CREAT. MATH. INFORM. **Online version at https://creative-mathematics.ubm.ro/**
30 (2021), No. 1, 97 - 106 **Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

Durrmeyer type modification of (p,q)-Szász Mirakjan operators and their quantitative estimates

HONEY SHARMA and RAMAPATI MAURYA

ABSTRACT. In the present paper, we introduce Durrmeyer type modification of (p,q)-Szász Mirakjan operators. We estimate the rate of convergence and local approximation behavior of the operators using modulus of continuity and Peetre's K-functional. We compute the quantitative estimates for difference of the proposed operators with (p,q)-Szász Mirakjan operators and (p,q)-Szász Mirakjan Kantorovich operators.

1. INTRODUCTION

In the last two decades, quantum calculus (q-calculus) was studied extensively in the field of approximation theory and quantum generalizations of several positive linear operators have been discussed, for details one may refer to [1]. Further in 2015, Mursaleen et al. [13] introduced the (p,q)-Bernstien operators (Post Quantum generalization of Bernstien operators) and discussed their approximation properties. Following [13], many authors have studied the post quantum generalizations of various operators and investigated about their approximation properties, for details, readers may refer to [10, 12, 15, 16, 17, 19].
Throughout the paper, we employ the following standard notations of (p,q)-calculus:
Let $0<q<p<1$. The (p,q)-integer $[n]_{p,q}$ is defined as

$$[n]_{p,q}=\frac{p^n-q^n}{p-q} \quad n=0,1,2....$$

and (p,q)-factorial is defined by

$$[n]_{p,q}!=\begin{cases}[1]_{p,q}[2]_{p,q}.........[n]_{p,q}, & \text{n} \geq 1\\ 1, & \text{n=0}\end{cases}.$$

For integers $0\leq k\leq n$, (p,q)-binomial coefficient is defined as

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The (p,q)-binomials expansion is expressed as:

$$(x+y)^n_{p,q}=\prod_{j=0}^{n-1}(p^jx+q^jy).$$

(p,q)-form of the exponential function is defined as follows:

$$e_{p,q}(x)=\sum_{n=0}^{\infty}\frac{p^{n(n-1)/2}x^n}{[n]_{p,q}!}$$

Received: 09.04.2020. In revised form: 15.09.2020. Accepted: 22.09.2020
2010 *Mathematics Subject Classification.* 41A25, 41A35.
Key words and phrases. *(p,q)-Calculus, (p,q)-Szász Mirakjan operators, modulus of continuity, Peetre's K-functional.*
Corresponding author: Ramapati Maurya; ramapatimaurya@gmail.com

and

$$E_{p,q}(x)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}x^n}{[n]_{p,q}!}.$$

(p,q)-analog of the Gamma function is defined as

$$\Gamma_{p,q}(n)=\int_0^{\infty}\frac{p^{(n-1)(n-2)/2}x^{n-1}}{E_{p,q}(qx)}d_{p,q}x$$

and $\Gamma_{p,q}(n+1)=[n]_{p,q}!$ for $n\in\mathbb{N}$.
Recently, T. Acar [2] introduced (p,q)-Szász Mirakjan operators and investigated the rate of uniform convergence over bounded and unbounded intervals, weighted approximation and Voronovskaya type theorem for these operators. Sharma et al. [18] introduced the Kantorovich type modification of (p,q)-Szász Mirakjan operators and investigated about their approximation behavior. For more studies in the similar direction, see [3, 4, 5, 6, 7, 11, 14].
Inspired by the above studies, we introduce Durrmeyer type modification of (p,q)-Szász Mirakjan operators and study their approximation properties.

2. Construction of Operator

For $0<q<p<1$ and $f\in C[0,\infty)$, (p,q)-Szász Mirakjan Durrmeyer operators are defined as follows:

$$D_n^{p,q}(f;x)=\frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{k(k-1)/2}}{q^{k(k+1)/2}}\int_0^{\infty}s_{n,k}^{p,q}(t)f\left(\frac{p^k}{q^k}t\right)d_{p,q}t,$$

where $s_{n,k}^{p,q}(.)$ is base function defined as

$$s_{n,k}^{p,q}(x)=\frac{q^{k(k-1)/2}}{E_{p,q}([n]_{p,q}x)}\frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}$$

and $E_{p,q}$ is (p,q)-exponential function.
By using definition of (p,q)-exponential function, we directly get $\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)=1$.

