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Slice ranks: lines in hypersurfaces

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ABSTRACT. Motivated by the notion of slice ranks of multivariate forms we study families of hypersurfaces of $\mathbb{P}^n$, mainly for $n = 3, 4$, containing a prescribed number of lines or having infinitely many lines. In many cases we compute their dimension and describe their irreducible components with maximal dimension.

1. INTRODUCTION

Let $\mathbb{K}$ be an algebraically closed base field with characteristic 0. For any $d \in \mathbb{K}$ let $\mathbb{K}[x_0, \dots, x_n]_d$ denote the $\binom{n+d}{n}$-dimensional $\mathbb{K}$-vector space of all degree d forms in the variables $x_0, \dots, x_n$. For any $f \in \mathbb{K}[x_0, \dots, x_n]_d$, $f \neq 0$, $d \geq 2$, the *slice rank* $\mathrm{sl}(f)$ of f is the minimal integer r such that $f = \sum_{i=1}^{r} f_i g_i$ for some $f_i \in \mathbb{K}[x_0, \dots, x_n]_1$ and some $g_i \in \mathbb{K}[x_0, \dots, x_n]_{d-1}$ ([1, 2, 6, 7, 8, 12]). There are very strong reasons for these studies which allowed to get insights and proofs of old conjectures and promise further extensions of classical papers ([15, 20]). Set $X := \{f = 0\} \in |\mathcal{O}_{\mathbb{P}^n}(d)|$. Since $\mathrm{sl}(f) = \mathrm{sl}(cf)$ for all $c \in \mathbb{K} \setminus \{0\}$, the integer $\mathrm{sl}(X) := \mathrm{sl}(f)$ is well-defined. The integer $\mathrm{sl}(X)$ is called the slice rank of X. It is easy to check that $\mathrm{sl}(X) = n - v$, where v is the maximal dimension of a linear space $L \subset X$ ([1, 6, 7, 8, 12]). Thus to study the integers $\mathrm{sl}(X)$, $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$, one can use several papers devoted to the study of linear spaces contained in a specific hypersurface. For any hypersurface $X \subset \mathbb{P}^n$ with $\mathrm{sl}(X) = n-v$ let $\mathcal{S}(X) \subset G(v+1, n+1)$ denote the set of all of v-dimensional linear subspaces contained in v. The papers [14, 22] by O. Debarre and L. Manivel are important to use this observation. The first one gives $\mathrm{sl}(X)$ for a general $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ for all n ands all d. In particular if $d \geq 2n-2$ a general $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ contains no positive dimensional linear subspace ([14]). L. Manivel studied the case in which X is not general ([22]) and this is the case considered in this paper. More precisely, for each $d \geq 2n-2$ a general $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ contains no positive dimensional linear subspace ([14]). We will always work in the range $d \geq 2n-2$ and consider hypersurfaces $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ such that $\mathrm{sl}(X) = n-1$. For any X with $\mathrm{sl}(X) = n-1$ set $Y_X := \cup_{L \in \mathcal{S}(X)} L \subseteq X$. Since the Grassmannian $G(2, n+1)$ is a projective variety and X is closed in $\mathbb{P}^n$, $\mathcal{S}(X)$ and $Y_X \subset X$ are projective algebraic sets.

In this paper we consider the following questions:

(1) Fix a closed algebraic subset $T \subset \mathbb{P}^n$ which is a union of lines, but contains no plane. Fix an integer $d \geq 2n-2$ such that $h^0(\mathcal{I}_T(d)) \neq 0$. Is $Y_X = T$ for a general $X \in |\mathcal{I}_T(d)|$?
(2) Fix the numerical invariants for T; for instance if T is finite fix the integer $\#T$ and, maybe, the arithmetic genus of the curve $\cup_{L \in T} L$. Compute or give lower/upper

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bounds for the dimension of all $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ such that $\mathrm{sl}(X) = n-1$ and Y_X has the numerical invariants of T.

We now list our main results and introduce the notation needed to state them.

Theorem 1.1. *Fix integers y, d and x such that $0 \le x \le d^2$ and $y \ge d \ge 5$. If $x = dk + e$ with $k \in \{0, \dots, d-1\}$ and $0 < e < d$ assume $y \ge \max\{d, d+k-e-1\}$. There is a union $E_x \subset \mathbb{P}^3$ of x distinct lines such that $h^1(\mathcal{I}_{E_x}(t)) = 0$ for all $t \in \mathbb{Z}$, $h^0(\mathcal{I}_{E_x}(d)) \ge 2$ and for each integer $y \ge d$ a general $S \in |\mathcal{I}_{E_x}(y)|$ is irreducible and contains exactly x lines.*

Remark 1.1. Fix integers $n \ge 3$ and $d \ge 2$, $i > 0$. Fix $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$. O. Debarre and L. Manivel computed the dimension $a(n,d,i)$ of the set $A(n,d,i)$ of all $X \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ with $\mathrm{sl}(X) = i$ ([14, 22]) and L. Manivel proved that if $A(n,d,i) \neq \emptyset$ and $a(n,d,i) < \binom{n+d}{n} - 1$, then a general $X \in A(n,d,i)$ contains a unique $(n-i)$-dimensional linear subspace ([22, part 2 of Theorem at p. 307]). Let $A(n,d,i,k)$, $k \ge 0$, be the set of $X \in A(n,d,i)$ such that $\dim \mathcal{S}(X) = k$.

