10] because of the relation between the matrices Q_g and Q_g^n for all $n \in \mathbb{Z}$. We shall investigate some 3×3 dimensional matrices whose eigenvalues are α, β, and 1. Since q is nonzero, it is obvious that α and β are nonzero, too. So, all the eigenvalues of the matrices which will be obtained will also be nonzero.

It is a well-known fact that there are many applications of Fibonacci sequences in real life, see, for instance, [14, 17]. Therefore, the topic is important not only in terms of the theory but also in terms of its applications. On the other hand, if there is a real physical problem that needs to be solved, then it must first be modeled mathematically. It is noteworthy the words of the Russian mathematician Nicholas Lobachevsk: "*There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world*". So, we believe that the considered problem will probably corresponds to a real physical problem in the future.

2. Main results

As mentioned before, the main aim of the work is to present a procedure to find some 3×3 dimensional matrices whose powers are related to generalized Fibonacci numbers.

Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}$$

be a 3×3 matrix having the eigenvalues $\lambda_1 = \alpha$, $\lambda_2 = \beta$, and $\lambda_3 = 1$. It is easily seen that the eigenvalues λ_1, λ_2, and λ_3 become mutually different if $p^2 + 4q > 0$ and $p + q \neq 1$. Thereafter, it will be continued under these assumptions.

Let us assume that the vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and $\mathbf{z} = (z_1, z_2, z_3)$ are the eigenvectors corresponding to the eigenvalues λ_1, λ_2, and λ_3, respectively. So, we have the systems of equations $A\mathbf{x} = \lambda_1\mathbf{x}$, $A\mathbf{y} = \lambda_2\mathbf{y}$, and $A\mathbf{z} = \lambda_3\mathbf{z}$. The matrix A is diagonalizable since all the eigenvalues are mutually different. So, without loss of the generality, we can write

$$A = S\Lambda S^{-1}$$

where $\Lambda = diag(\alpha, \beta, 1)$ and S is the matrix having the columns $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, respectively. Hence, we obtain

$$A^n = S\Lambda^n S^{-1}$$

for all $n \in \mathbb{Z}$. Thus, considering the equalities (1.2), we get

$$A^n = S \begin{pmatrix} \alpha^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} = S \begin{pmatrix} \alpha G_n + qG_{n-1} & 0 & 0 \\ 0 & \beta G_n + qG_{n-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1},$$

or equivalently,

$$A^n = G_n A + qG_{n-1} I_3 + (1 - G_n - qG_{n-1})D, \tag{2.3}$$

where

$$D = S \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}.$$

It is obvious that we have

$$\det(S) = z_1 (x_2 y_3 - y_2 x_3) + z_2 (x_3 y_1 - x_1 y_3) + z_3 (x_1 y_2 - y_1 x_2) \tag{2.4}$$

and

$$D = \frac{1}{\det(S)} \begin{pmatrix} z_1 (x_2 y_3 - y_2 x_3) & z_1 (x_3 y_1 - x_1 y_3) & z_1 (x_1 y_2 - y_1 x_2) \\ z_2 (x_2 y_3 - y_2 x_3) & z_2 (x_3 y_1 - x_1 y_3) & z_2 (x_1 y_2 - y_1 x_2) \\ z_3 (x_2 y_3 - y_2 x_3) & z_3 (x_3 y_1 - x_1 y_3) & z_3 (x_1 y_2 - y_1 x_2) \end{pmatrix}. \tag{2.5}$$