Lemma 2.1. *For $m\in\mathbb{N}$ and $I_m(k)=\frac{p^{k(k-1)/2}}{q^{k(k+1)/2}}\int_0^{\infty}s_{n,k}^{p,q}(t)\frac{p^{mk}}{q^{mk}}t^m d_{p,q}t$, we have*

$$I_m(k)\quad=\quad\frac{[k+1]_{p,q}[k+2]_{p,q}......[k+m]_{p,q}}{q^{(mk-m-1)}\ p^{(m(m-1)/2)}\ [n]_{p,q}^{m+1}}.$$

Proof. Using definition of (p,q)-Gamma function and the equality $\Gamma_{p,q}(n+1)=[n]_{p,q}!$ for $n\in\mathbb{N}$, we have

$$I_m(k)=\frac{p^{k(k-1)/2}}{q^{k(k+1)/2}}\int_0^{\infty}s_{n,k}^{p,q}(t)\frac{p^{mk}}{q^{mk}}t^m d_{p,q}t=\frac{p^{k(k-1)/2}}{q^{k(k+1)/2}}\frac{p^{mk}}{q^{mk}}\int_0^{\infty}\frac{q^{k(k-1)/2}}{E_{p,q}([n]_{p,q}t)}\frac{[n]_{p,q}^k}{[k]_{p,q}!}t^{m+k}d_{p,q}t$$

$$=p^{k(k-1)/2}\frac{p^{mk}}{q^{mk}[k]_{p,q}!}\frac{q^{m+1}}{[n]_{p,q}^{m+1}}\int_0^{\infty}\frac{1}{E_{p,q}(qt)}t^{m+k}d_{p,q}t$$

$$=\frac{p^{(2mk+k^2-k)/2}}{q^{mk-m-1}[k]_{p,q}!}\frac{1}{[n]_{p,q}^{m+1}}\frac{[k+m]_{p,q}!}{p^{(k+m)(k+m-1)/2}}=\frac{[k+1]_{p,q}[k+2]_{p,q}......[k+m]_{p,q}}{q^{(mk-m-1)}\ p^{m(m-1)/2}\ [n]_{p,q}^{m+1}}.$$

□

Lemma 2.2. *We have*

$$\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}}=x;\ \sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\left(\frac{[k]_{p,q}}{q^{(k-1)}[n]_{p,q}}\right)^2=\frac{x}{[n]_{p,q}}+\frac{p}{q}x^2.$$

Lemma 2.3. *For $e_i = t^i,\ i = 0,1,2$, the moments of the proposed operators are estimated as follows:*
(i) $D_n^{p,q}(e_0;x) = 1$,
(ii) $D_n^{p,q}(e_1;x) = \frac{q}{[n]_{p,q}} + px$,
(iii) $D_n^{p,q}(e_2;x) = \frac{q^2(q+p)}{p[n]_{p,q}^2} + \frac{(2pq+q^2+p^2)}{[n]_{p,q}}x + \frac{p^3}{q}x^2$.

Proof. For $f(t) = t^m,\ m = 0,1,2$, the proposed operators can be expressed as:

$$D_n^{p,q}(t^m;x) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) I_m(k).$$

For $m = 0$, we have

$$D_n^{p,q}(e_0;x) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) I_0(k) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{q}{[n]_{p,q}} = 1.$$

Using Lemma 2.1, Lemma 2.2 and identity $[k+1]_{p,q} = q^k + p[k]_{p,q}$, first moment can be obtained as follows:

$$D_n^{p,q}(e_1;x) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) I_1(k) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{[k+1]_{p,q}}{q^{k-2}[n]_{p,q}^2}$$

$$= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\left(\frac{q}{[n]_{p,q}} + \frac{p[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right) = \frac{q}{[n]_{p,q}} + px.$$

Using Lemma 2.1, 2.2, the second moment can be obtained as follows:

$$D_n^{p,q}(e_2;x) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) I_2(k) = \frac{[n]_{p,q}}{q}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{p^{2k}[k+1]_{p,q}[k+2]_{p,q}}{q^{2k-3}p\ [n]_{p,q}^3}$$

$$= \frac{q^2}{[n]_{p,q}^2}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{1}{pq^{2k}}\left(q^{2k+1} + pq^{2k} + 2p^2q^k[k]_{p,q} + pq^{k+1}[k]_{p,q} + p^3[k]_{p,q}^2\right)$$

$$= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\left(\frac{q^2(p+q)}{p[n]_{p,q}^2} + \frac{q(2p+q)}{[n]_{p,q}}\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}} + \frac{p^2[k]_{p,q}^2}{q^{2(k-1)}[n]_{p,q}^2}\right)$$

$$= \frac{q^2(p+q)}{p[n]_{p,q}^2} + \frac{(2pq+q^2+p^2)}{[n]_{p,q}}x + \frac{p^3}{q}x^2.$$