Theorem 1.2. *Fix an integer $d \ge 2$. Then:*

(1) $a(4,d,3,x) = 0$ for all $x > 3$;
(2) $a(4,d,3,3) = \binom{4+d}{4} - \binom{d+2}{4}$, $A(4,d,3,3)$ is irreducible and $X \in A(4,d,3,3)$ if and only if X contains no plane, but it has a smooth quadric hypersurface as an an irreducible component.

Theorem 1.3. *Fix an integer $d \ge 4$ and set $x := \lceil 3d/2 \rceil$. Let $\mathcal{J} \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be an irreducible family such that all $X \in \mathcal{J}$ contain infinitely many lines, but no irreducible component of X is a plane.*

(a) $\dim \mathcal{J} \le \binom{d+1}{3} + 8$ and equality holds if and only if all elements of $\mathcal{J}$ have a quadric as a component.

(b) Assume that no element of $\mathcal{J}$ has a quadric as a component. Then

$$\dim \mathcal{J} \le \binom{d}{3} + 16.$$

For all positive integer d and t let $\beta(d,t)$ (resp. $\beta_1(d,t)$) denote the dimension of all $X \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ (resp. all integral $X \in |\mathcal{O}_{\mathbb{P}^3}(d)|$) such that $\#Y_X = t$, with the convention $\dim \emptyset = -\infty$. Obviously $\beta(d,t) = \beta_1(d,t) = -\infty$ for $d = 1,2$. It is classically known that $\beta(3,t) = \beta_1(3,t) = -\infty$ for all $t > 27$. Let $E(d,t)$ (resp. $E_1(d,t)|$) be the set of all $X \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ (resp. of all integral $X \in |\mathcal{O}_{\mathbb{P}^3}(d)|$) such that $\#\mathcal{S}(X) = t$.

Proposition 1.1. *Fix an integer $d \ge 4$.*

(a) $\beta(d,1) = \beta_1(d,1) = \binom{d+3}{3} - d - 2$ and $E(d,1)$ and $E_1(d,1)$ are irreducible.

(b) $\beta(d,2) = \beta_1(d,1) = \binom{d+3}{3} - 2d + 1$ and $E(d,2)$ and $E_1(d,2)$ have two irreducible components, both of maximal dimension, one formed by the surfaces containing a reducible conic and the other one by the surfaces containing 2 disjoint lines.

(c) $E(d,3)$ and $E_1(d,3)$ have 4 irreducible components, all of dimension $\binom{d+3}{3} - 3d + 8$, distinguished by the set Y_X of any X in the irreducible family:

(1) 3 disjoint lines;
(2) a reducible conic and a disjoint line;
(3) a planar union of 3 lines;
(4) a connected union of 3 lines with arithmetic genus 0.

Several papers studied finite unions $T \subset \mathbb{P}^3$ of lines ([3, 9, 10, 16, 24, 26]). For general unions of a prescribed number of lines in $\mathbb{P}^n$, $n \ge 3$, see [19]. As far as we know our is the first systematic attempt to study the two questions we raised in the introduction.

We conclude the introduction with the following questions.
Take $S, S' \in |\mathcal{O}_{\mathbb{P}^n}(d)|$ such that $\mathcal{S}(S) = \mathcal{S}(S')$.

(1) Under which assumptions on n, d and $\mathcal{S}(S)$ are $S = S'$?
(2) Under which assumptions on n and d the hypersurfaces S and S' are related in a certain way, e.g. there are an integer d', $E \in |\mathcal{O}_{\mathbb{P}^n}(d')|$ and $W, W' \in |\mathcal{O}_{\mathbb{P}^n}(d-d')|$ such that $\mathrm{sl}(E) = \mathcal{S}(E)$, $S = E + W$, $S' = E + W'$ and either $\mathrm{sl}(W) < \mathrm{sl}(S)$ or $\mathrm{sl}(W) = \mathrm{sl}(S)$ and $\mathrm{sl}(W) \subseteq \mathrm{sl}(S)$, either $\mathrm{sl}(W') < \mathrm{sl}(S)$ or $\mathrm{sl}(W') = \mathrm{sl}(S)$ and $\mathrm{sl}(W') \subseteq \mathrm{sl}(S)$?

Of course, we may exclude the latter possibility if we assume that at least one among S and S' is irreducible.