From now on, without loss of the generality, we will proceed choosing $\mathbf{x} = (\alpha, \beta, -1)$ and $\mathbf{y} = (\beta, \alpha, -1)$. Under these assumptions, the identities

$$\begin{aligned} a\left(\tfrac{p+\sqrt{p^2+4q}}{2}\right) + b\left(\tfrac{p-\sqrt{p^2+4q}}{2}\right) - c = \alpha^2, \\ a\left(\tfrac{p-\sqrt{p^2+4q}}{2}\right) + b\left(\tfrac{p+\sqrt{p^2+4q}}{2}\right) - c = \beta^2 \end{aligned} \tag{2.6}$$

can be obtained from the first equations of $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$, respectively. Adding and subtracting the terms of the equalities (2.6) side by side lead to the equalities

$$\begin{aligned} &ap + bp - 2c = \alpha^2 + \beta^2 = p^2 + 2q, \\ &a\left(\sqrt{p^2+4q}\right) - b\left(\sqrt{p^2+4q}\right) = \alpha^2 - \beta^2 = p\sqrt{p^2+4q}, \end{aligned} \tag{2.7}$$

respectively.

Similarly, from the second and third equations of $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$, proceeding in exactly the same way, we get

$$\begin{aligned} &dp + ep - 2f = -2q, \\ &d\sqrt{p^2+4q} - e\sqrt{p^2+4q} = 0 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &gp + hp - 2l = -p, \\ &g\sqrt{p^2+4q} - h\sqrt{p^2+4q} = -\sqrt{p^2+4q}. \end{aligned} \tag{2.9}$$

So, to satisfy the systems of equations $A\mathbf{x} = \lambda_1 \mathbf{x}$ and $A\mathbf{y} = \lambda_2 \mathbf{y}$, it is necessary to satisfy the systems of equations (2.7), (2.8), and (2.9), or equivalently, the system of equations

$$\begin{aligned} &pa + pb - 2c = p^2 + 2q, a - b = p, \\ &pd + pe - 2f = -2q, d - e = 0, \\ &pg + ph - 2l = -p, g - h = -1. \end{aligned} \tag{2.10}$$

Taking into account the choices of the vectors $\mathbf{x}$ and $\mathbf{y}$, it is seen that we have the equalities

$$x_2 y_3 - y_2 x_3 = x_3 y_1 - x_1 y_3 = \sqrt{p^2+4q} \quad \text{and} \quad x_1 y_2 - y_1 x_2 = p\sqrt{p^2+4q}. \tag{2.11}$$

Now, we can find different matrices A satisfying all of the systems $A\mathbf{x} = \lambda_1 \mathbf{x}$, $A\mathbf{y} = \lambda_2 \mathbf{y}$, and $A\mathbf{z} = \lambda_3 \mathbf{z}$ by doing appropriate choices for the vector $\mathbf{z}$:

Firstly, let $\mathbf{z} = (z_1, z_2, z_3)$ be a vector such that $z_1 = k, z_2 = -k, z_3 = k$ with k being a positive integer. In this case, from (2.4) and (2.11), it is obtained that

$$\det(S) = kp\sqrt{p^2+4q}.$$

So, we get

$$D = \frac{1}{p}\begin{pmatrix} 1 & 1 & p \\ -1 & -1 & -p \\ 1 & 1 & p \end{pmatrix}$$

from (2.5). In addition, the system of equations $A\mathbf{z} = \lambda_3\mathbf{z}$ turns into the system of equations

$$\begin{aligned} a - b + c &= 1 \\ d - e + f &= -1 \\ g - h + l &= 1. \end{aligned} \tag{2.12}$$

From the common solution of the systems (2.10) and (2.12), we obtain

$$A = \frac{1}{p}\begin{pmatrix} p^2 - p + q + 1 & -p + q + 1 & -p^2 + p \\ -1 - q & -1 - q & -p \\ 2 - p & 2 & 2p \end{pmatrix}. \tag{2.13}$$

So, from (2.3), we get

$$A^n = \frac{1}{p}\begin{pmatrix} (p^2 - p + q)G_n + q(p-1)G_{n-1} + 1 & (-p+q)G_n - qG_{n-1} + 1 & -p^2G_n - pqG_{n-1} + p \\ -qG_n + qG_{n-1} - 1 & -qG_n + q(1+p)G_{n-1} - 1 & -p + pqG_{n-1} \\ (1-p)G_n - qG_{n-1} + 1 & G_n - qG_{n-1} + 1 & pG_n + p \end{pmatrix}$$

for all $n \in \mathbb{Z}$. Thus, we have proved the following result.