□

Lemma 2.4. *Using Lemma 2.3, the central moments are computed as follows:*

$$D_n^{p,q}(t-x;x) = \frac{q}{[n]_{p,q}} + (p-1)x.$$

$$D_n^{p,q}((t-x)^2;x) = \frac{q^2(q+p)}{p[n]_{p,q}^2} + \frac{(q+p)^2-2q}{[n]_{p,q}}x + \left(1-2p+\frac{p^3}{q}\right)x^2.$$

Remark 2.1. For $0 < q < p \le 1$, one can observe that $\lim_{n\to\infty}[n]_{p,q} = \frac{1}{p-q}$, which restricts the operators $\{D_n^{p,q}\}$ to involve in the approximation process. We proposed the sequence $\{p_n\}$, $\{q_n\}$ such that $0 < q_n < p_n \le 1$ and $q_n \to 1,\ p_n \to 1,\ q_n^n \to a,\ p_n^n \to b$ as $n \to \infty$. Such sequences always exists, for instance, let $a,\ b \in \mathbb{R}^+$, $a > b$ and $q_n = \left(1+\frac{a}{n}\right)^{-1}$, $p_n =$

$\left(1+\frac{b}{n}\right)^{-1}$ then $0<q_n<p_n\le 1$ and $q_n\to 1,\ p_n\to 1,\ q_n^n\to e^a,\ p_n^n\to e^b, [n]_{p_n,q_n}\to\infty$ as $n\to\infty$. For the above type of sequences, we may consider the following:

$$\lim_{n\to\infty}[n]_{p_n,q_n}(p_n-1)=\alpha,\ \lim_{n\to\infty}[n]_{p_n,q_n}\left(1-2p_n+\frac{p_n^3}{q_n}\right)=\beta.$$

Theorem 2.1. *For the sequence $\{p_n\}$ and $\{q_n\}$ as considered in Remark 2.1 and $f\in C[0,\infty)$, the sequence of operators $\{D_n^{p_n,q_n}(f;.)\}$ converges uniformly to f.*

Proof. By using well-known Korovkin theorem along with Lemma 2.3, result can be obtained directly. □

3. Convergence properties of operators

In this section, we will present some direct results and Voronovskaya type theorem to estimate the rate of convergence of proposed operators.

Theorem 3.2 (Local approximation theorem). *For $f\in C_B[0,\infty)$ be the space of real valued continuous and bounded functions defined on $[0,\infty)$ equipped with supremum norm and sequences $\{p_n\}$ and $\{q_n\}$ as assumed in Remark 2.1, there exists an absolute constant $C>0$ such that*

$$|D_n^{p_n,q_n}(f;x)-f(x)|\le C\omega_2(f,\delta_n(x))+\omega\left(f,\left|\frac{q_n}{[n]_{p_n,q_n}}-(1-p_n)x\right|\right),\forall x\in[0,\infty),$$

where $\delta_n^2(x)=\frac{3q_n^2}{p_n[n]_{p_n,q_n}^2}+\frac{(q_n+p_n)^2+2p_nq_n-4q_n}{[n]_{p_n,q_n}}x+2\left(1-2p_n+\frac{p_n^3}{q_n}\right)x^2.$

Proof. To prove the result, we shall employ the properties of modulus of continuity and Peetre's K-functional. For details readers may refer to [8, Theorem 2.4].
For $f\in C_B[0,\infty)$, we define

$$\tilde{D}_n^{p_n,q_n}(f;x)=D_n^{p_n,q_n}(f;x)+f(x)-f\left(\frac{q_n}{[n]_{p_n,q_n}}+p_nx\right).$$

Using Lemma 2.3, we immediately get, $\tilde{D}_n^{p_n,q_n}(t-x;x)=0$.
For $g\in W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$, by using Taylor's formula, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Therefore, we have

$$\tilde{D}_n^{p_n,q_n}(g;x)=g(x)+\tilde{D}_n^{p_n,q_n}\left(\int_x^t(t-u)g''(u)du;x\right)$$

$$=g(x)+D_n^{p_n,q_n}\left(\int_x^t(t-u)g''(u)du;x\right)-\int_x^{\frac{q_n}{[n]_{p_n,q_n}}+p_nx}\left(\frac{q_n}{[n]_{p_n,q_n}}+p_nx-u\right)g''(u)du.$$