2. Proofs

Let $A \subset \mathbb{P}^n$ be a closed subscheme. Fix any $Q \in |\mathcal{O}_{\mathbb{P}^n}(m)|$, $m > 0$. The *residual scheme* $\mathrm{Res}_Q(A)$ of A with respect to Q is the closed subscheme of $\mathbb{P}^n$ with $\mathcal{I}_A : \mathcal{I}_Q$ as it ideal sheaf. If A is a reduced scheme, then $\mathrm{Res}_Q(A)$ is the closure of $A \setminus A \cap Q$ in $\mathbb{P}^n$, i.e. the union of all irreducible components of A not contained in Q. For any $t \in \mathbb{Z}$ there is an exact sequence of coherent sheaves:

$$0 \to \mathcal{I}_{\mathrm{Res}_Q(A)}(t-m) \to \mathcal{I}_A(t) \to \mathcal{I}_{A\cap Q,Q}(t) \to 0 \tag{2.1}$$

Proof of Theorem 1.2: Fix any $X \in A(4,d,3,x)$ with $x \geq 3$. Since X contains no plane, any surface $S \subset X$ contains at most ∞^1 lines. Since X contains no plane, if X is a cone (i.e. if there is $p \in X$ contained in ∞^2 lines contained in X), then a general hyperplane section of X contains no line. Thus the assumption $x \geq 3$ and the properness of the Grassmannian $G(2,5)$ imply that X is not a cone, that $x = 3$ and that there is an irreducible component W of X such that each $p \in W$ is contained in exactly ∞^1 lines contained in W. Set $t := \deg(W)$. To conclude the proof of the theorem it is sufficient to prove that $t = 2$. Fix $p \in X_{\mathrm{reg}} \cap W$, so that $T_pX = T_pW$ is a 3-dimensional linear space. Each line $L \subset W$ containing p is contained in T_pW. Thus $T_pW \cap W$ contains a 2-dimensional cone C_p with vertex p. Set $e := \deg(C_p) \leq t$.

Assume for the moment that the dual variety $W^\vee \subset \mathbb{P}^{4\vee}$ of W is a hypersurface. Since we are in characteristic zero and p is general in W, T_pW is tangent to W only at p and $T_pW \cap T_pX$ has a quadratic singularity. Thus $e = 2$ and there is an open neighborhood U of $p \in W$ such that the scheme-theoretic intersection $T_pW \cap W$ is smooth at all points of $U \setminus \{p\}$. Thus $t = e$ and hence $t = 2$.

Now assume that the dual variety $W^\vee$ has dimension b for some $1 \leq b \leq 2$. Since we are in characteristic zero, this implies that T_pW is tangent to W along a $3-b$ linear space. Since X contains no plane, we get $b = 2$. Thus (again because we are in characteristic 0) the closure in W of a general fiber of the Gaussian map $\gamma_W : W_{\mathrm{reg}} \to G(4,5)$ is a line. Call L_p the closure of the fiber of γ_W containing p. For a general $q \in L_p$, $q \in W_{\mathrm{reg}}$ and $T_qW = T_pW$. Thus $C_q = C_p$. Hence C_p is a cone with vertex containing a line. Thus the surface C_p contains a plane. Since $C_p \subset W \subset X$, we obtained a contradiction. $\square$

Proof of Proposition 1.1: Let A_t be a plane curve of degree t. Let E_1 (resp. E_2) be a union of 2 (resp. 3) disjoint lines. Let E_3 be the union of a reducible conic and a line disjoint from it. Since $d \geq 4$, $h^1(\mathcal{I}_{A_t}(d)) = h^1(\mathcal{I}_{E_i}(d)) = 0$, $i = 1,2,3$. Thus $h^0(\mathcal{I}_{A_t}(d)) = \binom{d+3}{3} - \binom{d+2}{2} + \binom{d-t+2}{2}$, $h^0(\mathcal{I}_{E_1}(d)) = \binom{d+3}{3} - d - 1$, $h^0(\mathcal{I}_{E_2}(d)) = \binom{d+3}{3} - 3d - 3$, and $h^0(\mathcal{I}_{E_3}(d)) = \binom{d+3}{3} - 2d - 2$. Note that $\dim G(2,3) = 2$, that $\dim G(2,4) = 4$ and that the set of all lines of $\mathbb{P}^3$ intersecting a given line has dimension 3. Part (a) follows considering all $L \neq A_1$.

Now we prove part (c), leaving the similar (but shorter) proof of part (b) to the reader. A union $T \subset \mathbb{P}^3$ of 3 distinct lines may have 3, 2 or 1 connected components and in the latter case we need to distinguish if T is contained in a plane or not. We see that $E(d,3)$ and $E_1(d,3)$ are the union of 4 disjoint algebraic sets and we first study them separately.

(a) Take T with 3 connected components. The set of all such T has dimension 12. In this case $h^0(\mathcal{I}_T(d)) = \binom{d+3}{3} - 3d - 3$. Take a line $L \subset \mathbb{P}^3$ such that $L \nsubseteq T$ and set $z := \deg(L \cap T)$. Let $U(z)$ be the set of of all lines L with $\deg(L \cap T) = z$. It is easy to check that $\dim U(z) = 4 - z$ for all z. Since $d \geq 4$, in all cases the residual exact sequence of a general plane containing L gives $h^1(\mathcal{I}_{T\cup L}(d)) = 0$ and hence $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d - 1 + z$. Since in all cases we have $d + 1 - z > \dim U(z)$, the part of $E(d,3)$ and $E_1(d,3)$ associated to a fixed T in case (a) is irreducible and of codimension $3d+3$. Hence the part of $E(d,3)$ and $E_1(d,3)$ associated to some T in case (a) is irreducible and of dimension $\binom{d+3}{3} - 3d + 8$.