Theorem 2.1. *If* $A = \frac{1}{p}\begin{pmatrix} p^2 - p + q + 1 & -p + q + 1 & -p^2 + p \\ -1 - q & -1 - q & -p \\ 2 - p & 2 & 2p \end{pmatrix}$*, where p and q are nonzero real numbers such that* $p + q \neq 1$ *and* $p^2 + 4q > 0$*, then*

$$A^n = \frac{1}{p}\begin{pmatrix} (p^2 - p + q)G_n + q(p-1)G_{n-1} + 1 & (-p+q)G_n - qG_{n-1} + 1 & -p^2G_n - pqG_{n-1} + p \\ -qG_n + qG_{n-1} - 1 & -qG_n + q(1+p)G_{n-1} - 1 & -p + pqG_{n-1} \\ (1-p)G_n - qG_{n-1} + 1 & G_n - qG_{n-1} + 1 & pG_n + p \end{pmatrix}$$

for all $n \in \mathbb{Z}$.

□

Secondly, let $\mathbf{z} = (z_1, z_2, z_3)$ be a vector such that $z_1 = k, z_2 = k, z_3 = -k$ with k being a positive integer. From (2.4) and (2.11), it is clear that $\det(S) = (2k - kp)\sqrt{p^2 + 4q}$. Now, under the assumption $p \neq 2$, the matrix

$$D = \frac{1}{2-p}\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & p \\ -1 & -1 & -p \end{pmatrix}$$

is obtained from (2.5). On the other hand, the system of equations $A\mathbf{z} = \lambda_3\mathbf{z}$ turns into the system of equations

$$\begin{aligned} a + b - c &= 1 \\ d + e - f &= 1 \\ g + h - l &= -1. \end{aligned} \tag{2.14}$$

Common solution of (2.10) and (2.14) leads to the matrix

$$A = \frac{1}{p-2}\begin{pmatrix} p^2 + q - p - 1 & q + p - 1 & p^2 + 2q - p \\ -q - 1 & -q - 1 & -2q - p \\ 2 - p & 0 & 0 \end{pmatrix}. \tag{2.15}$$

So, from (2.3), we get

$$A^n = \frac{1}{p-2}\begin{pmatrix} (p^2+q-p)G_n+q(p-1)G_{n-1}-1 & (q+p)G_n+qG_{n-1}-1 & (p^2+2q)G_n+pqG_{n-1}-p \\ -qG_n+qG_{n-1}-1 & -qG_n+q(p-1)G_{n-1}-1 & -2qG_n+pqG_{n-1}-p \\ (1-p)G_n-qG_{n-1}+1 & 1-G_n-qG_{n-1} & -pG_n+q(p-3)G_{n-1}+p \end{pmatrix}$$

for all $n \in \mathbb{Z}$. Thus, we have proved the following theorem.

Theorem 2.2. *If* $A = \frac{1}{p-2}\begin{pmatrix} p^2+q-p-1 & q+p-1 & p^2+2q-p \\ -q-1 & -q-1 & -2q-p \\ 2-p & 0 & 0 \end{pmatrix}$*, where* p *and* q *are nonzero real numbers such that* $p+q \neq 1$, $p^2+4q > 0$, *and* $p \neq 2$, *then*

$$A^n = \frac{1}{p-2}\begin{pmatrix} (p^2+q-p)G_n+q(p-1)G_{n-1}-1 & (q+p)G_n+qG_{n-1}-1 & (p^2+2q)G_n+pqG_{n-1}-p \\ -qG_n+qG_{n-1}-1 & -qG_n+q(p-1)G_{n-1}-1 & -2qG_n+pqG_{n-1}-p \\ (1-p)G_n-qG_{n-1}+1 & 1-G_n-qG_{n-1} & -pG_n+q(p-3)G_{n-1}+p \end{pmatrix}$$

for all $n \in \mathbb{Z}$.