Thus,

$$\begin{aligned}|\tilde{D}_n^{p_n,q_n}(g;x)-g(x)| &\le D_n^{p_n,q_n}\left(\int_x^t|t-u||g''(u)|du;x\right)\\ &+\int_x^{\frac{q_n}{[n]_{p_n,q_n}}+p_nx}\left|\frac{q_n}{[n]_{p_n,q_n}}+p_nx-u\right||g''(u)|du\\ &\le D_n^{p_n,q_n}((t-x)^2;x)\|g''\|\\ &+\left(\frac{q_n}{[n]_{p_n,q_n}}+p_nx-x\right)^2\|g''\|.\end{aligned}$$

Using central moments, we get

$$\begin{aligned}|\tilde{D}_n^{p_n,q_n}(g;x)-g(x)| &\leq \left(\frac{3q_n^2}{p_n[n]_{p_n,q_n}^2}+\frac{(q_n+p_n)^2+2p_nq_n-4q_n}{[n]_{p_n,q_n}}x\right.\\ &+ \left.2\left(1-2p_n+\frac{p_n^3}{q_n}\right)x^2\right)\|g''\|\\ &= \delta_n^2(x).\|g''\|.\end{aligned}$$

Also, $|\tilde{D}_n^{p_n,q_n}(f;x)| \leq 3\|f\|$. We observe that, for $f \in C_B[0,\infty)$ and $g \in W^2$, we have

$$\begin{aligned}&|D_n^{p_n,q_n}(f;x)-f(x)|\\ &= \left|\tilde{D}_n^{p_n,q_n}(f;x)-f(x)+f\left(\frac{q_n}{[n]_{p_n,q_n}}+p_nx\right)-f(x)\right|\\ &\leq |\tilde{D}_n^{p_n,q_n}(f-g;x)|+|\tilde{D}_n^{p_n,q_n}(g;x)-g(x)|+|g(x)-f(x)|\\ &\ +\left|f\left(\frac{q_n}{[n]_{p_n,q_n}}+p_nx\right)-f(x)\right|\\ &\leq 4\|f-g\|+\delta_n^2(x)\|g''\|\\ &+\omega\left(f,\left|\frac{q_n}{[n]_{p_n,q_n}}-(1-p_n)x\right|\right)\\ &\leq 4\left(\|f-g\|+\delta_n^2(x)\|g''\|\right)+\omega\left(f,\left|\frac{q_n}{[n]_{p_n,q_n}}-(1-p_n)x\right|\right).\end{aligned}$$

Now, taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$|D_n^{p_n,q_n}(f;x)-f(x)| \leq 4K_2\left(f,\delta_n^2(x)\right)+\omega\left(f,\left|\frac{q_n}{[n]_{p_n,q_n}}-(1-p_n)x\right|\right).$$

Finally, we have

$$|D_n^{p_n,q_n}(f;x)-f(x)| \leq C\omega_2\left(f,\delta_n(x)\right)+\omega\left(f,\left|\frac{q_n}{[n]_{p_n,q_n}}-(1-p_n)x\right|\right).$$

This is the required result. □

Let $B_2[0,\infty)$ be the space of all functions f, for which $|f(x)| \leq M_f(1+x^2)$, where M_f is a positive constant that may depend on the function f only. Clearly, $B_2[0,\infty)$ is a linear normed space, equipped with the norm $\|f\|_2 = \sup_{x\geq 0}\frac{|f(x)|}{1+x^2}$. By $C_2[0,\infty)$, we denote the subspace of all continuous functions $f \in B_2[0,\infty)$ and $C_2^*[0,\infty)$ denotes the subspace of all functions $f \in C_2[0,\infty)$ satisfying

$$\lim_{x\to\infty}\frac{f(x)}{1+x^2} = C,$$

here C is a constant depending on f only.

Theorem 3.3. *For $\{p_n\}$ and $\{q_n\}$ be the sequence as considered in Remark 2.1 and $f \in C_2[0,\infty)$, we have*

$$||D_n^{p_n,q_n}\big(|f(t)-f(x)|;x\big)||_{C[0,a]} \leq 6M_f(1+a^2)\delta_n^2+2\omega_{a+1}\big(f,\delta_n\big),$$

here, $\omega_a(f,\delta)$ is modulus of continuity on the interval $[0,a]$ and

$$\delta_n = \sqrt{D_n^{p_n,q_n}\big((t-x)^2;x\big)}.$$

We are here omitting the proof of the theorem, for details readers may refer to [18, Theorem 4].

Theorem 3.4 (Voronovskaya Type Theorem). *Let $\{p_n\}$ and $\{q_n\}$ be the sequence as considered in Remark 2.1 and $f, f', \ f'' \in C_2^*[0,\infty)$, then*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\big(D_n^{p_n,q_n}(f;x)-f(x)\big)=(1+\alpha x)f'(x)+(x+\frac{\beta}{2}x^2)f''(x)$$

uniformly on any $[0,K], \ K>0$.