(b) Take T with 2 connected components, a reducible conic A and a line R. The set of all such T has dimension 11. In this case $h^0(\mathcal{I}_T(d)) = \binom{d+3}{3} - 3d - 2$. Call $U(a,b)$ the set of all lines $L \subset \mathbb{P}^3$ such that $L \nsubseteq T$, $\deg(L \cap R) = a$ and $\deg(L \cap R) = b$. Obviously $a \in \{0,1,2\}$ and $b \in \{0,1\}$. Let H be the plane spanned by R. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{\mathrm{Res}_H(T\cup L)}(d-1) \to \mathcal{I}_{T\cup L}(d) \to \mathcal{I}_{H\cap(T\cup L),H}(d) \to 0 \tag{2.2}$$

Since $\mathrm{Res}_H(T \cup L)$ is either R or $R \cup L$, $h^1(\mathcal{I}_{\mathrm{Res}_H(T\cup L)}(d-1)) = 0$. Since $H \cap (T \cup L)$ is either A or $A \cup L$ or the union of A and a point, $h^1(H, \mathcal{I}_{H\cap(T\cup L),H}(d)) = 0$. Thus $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d - 1 + a + b$. We have $\dim U(2,1) = 1$ (all lines of H passing through $R \cap H$) $\dim U(2,0) = 2$, $\dim U(1,1) = 2$, $\dim U(1,0) = 3$, $\dim U(0,1) = 3$ and $\dim U(0,0)$. Thus in all cases $d + 1 > a + b + \dim U(a,b)$. Hence the part of $E(d,3)$ and $E_1(d,3)$ associated to a fixed T in case (b) is irreducible and of codimension $3d+2$. Hence the part of $E(d,3)$ and $E_1(d,3)$ associated to some T in case (b) is irreducible and of dimension $\binom{d+3}{3} - 3d - 8$.

(c) Assume that T is connected. This is divided into two cases, T contained in a plane or not.

(c1) Assume that T is contained in a plane H. In this case $h^0(\mathcal{I}_T(d)) = \binom{d+3}{3} - 3d$. Since $\dim G(3,4) = 3$, the set of all such degree 3 curves T is irreducible and of dimension 9. Fix a line $L \subseteq T$ and set $z := \deg(L \cap T)$. If $L \subset H$, then $z = 3$. If $L \nsubseteq H$, then $z \in \{0,1\}$. Consider the residual exact sequence (2.2) of H. If $L \subset H$, then $T \cup L$ is a plane curve, $h^1(\mathcal{I}_{T\cup L}(d)) = 0$ and $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d + 1$. Note that the family of possible L's has dimension 2. Now assume $L \nsubseteq H$ and hence $z \leq 1$, $\mathrm{Res}_H(T \cup L) = L$ and again $h^1(\mathcal{I}_{T\cup L}(d)) = 0$, i.e. $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d - 1 + z$. The family of all L has dimension $4 - z$. Using both cases for L we get $Y_X = T$ for a general $X \in |\mathcal{I}_T(d)|$. Varying T, this case gives an irreducible family of $E(d,3)$ and $E_1(d,3)$ with dimension $\binom{d+3}{3} - 3d + 8$.

(c2) Assume that T is not contained in a plane.

(c2.1) Assume that T is nodal. Thus we may order the irreducible components R_1, R_2, R_3 of T so that $R_1 \cap R_3 = \emptyset$. The set of all such T is irreducible and of dimension 10. Fix a line $L \nsubseteq T$ and set $z := \deg(L \cap T)$. Since T is scheme-theoretically cut out by reducible quadrics, $z \leq 2$. Note that $z = 2$ if and only if either L is contained in one of the plane spanned by $R_1 \cup R_2$ or $R_2 \cup R_3$ or $T \cup L$ is the complete intersection of 2 reducible quadric surfaces. In the latter case we have $h^1(\mathcal{I}_{T\cup L}(d)) = 0$. In the other cases, even the ones with $z \leq 1$, we get $h^1(\mathcal{I}_{T\cup L}(d)) = 0$ using the residual exact sequence of one of the planes containing 2 irreducible components of T. Hence $Y_X = T$ for a general $X \in |\mathcal{I}_T(d)|$.