□

Finally, let $\mathbf{z} = (z_1, z_2, z_3)$ be a vector such that $z_1 = -k, z_2 = k$, and $z_3 = k$ with k being a positive integer. In view of the former discussions, from (2.4) and (2.11) again, it is obtained that $\det(S) = kp\sqrt{p^2+4q}$. So, from (2.5), we get

$$D = \frac{1}{p}\begin{pmatrix} -1 & -1 & -p \\ 1 & 1 & p \\ 1 & 1 & p \end{pmatrix}.$$

Moreover, the system of linear equations $A\mathbf{z} = \lambda_3\mathbf{z}$ turns into the system

$$\begin{aligned} -a+b+c &= -1 \\ -d+e+f &= 1 \\ -g+h+l &= 1. \end{aligned} \tag{2.16}$$

From the common solution of (2.10) and (2.16), it is obtained that

$$A = \frac{1}{p}\begin{pmatrix} p^2+p+q-1 & p+q-1 & p^2-p \\ -q+1 & -q+1 & p \\ -p & 0 & 0 \end{pmatrix}. \tag{2.17}$$

So, from (2.3), we obtain

$$A^n = \frac{1}{p}\begin{pmatrix} (p^2+p+q)G_n+(pq+q)G_{n-1}-1 & (p+q)G_n+qG_{n-1}-1 & p^2G_n+pqG_{n-1}-p \\ -qG_n-qG_{n-1}+1 & -qG_n+(pq-q)G_{n-1}+1 & p-pqG_{n-1} \\ (-p-1)G_n-qG_{n-1}+1 & 1-G_n-qG_{n-1} & p-pG_n \end{pmatrix}$$

for all $n \in \mathbb{Z}$. Thus, we have the following result.

Theorem 2.3. *If* $A = \frac{1}{p}\begin{pmatrix} p^2+p+q-1 & p+q-1 & p^2-p \\ -q+1 & -q+1 & p \\ -p & 0 & 0 \end{pmatrix}$*, where* p *and* q *are nonzero real numbers such that* $p+q \neq 1$ *and* $p^2+4q > 0$ *, then*

$$A^n = \frac{1}{p}\begin{pmatrix} (p^2+p+q)G_n+(pq+q)G_{n-1}-1 & (p+q)G_n+qG_{n-1}-1 & p^2G_n+pqG_{n-1}-p \\ -qG_n-qG_{n-1}+1 & -qG_n+(pq-q)G_{n-1}+1 & p-pqG_{n-1} \\ (-p-1)G_n-qG_{n-1}+1 & 1-G_n-qG_{n-1} & p-pG_n \end{pmatrix}$$

for all $n \in \mathbb{Z}$.

□

Recall that the Fibonacci Q matrix is defined by

$$Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

As we have already pointed out in Introduction. For all $n \in \mathbb{Z}$, the equality

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

is well known, where F_n is n-th Fibonacci number [11,13,14]. Notice that the eigenvalues of the matrix Q are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Based on this fact, in [10], the authors gave a procedure to find some 3×3 dimensional matrices whose eigenvalues are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, and 0. Since the matrices obtained in [10] are singular, the results are only valid for nonnegative integers n. Even so, the procedure established in this work is similar to the one in [10]. Since the procedure in this work contains 3×3 dimensional nonsingular matrices having the eigenvalues $\frac{p+\sqrt{p^2+4q}}{2}$, $\frac{p-\sqrt{p^2+4q}}{2}$, and 1, where p and q are nonzero real numbers with $p+q \neq 1$ and $p^2+4q > 0$, the results given here are valid all integers n. In addition, if it is taken $p = q = 1$, then the results obtained are reduced to the nonsingular matrices whose powers are related to Fibonacci numbers. Finally, note that for the sake of simplicity and without loss of the generality, the third nonzero eigenvalue of the matrix A is selected as 1.