We are here omitting the proof of the theorem, for details readers may refer to [18, Theorem 5].

4. Quantitative estimates

In this section, we give some quantitative estimates for the difference of the proposed operators with the operators having similar basis function.
Let $C(I)$ be the space of all real valued continuous functions defined on interval $I\subset\mathbb{R}$ and $C_B(I)=\{f\in C(I): ||f||=sup_{x\in I}|f(x)|<\infty\}$.
Further, we consider a positive linear functional $F: C(I)\to\mathbb{R}$ such that $F(e_0)=1$. We set $b^F=F(e_1)$ and $\mu_r^F=F(e_1-b^Fe_0)^r, \ r\in\mathbb{N}$.

Theorem 4.5. *([9, Theorem 1]) Let $U_n=\sum_{k=0}^{\infty}p_{n,k}(x)F_{n,k}(f)$ and $V_n=\sum_{k=0}^{\infty}p_{n,k}(x)G_{n,k}(f)$ be positive linear operators having their basis as $p_{n,k}(x)$. If $f\in D(I)$ with $f''\in C_B(I)$, then*

$$|(U_n-V_n)(f;x)|\le\alpha(x)||f''||+2\omega(f,\delta(x)),$$

where

$$\begin{aligned}\alpha(x) &= \frac{1}{2}\sum_{k=0}^{\infty}p_{n,k}(x)\Big(\mu_2^{F_{n,k}}+\mu_2^{G_{n,k}}\Big),\\ \delta^2(x) &= \sum_{k=0}^{\infty}p_{n,k}(x)\Big(b^{F_{n,k}}-b^{G_{n,k}}\Big)^2.\end{aligned}$$

Using the above assumptions, the proposed operators $D_n^{p,q}$ can be expressed as

$$D_n^{p,q}(f;x)=\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)F_{n,k}^{p,q}(f),$$

where

$$s_{n,k}^{p,q}(x)=\frac{q^{k(k-1)/2}}{E_{p,q}([n]_{p,q}x)}\frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}$$

and

$$F_{n,k}^{p,q}(f)=\frac{[n]_{p,q}}{q}\frac{p^{k(k-1)/2}}{q^{k(k+1)/2}}\int_0^{\infty}s_{n,k}^{p,q}(t)f\left(\frac{p^k}{q^k}t\right)d_{p,q}t.$$

Remark 4.2. By using Lemma 2.1, we have

$$F_{n,k}^{p,q}(e_m)=\frac{[k+1]_{p,q}[k+2]_{p,q}......[k+m]_{p,q}}{q^{m(k-1)}\ p^{m(m-1)/2}\ [n]_{p,q}^m}, \ m=0,1,2,\dots.$$

Following the above equation, we get

$$b^{F_{n,k}^{p,q}}=\frac{[k+1]_{p,q}}{q^{(k-1)}[n]_{p,q}}$$

and

$$\begin{aligned}
\mu_2^{F_{n,k}^{p,q}} &= F_{n,k}^{p,q}(e_1 - b^{F_{n,k}^{p,q}} e_0)^2 \\
&= F_{n,k}^{p,q}(e_2) - (F_{n,k}^{p,q}(e_1))^2 \\
&= \frac{[k+1]_{p,q}[k+2]_{p,q}}{q^{2(k-1)}[n]_{p,q}^2} - \frac{[k+1]_{p,q}^2}{q^{2(k-1)}[n]_{p,q}^2} \\
&= \frac{[k+1]_{p,q}}{pq^{(k-3)}[n]_{p,q}^2}.
\end{aligned}$$

4.1. **Difference with (p,q)-Szász-Mirakyan operators.** For $f:[0,\infty)\to\mathbb{R}$, (p,q)-Szász-Mirakyan operators defined by T. Acar [2], can be expressed as:

$$S_{n,p,q}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) G_{n,k}^{p,q}(f),$$

where

$$s_{n,k}^{p,q}(x) = \frac{q^{k(k-1)/2}}{E_{p,q}([n]_{p,q}x)} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}$$

and

$$G_{n,k}^{p,q}(f) = f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right).$$

Remark 4.3. *By above representation of the operators $S_{n,p,q}$, we have*

$$\begin{aligned}
b^{G_{n,k}^{p,q}} &= G_{n,k}^{p,q}(e_1) \\
&= \frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}
\end{aligned}$$

and hence, for $r\in\mathbb{N}$, we get

$$\mu_r^{G_{n,k}^{p,q}} = G_{n,k}^{p,q}(e_1 - b^{G_{n,k}^{p,q}} e_0)^r = 0.$$

Now, we give the quantitative estimates for the difference of the proposed operators with (p,q)-Szász-Mirakyan operators.