(c2.2) Assume that T is not nodal. Thus there is $o \in \mathbb{P}^3$ such that $T = R_1 \cup R_2 \cup R_3$ with R_1, R_2, R_3 distinct lines containing o and no plane contains T. Varying o we get a family Δ of unions of lines with $\dim \Delta = 9$. Call H_{ij} the plane spanned by $R_i \cup R_j$, $i \neq j$. Fix a line $L \nsubseteq T$. First assume $L \subset H_{ij}$ for some i, j. Set $H := H_{ij}$ and write $\{k\} := \{1,2,3\} \setminus \{i,j\}$. The residual exact sequence (2.2) of H has $\mathrm{Res}_H(T \cup L) = R_k$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_H(T\cup L)}(d-1)) = 0$. Since $T \cup L$ is a planar curve, $h^1(H, \mathcal{I}_{H\cap(T\cup L),H}(d)) = 0$. Thus $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d + 1$. This case is in the closure of the case described in step (c1.1) and so, although it occurs, it does not give an irreducible component of $E(d,3)$ and $E_1(d,3)$.

Now assume that L is contained in no plane H_{ij}. The set of such T (for any $o \in \mathbb{P}^3$) has dimension 9. Set $z := \deg(L \cap T)$. Note that $z \leq 1$ if $o \notin L$ and that the set of all L with $z = 1$ has dimension 3. The case $o \notin L$ is done taking the residual exact sequence of $H := H_{12}$. Now assume $o \in L$. In this case $p_a(T \cup L) = 1$ and hence $T \cup L$ is a complete intersection of 2 reducible quadrics. Thus $h^1(\mathcal{I}_{T\cup L}(d)) = 0$ and $h^0(\mathcal{I}_{T\cup L}(d)) = h^0(\mathcal{I}_T(d)) - d + 1$. Since the set of such lines have dimension 2, we see that $Y_X = T$ for a general $X \in |\mathcal{I}_T(d)|$. But this case is in the boundary of the one described in step (c2.1): instead of $R_1 \cup R_2 \cup R_3$ take a family of all curves $R_1 \cup R_2 \cup E$ with E a general line meeting R_2. □

Proof of Theorem 1.1: Fix $2d$ general hyperplanes $H_1, \dots, H_d, M_1, \dots, M_d$ and let T be the union of the d^2 lines $L_{i,j} := H_i \cap M_j$, $1 \leq i \leq d$, $1 \leq j \leq d$. Set $X_1 := H_1 \cup \cdots \cup H_d$ and $X_2 := M_1 \cup \cdots \cup M_d$. Since T is the complete intersection of the two degree d surfaces X_1 and X_2, $h^1(\mathcal{I}_T(t)) = 0$ for all $t \in \mathbb{Z}$ and the homogeneous ideal of T is generated by 2 forms with zero-loci X_1 and X_2. Take for the moment any integer $y \geq d$ and let S be a general element of $|\mathcal{I}_T(y)|$.

Claim 1: S is irreducible.

Proof of Claim 1: Assume S reducible, say $S = S_1 \cup S_2$ with $\deg(S_1) = a$ and $\deg(S_2) = y - a$. Thus $\mathrm{Sing}(S) \supseteq S_1 \cap S_2$. Since $\mathcal{I}_T(d)$ is globally generated and $y \geq d$, $\mathcal{I}_T(y)$ is globally generated. Bertini's theorem gives $\mathrm{Sing}(S) \subseteq T$. Since $\dim(S_1 \cap S_2) \geq 1$, there is a line $L := L_{i,j} = H_i \cap M_j \subseteq \mathrm{Sing}(S)$. Since T has finitely many irreducible components and S is general in the irreducible variety $|\mathcal{O}_T(y)|$, L is contained in all $X \in |\mathcal{I}_T(y)|$. If $y = d$, this is contradicted by X_1, whose singular locus is the union of all lines $H_u \cap H_v$, $u \neq v$, while by the generality of $H_1, \dots, H_d, M_1, \dots, M_d$, $H_u \cap H_v \cap M_j$ is a unique point for all $u \neq v$. If $y > d$ instead of X_1 we take $W \cup X_1$ with W a general surface of degree $y - k$ (which does not contain L), concluding the proof of Claim 1.

The multiplication by the equations of X_1 and X_2 induces the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(t-2d) \to \mathcal{O}_{\mathbb{P}^3}(t-d)^{\oplus 2} \to \mathcal{I}_T(t) \to 0 \tag{2.3}$$

Observation 1: From (2.3) we get $h^0(\mathcal{I}_T(t)) = 2\binom{t-d+3}{3}$ for $t \leq 2d-1$. Restricting (2.3) to any plane $H \subset \mathbb{P}^3$ containing no irreducible component of T we get the exact sequence

$$0 \to \mathcal{O}_H(t-2d) \to \mathcal{O}_H(t-d)^{\oplus 2} \to \mathcal{I}_{T\cap H,H}(t) \to 0 \tag{2.4}$$

Observation 2: From (2.4) we get $h^0(H, \mathcal{I}_{T\cap H,H}(t)) = 2\binom{t-d+2}{2}$ for $t \leq 2d-1$.

(a) In this step we prove the theorem for $x = d^2$ taking $E_{d^2} := T$.