Let's close the work giving an illustrating example. For example, suppose that $p = q = 1$. So, as pointed out in Theorem 2.1, if $A = \begin{pmatrix} 2 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$, then

$$A^n = \begin{pmatrix} F_n+1 & -F_{n-1}+1 & -F_n-F_{n-1}+1 \\ -F_n+F_{n-1}-1 & -F_n+2F_{n-1}-1 & -1+F_{n-1} \\ 1-F_{n-1} & F_n-F_{n-1}+1 & F_n+1 \end{pmatrix}$$

for all $n \in \mathbb{Z}$.

Acknowledgements. The authors are very thankful to two anonymous referees who made a very careful reading and many valuable comments, corrections, and suggestions.

REFERENCES

[1] Cerda-Morales, G., *On generalized Fibonacci and Lucas numbers by matrix methods*, Hacet. J. Math. Stat, **42** (2013), No. 2, 173–179

[2] Dunlap, R. A., *The Golden Ratio and Fibonacci Numbers*, World Scientific, 1997

[3] Gupta, V. K., Panwar, K. Y., and Sikhwal, O., *Generalized Fibonacci Sequences*, Theoretical Mathematics & Applications, **2** (2012), No. 2, 115–124

[4] Hoggatt, JR., V. E., *Fibonacci and Lucas Numbers*, Fibonacci Association, University of Santa Clara, 1969
[5] Horadam, A. F., *A Generalized Fibonacci Sequence*, Amer. Math. Montly, **68** (1961), No. 5, 455–459
[6] Horadam, A. F., *Basic Properties of a Certain Generalized Sequence of Numbers*, Fibonacci Quart., **3** (1965), No. 3, 161–176
[7] Horadam, A. F., *Generating Function for Powers of a Certain Generalised Sequence of Numbers*, Duke Math. J., **32** (1965), No. 3, 437–446
[8] Horadam, A. F., *Special Properties of the Sequence* $W_n(a, b; p, q)$, Fibonacci Quart., **5** (1967), No. 5, 424–434
[9] Kalman D. and Mena, R., *The Fibonacci numbers-exposed*, Math. Mag., **76** (2003), No. 3, 167–181
[10] Karakaya, S., Özdemir, H., and Petik, T., 3×3 *Dimensional Special Matrices Associated with Fibonacci and Lucas numbers*, Sakarya University Journal of Science, **22** (2018), No. 6, 1917–1922
[11] Keskin, R. and Demirtürk, B., *Some new Fibonacci and Lucas identities by matrix methods*, Internat. J. Math. Ed. Sci. Tech., **41** (2010), No. 3, 379–387
[12] Kılıç, E., *Sums of Generalized Fibonacci Numbers by Matrix Methods*, Ars Combin., **84** (2007), 23–31
[13] King, C.H., *Some Properties of the Fibonacci Numbers, Master's Thesis*, San Jose State College, June, 1960
[14] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, Wiley Interscience, 2001
[15] Panwar, Y. K., Singh, B., and Gupta, V. K., *Generalized Fibonacci Sequences and Its Properties*, Palest. J. Math., **3** (2014), No. 1, 141–147
[16] Panwar, Y. K. and Singh, M., *Certain Properties of Generalized Fibonacci Sequence*, Turkish Journal of Analysis and Number Theory, **2** (2014), No. 1, 6–8
[17] Ribenboim, P., *My Numbers, My Friends: Popular Lectures on Number Theory*, Springer-Verlag New York, 2000
[18] Şiar, Z. and Keskin, R., *Some new identities concerning generalized Fibonacci and Lucas numbers*, Hacet. J. Math. Stat., **42** (2013), No. 3, 211–222
[19] Vajda, S., *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications*, Courier Corporation, 1989

DEPARTMENT OF MATHEMATICS
SAKARYA UNIVERSITY
SERDIVAN, 54187, SAKARYA, TURKEY
Email address: tpetik@sakarya.edu.tr
Email address: hckmk58@gmail.com
Email address: hozdemir@sakarya.edu.tr