Proposition 4.1. *Let $f, f', f'' \in C_B[0,\infty)$ and $x\in[0,\infty)$. Then for $n\in\mathbb{N}$, we have*

$$|(D_n^{p,q} - S_{n,p,q})(f;x)| \le \alpha_1(x)||f''|| + 2\omega(f,\delta_1(x)),$$

where

$$\alpha_1(x) = \frac{1}{2}\frac{q^2}{[n]_{p,q}}\left(\frac{q}{p[n]_{p,q}} + x\right),\ \delta_1^2(x) = \frac{p}{q}(p-q)^2x^2 + \frac{(p^2-q^2)}{[n]_{p,q}}x + \frac{q^2}{[n]_{p,q}^2}.$$

Proof. Here, we observe that Theorem 4.5 validates the Proposition. It remains only to compute the values of $\alpha_1(x)$ and $\delta_1(x)$ accordingly.
Using Remark 4.2 and Remark 4.3, we have

$$\begin{aligned}
\alpha_1(x) &= \frac{1}{2}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\left(\mu_2^{F_{n,k}^{p,q}} + \mu_2^{G_{n,k}^{p,q}}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\left(\frac{[k+1]_{p,q}}{pq^{(k-3)}[n]_{p,q}^2}\right).
\end{aligned}$$

Now, using the identity $[k+1]_{p,q} = q^k + p[k]_{p,q}$ and Lemma 2.2 in above, we get

$$\alpha_1(x) = \frac{1}{2}\frac{q^2}{[n]_{p,q}}\bigg(\frac{q}{p[n]_{p,q}} + x\bigg).$$

Similarly,

$$\begin{aligned}
\delta_1^2(x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\bigg(b^{F_{n,k}^{p,q}} - b^{G_{n,k}^{p,q}}\bigg)^2 \\
&= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\bigg(\frac{[k+1]_{p,q}}{q^{k-1}[n]_{p,q}} - \frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\bigg)^2 \\
&= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\bigg((p-q)^2\frac{[k]_{p,q}^2}{q^{2(k-1)}[n]_{p,q}^2} + 2q(p-q)\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}^2} + \frac{q^2}{[n]_{p,q}}\bigg) \\
&= (p-q)^2\bigg(\frac{p}{q}x^2 + \frac{x}{[n]_{p,q}}\bigg) + \frac{2q(p-q)}{[n]_{p,q}}x + \frac{q^2}{[n]_{p,q}^2} \\
&= \frac{p}{q}(p-q)^2x^2 + \frac{(p^2-q^2)}{[n]_{p,q}}x + \frac{q^2}{[n]_{p,q}^2}.
\end{aligned}$$

□

4.2. **Difference with (p,q)-Szász-Mirakyan Kantorovich operators.** For $n \in \mathbb{N}$ and $f : [0,\infty) \to \mathbb{R}$, (p,q)-Szász-Mirakyan Kantorovich operators [18], can be expressed as

$$K_n^{p,q}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) H_{n,k}^{p,q}(f),$$

here,

$$s_{n,k}^{p,q}(x) = \frac{q^{k(k-1)/2}}{E_{p,q}([n]_{p,q}x)}\frac{[n]_{p,q}^k x^k}{[k]_{p,q}!} \text{ and } H_{n,k}^{p,q}(f) = [n]_{p,q}\frac{q^{k-2}}{p^k}\int_{\frac{[k]_{p,q}}{q^{k-3}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^{k-2}[n]_{p,q}}} f(t)d_{p,q}t.$$

Lemma 4.5. *For $H_{n,k}^{p,q}(f) = [n]_{p,q}\frac{q^{k-2}}{p^k}\int_{\frac{[k]_{p,q}}{q^{k-3}[n]_{p,q}}}^{\frac{[k+1]_{p,q}}{q^{k-2}[n]_{p,q}}} f(t)d_{p,q}t$, we have*

$$\begin{aligned}
(i)\ H_{n,k}^{p,q}(e_0) &= 1, \\
(ii)\ H_{n,k}^{p,q}(e_1) &= \frac{[k+1]_{p,q} + q[k]_{p,q}}{q^{k-2}(p+q)[n]_{p,q}}, \\
(iii)\ H_{n,k}^{p,q}(e_2) &= \frac{[k+1]_{p,q}^2 + q[k]_{p,q}[k+1]_{p,q} + q^2[k]_{p,q}^2}{q^{2(k-2)}(p^2+pq+q^2)[n]_{p,q}^2}.
\end{aligned}$$