Claim 2: Fix any $y \geq d$ such that $y \neq d+1$. For a general T a general $S \in |\mathcal{I}_T(y)|$ contains no line $L \nsubseteq T$ such that there is $i \in \{1, \dots, d\}$ with $L \subset H_i \cup M_i$. This is false for $y = d+1$. A general $S \in |\mathcal{I}_T(y+1)|$ contains exactly $2d$ lines $L \nsubseteq T$ such that there is $i \in \{1, \dots, d\}$ with $L \subset H_i \cup M_i$ and for each i there is exactly one $L_i \subset H_i$ and exactly one $R_i \subset M_i$.

Proof of Claim 2: Assume the existence of L for a general (T, S), $S \in |\mathcal{I}_T(y)|$. First assume $y = d$. Just to fix the notation we assume $L \subset H_i$ and hence $L \subset X_1$. Since $L \nsubseteq X_2$,

$T \cap L_i$ is a degree $d+1$ plane curve. Hence $H_i \subset S$. Thus S is reducible, contradicting the generality of S. If $y \ge 2d$ fix an integral $S' \in |\mathcal{I}_T(d)|$ (Claim 1) and take $S' \cup W$ with W a general surface of degree $y-d \ge 2$. Now assume $y = d+1$ and fix a general $S \in |\mathcal{I}_T(y+1)|$. By Claim 1 S is irreducible. Hence $S \cap H_i \ne H_i$. Thus $S \cap H_i$ is a degree $d+1$ plane curve containing the d lines $T \cap H_i$. Thus either $S \cap H_i$ has a line $H_i \cap M_j$ appearing with multiplicity 2 or $S \cap H_i$ contains a line $L_i \nsubseteq T$. The former case is excluded by a dimensional count and the fact that $\deg(T)$ is finite; indeed, for each $i, j \in \{1, \dots, d\}$ the set of all irreducible $S \in |\mathcal{I}_T(d+1)|$ with $L_{i,j}$ appearing with multiplicity 2 in $S \cap H_i$ has codimension 2 in $|\mathcal{I}_T(d+1)|$, because $\mathcal{I}_T(d)$ is globally generated. The same proof works for the planes M_i and gives a unique line $R_i \subset M_i$, $R_i \nsubseteq T$. Since $L \nsubseteq T$, there are no i, j such $L \subseteq H_i \cap M_j$, i.e. $\#\{L_1, \dots, L_d, R_1, \dots, R_d\} = 2d$.

(a1) Assume $y = d$. Take a general $S \in |\mathcal{I}_T(d)|$. Fix any line $L_S \subset S$ such that $L_S \nsubseteq T$. After an étale covering of a non-empty open subset of $|\mathcal{I}_T(d)|$ we may assume that L_S depends algebraically on S even if S contains more than d^2+1 lines.

Claim 3: $L_S \cap \mathrm{Sing}(T) = \emptyset$.

Proof of Claim 3: We first prove that $\#(L_S \cap \mathrm{Sing}(T)) \le 1$. Assume $\#(L_S \cap \mathrm{Sing}(T)) > 1$. Thus L_S is the line spanned by two different elements of $\mathrm{Sing}(T)$. Since $\mathrm{Sing}(T)$ is finite, we would get $L_S = L_{S'}$ for a general $(S, S') \in |\mathcal{I}_T(d)|^2$. Since $G(2,4)$ is projective, we would get $L_S \subset X_i$ for $i = 1, 2$ and hence $L_S \subset X_1 \cap X_2 = T$, contradicting one of our assumptions. Now assume $\#(L_S \cap \mathrm{Sing}(T)) = 1$, say $L_S \cap \mathrm{Sing}(T) = \{p_S\}$. Since $H_1, \dots, H_d, M_1, \dots, M_d$ are general, the set $\mathrm{Sing}(T)$ is the union of the set $\Sigma_1 := \{H_u \cap H_v \cap M_j\}_{1 \le u < v \le d, 1 \le j \le d}$ and the set $\Sigma_2 := \{M_u \cap M_v \cap H_j\}_{1 \le u < v \le d, 1 \le j \le d}$. Note that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Thus there is a unique $i \in \{1, 2\}$ such that $p_S \in \Sigma_i$. Now we move the $2d$-ple of planes $(H_1, \dots, H_d, M_1, \dots, M_d) \in G(2,4)^d$ and come back to the same set of $2d$ planes with a different ordering, e.g. exchanging each H_i with M_i. We exchange Σ_1 and Σ_2 and get a contradiction.

Claim 4: $\#(L_S \cap T) = d$ and L_S meets each H_i (resp. each M_j) at a unique point, $p_{i,S}$ (resp. $q_{j,S}$) and $p_{i,S} \in T$ (resp. $q_{j,S} \in T$).

Proof of Claim 4: Assume for instance $L_S \subset H_i$. We would get that $H_i \cap S$ contains a degree $d+1$ plane curve and hence $H_i \subset S$, contradicting the irreducibility of S. Thus $\#(L_S \cap H_i) = 1$, say $L_S \cap H_i = \{p_{i,S}\}$. Since $S \cap H_i$ is a degree plane curve and H_i is not an irreducible component of S, $p_{i,S} \in T \cap H_i$. The proof for M_j is similar.

Since each $p_{i,S}$ and each $q_{j,S}$ is a smooth point of T, $\#(\{p_{1,S}, \dots, p_{d,S}\}) = \#(\{q_{1,S}, \dots, q_{d,S}\}) = d$.

Claims 3 and 4 gives $\{p_{1,S}, \dots, p_{d,S}, q_{1,S}, \dots, q_{d,S}\} \subseteq T \cap L_S$ and $\#(L_S \cap X_1) = \#(L_S \cap X_2) = d$. Since T is the complete intersection of 2 degree d surfaces and $L_S \nsubseteq S$, we get $\#(\{p_{1,S}, \dots, p_{d,S}, q_{1,S}, \dots, q_{d,S}\}) \le d$. Thus for each $i \in \{1, \dots, d\}$ there is a unique $\sigma(i) \in \{1, \dots, d\}$ such that $q_{\sigma(i),S} = p_{i,S}$.

Observation 3: Any 3 disjoint lines of $\mathbb{P}^3$ are contained in a unique quadric $Q \subset \mathbb{P}^3$ and this quadric Q is smooth. The 3 disjoint lines belong to the same ruling of Q. By Bezout a line meets all these 3 lines if and only if it is an element of the other ruling of Q.

Observation 4: Let $Q, Q' \subset \mathbb{P}^3$ be smooth quadrics such that $Q \cap Q'$ contains 2 disjoint lines L', L''. The scheme $Q \cap Q'$ is a divisor of bidegree $(2,2)$ of both Q and Q'. Thus $Q \cap Q'$ is the union of $L' \cup L''$ and either two disjoint lines R and D in the rulings of Q and Q' not containing L' or the divisor $2R$ in the rulings of Q and Q' not containing L'.

Now we use that $d \ge 5$. The lines $L_{1,\sigma(1)}, L_{2,\sigma(2)}, L_{3,\sigma(3)}, L_{4,\sigma(4)}$ are pairwise disjoint. Thus for each $E \subset \{1,2,3,4\}$ with $\#E = 3$ there is a unique quadric surface Q_E containing all lines $L_{i,\sigma(i)}$, $i \in E$, and this quadric is smooth. Q_E is the union of the lines of $\mathbb{P}^3$ intersecting all lines $L_{i,\sigma(i)}$, $i \in E$. Thus $L \subset Q_E$ for all E (Observation 1). For each $i \in \{1,2,3,4\}$ set $E_i := \{1,2,3,4\} \setminus \{i\}$. Thus we get 4 different smooth quadrics Q_{E_i},

$1 \le i \le 4$, with Q_{E_i} containing $\cup_{j\in E_i} L_{j,\sigma(j)}$ and each of these lines meets L. Thus $L \subset Q_{E_i}$ (Observation 1). For each $j \in E_i$ set $E_{ij} := \{1,2,3,4\}\setminus\{i,j\}$ and call R_{ij} the line $\ne L$ given by Observation 1 applied to the smooth quadrics Q_{E_i} and Q_{E_j} with the convention $R_{ij} = L$ if $Q_{E_i}\cap Q_{E_j}$ contains L with multiplicity 2. There are at most 2 lines in $Q_{E_1}\cap Q_{E_2}\cap Q_{E_3}$ in the same ruling of these quadric containing L, say L and L' with perhaps $L = L_5$. For a general H_5 and a general $M_{\sigma(5)}$ the line $L_{5,\sigma(5)}$ does not meet $R\cup R'$, a contradiction.

(a2) Assume $y > d$.

(a2.1) If $y \ge 4+d$ we take as S a generalization of $S'\cup W$, where S' is a general element of $|\mathcal{I}_T(d)|$ and W is a quartic surface containing no line. Thus we could assume $d+1 \le y \le d+3$. Assume $y \in \{d+2, d+3\}$. Taking a generalization of $X_1 \cup W$ with W a general surface of degree $y-d$ we see that $L_S \cap T = \emptyset$ for a general $S \in |\mathcal{I}_T(y)|$. Thus (2.3) and (2.4) give $h^0(\mathcal{I}_{T\cup L_S}(y)) = 2\binom{y-d+2}{3} + 2\binom{y-d+1}{2}$. Since $\dim G(2,4) = 4$ and a general $S \in |\mathcal{I}_T(y)|$ contains the line L_S, we have $h^0(\mathcal{I}_{T\cup L_S}(y)) \ge h^0(\mathcal{I}_T(y)) - 4$. Thus (2.3) gives $h^0(\mathcal{I}_T(y)) = 2\binom{y-d+3}{3} = 2\binom{y-d+2}{3} + 2\binom{y-d+2}{2}$. Thus $2(\binom{y+2-d}{2} - \binom{y+1-d}{2}) \le 4$, i.e. $2(y-d+1) \le 4$, a contradiction.