Remark 4.4. By using above lemma, we have

$$b^{H_{n,k}^{p,q}} = H_{n,k}^{p,q}(e_1) = \frac{[k+1]_{p,q} + q[k]_{p,q}}{q^{k-2}(p+q)[n]_{p,q}},$$

and

$$\begin{aligned}
\mu_2^{H^{p,q}_{n,k}} &= H^{p,q}_{n,k}(e_1 - b^{H^{p,q}_{n,k}}e_0)^2\\
&= H^{p,q}_{n,k}(e_2) - (H^{p,q}_{n,k}(e_1))^2\\
&= \frac{[k+1]^2_{p,q} + q[k]_{p,q}[k+1]_{p,q} + q^2[k]^2_{p,q}}{(p^2+pq+q^2)q^{2(k-2)}[n]^2_{p,q}} - \frac{([k+1]_{p,q}+q[k]_{p,q})^2}{(p+q)^2q^{2(k-2)}[n]^2_{p,q}}\\
&= \frac{q^4}{(p^2+pq+q^2)(p+q)^2}\left(\frac{(p^2-q^2)[k]_{p,q}}{q^{k-1}[n]^2_{p,q}} - \frac{pq}{[n]^2_{p,q}}\right).
\end{aligned}$$

Here, we present the quantitative estimates for the difference of the proposed operators with (p,q)-Szász-Mirakyan Kantorovich operators.

Proposition 4.2. *Let $f, f', f'' \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then for $n \in \mathbb{N}$, we have*

$$|(D^{p,q}_n - K^{p,q}_n)(f;x)| \le \alpha_2(x)||f''|| + 2\omega(f, \delta_2(x))$$

where

$$\begin{aligned}
\alpha_2(x) &= \alpha_1(x) + \frac{q^4}{(p^2+pq+q^2)(p+q)^2}\left(\frac{(p^2-q^2)}{[n]_{p,q}}x - \frac{pq}{[n]^2_{p,q}}\right)\\
\text{and}\qquad \delta_2^2(x) &= (p-q)^2\left(\frac{p}{q}x^2 + \frac{x}{[n]_{p,q}}\right) + \frac{2pq(p-q)}{p+q}\frac{x}{[n]_{p,q}} + \frac{p^2q^2}{(p+q)^2[n]^2_{p,q}}.
\end{aligned}$$

Proof. Here, we observe that Theorem 4.5 validates the Proposition. It remains only to compute the values of $\alpha_2(x)$ and $\delta_2(x)$.
By Remark 4.2, Remark 4.4 and Lemma 2.2, we have

$$\begin{aligned}
\alpha_2(x) &= \frac{1}{2}\sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\left(\mu_2^{F^{p,q}_{n,k}} + \mu_2^{H^{p,q}_{n,k}}\right)\\
&= \alpha_1(x) + \frac{1}{2}\sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\mu_2^{H^{p,q}_{n,k}}\\
&= \alpha_1(x) + \frac{1}{2}\sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\frac{q^4}{(p^2+pq+q^2)(p+q)^2}\left(\frac{(p^2-q^2)[k]_{p,q}}{q^{k-1}[n]^2_{p,q}} - \frac{pq}{[n]^2_{p,q}}\right)\\
&= \alpha_1(x) + \frac{q^4}{(p^2+pq+q^2)(p+q)^2}\left(\frac{(p^2-q^2)}{[n]_{p,q}}x - \frac{pq}{[n]^2_{p,q}}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
b^{F^{p,q}_{n,k}} - b^{H^{p,q}_{n,k}} &= \frac{[k+1]_{p,q}}{q^{(k-1)}[n]_{p,q}} - \frac{[k+1]_{p,q}+q[k]_{p,q}}{q^{k-2}(p+q)[n]_{p,q}}\\
&= (p-q)\frac{[k]_{p,q}}{q^{k-1}[n]_{p,q}} + \frac{pq}{(p+q)[n]_{p,q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta_2^2(x) &= \sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\left(b^{F^{p,q}_{n,k}} - b^{H^{p,q}_{n,k}}\right)^2\\
&= \sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\left((p-q)^2\frac{[k]^2_{p,q}}{q^{2(k-1)}[n]^2_{p,q}} + \frac{2pq(p-q)}{p+q}\frac{[k]_{p,q}}{q^{k-1}[n]^2_{p,q}} + \frac{p^2q^2}{(p+q)^2[n]^2_{p,q}}\right).
\end{aligned}$$

Finally, using Lemma 2.2, we have

$$\delta_2^2(x) = (p-q)^2\left(\frac{p}{q}x^2 + \frac{x}{[n]_{p,q}}\right) + \frac{2pq(p-q)}{p+q}\frac{x}{[n]_{p,q}} + \frac{p^2q^2}{(p+q)^2[n]_{p,q}^2}.$$

□

REFERENCES

[1] Aral, A., Gupta, V. and Agarwal, R. P., *Applications of q-Calculus in Operator Theory*, Springer (2013)
[2] Acar, T., *(p, q)-Generalization of Szász-Mirakyan operators*, Math. Methods Appl. Sci., **39** (2016), No. 10, 2685–2695
[3] Acar, T., *Quantitative q-Voronovskaya and q-Grüss-Voronovskaya-type results for q-Szász Operators*, Georgian Math. J., **23** (2016), No. 4, 459–468
[4] Acar, T., Agrawal, P. N. and Kumar, A. S., *On a Modification of (p, q)-Szász-Mirakyan Operators*, Complex Anal. Oper. Theory, **12** (2018), 155–167
[5] Acar, T., Mohiuddine, S. A. and Mursaleen, M., *Approximation by (p, q)-Baskakov-Durrmeyer-Stancu operators*, Complex Anal. Oper. Theory, **12** 2018, No. 6, 1453–1468
[6] Acar, T., Aral, A. and Mursaleen, M., *Approximation by Baskakov-Durrmeyer operators based on (p, q)-integers*, Math. Slovaca, **68** 2018, No. 4, 897–906
[7] Acar, T., Aral, A. and Rasa, I., *Positive Linear Operators Preserving τ and τ^2*, Constr. Math. Anal., **2** (2019), No. 3, 98–102
[8] Devore, R. A. and Lorentz, G. G., *Constructive Approximation*, Springer, Berlin, 1993
[9] Gupta, V., *General estimates for the difference of operators*, Comput. Math. Methods, **1** (2019), No. 2, e1018, 11 pp.
[10] Gupta, V. and Aral, A., *Bernstein Durmeyer operators based on Two Parameters*, SER. Math. Inform., **31** (2016), No. 1, 79–95
[11] Ilarslan, H. G. I. and Acar, T., *Approximation by bivariate (p, q)-Baskakov-Kantorovich operators*, Georgian Math. J., **25** (2018), No. 3, 397–407
[12] Khan, K. and Lobiyal, D. K., *Bezier curves based on Lupas (p, q)-analogue of Bernstein polynomials in CAGD*, arXiv: 1505.01810[cs.GR]
[13] Mursaleen, M., Ansari, K. J. and Khan, A., *On (p, q)-analogue of Bernstein Operators*, Appl. Math. Comput., **266** (2015), 874–882, (Erratum to: On (p, q)-analogue of Bernstein Operators, [Appl. Math. Comput., **266** (2015) 874—882] [MR3377604]. Appl. Math. Comput., **278** (2016), 70–71
[14] Mursaleen, M. and Nasiruzzaman, M., *Approximation of Modified Jakimovski-Leviatan-Beta Type Operators*, Constr. Math. Anal., **1** (2018), No. 2, 88–98
[15] Sadjang, P. N., *On the fundamental theorem of (p, q)-Taylor formulas*, arXiv: 1309.3934[math. QA]
[16] Sahai, V. and Yadav, S., *Representations of two parameter quantum algebras and (p, q)-special functions*, J. Math. Anal. Appl., **335** (2007), 268–279
[17] Sharma, H., *On Durrmeyer-type generalization of (p, q)-Bernstein operators*, Arab. J. Math. (Springer), **5** (2016), No. 4, 239–248
[18] Sharma, H. and Gupta, C., *On (p, q)-generalization of Szász-Mirakyan Kantorovich operators*, Boll. Unione Mat. Ital., **8** (2015), No. 3, 213–222
[19] Sharma, H., Maurya, R. and Gupta, C., *Approximation Properties of Kantorovich Type Modifications of (p, q)-Meyer-König-Zeller Operators*, Constr. Math. Anal., **1** (2018), No. 1, 58–72

GULZAR GROUP OF INSTITUTES
PUNJAB, INDIA
Email address: pro.sharma.h@gmail.com

I. K. G. PUNJAB TECHNICAL UNIVERSITY
PUNJAB, INDIA
MANAV RACHNA UNIVERSITY, HARYANA, INDIA
Email address: ramapatimaurya@gmail.com