(a2.2) Assume $y = d+1$. Assume for the moment $h^0(H, \mathcal{I}_{T\cap H\setminus T\cap L_S}(d)) = 2$. In this case the last inequality in step (a2.1) does not give a contradiction, but it is an equality. To prove that a general $S \in |\mathcal{I}_T(d+1)|$ contains no line $L \nsubseteq T$ with $h^0(H, \mathcal{I}_{T\cap H\setminus T\cap L}(d)) = 2$ it is sufficient to observe that there are ∞^2 degree $d+1$ surfaces containing T and each of them contains ∞^2 lines: the union of any $S' \in |\mathcal{I}_T(d)|$ and a plane. Let $\mathrm{Res}_L(T)$ denote the residual scheme of $T\cap H$ with respect to the Cartier divisor of H. If $H \cap T$ is reduced, then $\mathrm{Res}_L(T\cap H) = T\cap H \setminus L\cap T$ for the residual scheme of $T\cap H$ with respect to the Cartier divisor L of H. Thus we only need to exclude the lines $L \subset \mathbb{P}^3$ such that $h^0(H, \mathcal{I}_{\mathrm{Res}_L(T\cap H)}(d)) > 2$ for a general plane containing L. The schemes $L\cap T$ and $\mathrm{Res}_L(T\cap H)$ are linked by the complete intersection $T\cap H$ and hence there is a relation between the numerical invariants of these sets, as we will now explain. Set $z := \deg(T\cap L)$. Since $\deg(T\cap L) = z$ and $\deg(T\cap H) = d^2$, $\deg(\mathrm{Res}_L(T\cap H)) = d^2 - z$. Since $T\cap L$ is a degree z scheme contained in a line, $h^1(H, \mathcal{I}_{T\cap L,H}(t)) = 0$ and $h^0(H, \mathcal{I}_{T\cap L,H}(t)) = \binom{t+2}{2} - z$ for all $t \ge z-1$, while $h^0(H, \mathcal{I}_{T\cap L,H}(t)) = \binom{t+1}{2}$ and $h^1(H, \mathcal{I}_{T\cap L,H}(t)) = z-t-1$ for all $0 \le t \le z-1$. We have $h^0(H, \mathcal{I}_{\mathrm{Res}_L(H\cap T),H}(t)) = h^1(H, \mathcal{I}_{L\cap T,H}(2d-t+3))$ for all integers t ([25], the case in which $T\cap H$ is reduced is [**?**, Lemma at p. 199]). Thus $h^0(H, \mathcal{I}_{\mathrm{Res}_L(H),H}(d+1)) = 0$. Since $h^0(H, \mathcal{I}_{T\cap H,H}(d+1)) = 6$, we conclude the proof of this case.

(b) Now we take $x = dk$ for some integer k such that $0 < k < d$. We take as E_x the intersection of X_1 with the surface $M_1 \cup \cdots \cup M_k$. Since E_x is a complete intersection, $h^1(\mathcal{I}_{E_x}(t)) = 0$ for all t. Since $E_x \subsetneq T$ and $\mathcal{I}_T(y)$ is globally generated, $|\mathcal{I}_T(y)| \subsetneq |\mathcal{I}_{E_x}(y)|$. Thus it is sufficient to exclude the lines $L \subset T$ such that $L \nsubseteq E_{dk}$. Since there are only finitely many such lines, it is sufficient to prove $h^0(\mathcal{I}_{E_{dk}\cup L}(y)) < h^0(\mathcal{I}_{E_{dk}}(y))$ for each line $L \subset T$ such that $L \nsubseteq E_{dk}$. Fix any such a line L. Take any surface S' of degree $y - dk$ not containing L. The surface $M_1 \cup \cdots \cup M_k \cup S'$ gives $h^0(\mathcal{I}_{E_{dk}\cup L}(y)) < h^0(\mathcal{I}_{E_{dk}}(y))$.

(c) Now we take $x = dk + e$ with $0 \le k < d$ and $0 < e < d$. We take as E_x the union of all lines $L_{i,j}$ such that either $j \le k$ or $j = k+1$ and $1 \le i \le e$. As in step (b) we see that it is sufficient to prove that $h^0(\mathcal{I}_{E_x\cup L}(y)) < h^0(\mathcal{I}_{E_x\cup L}(y))$ for any line $L \subset T$ such that $L \nsubseteq E_x$. If $L \nsubseteq M_{k+1}$ (and hence $e \le d-2$) it is sufficient to take the union of $M_1\cup\cdots\cup M_{k+1}$ and a surface of degree $y-k+1$ not containing L. Thus we may assume $L \subset M_{k+1}$. The curve E_x (resp. $E_x \cup L$) is linked by the complete intersection of X_1 and $M_1 \cup \cdots \cup M_{d+1}$ to a plane curve E (resp. F) of degree $d-e$ (resp. degree $d-e-1$). By [23, Remarque III.1.3] $h^1(\mathcal{I}_{E_x}(t)) = h^1(\mathcal{I}_{E_x\cup L}(t)) = 0$ for all $t \in \mathbb{Z}$. Since $\deg(E_x \cup L) = \deg(E_x) + 1$, Riemann-Roch gives $h^0(\mathcal{I}_{E_x}(y)) - h^0(\mathcal{I}_{E_x\cup L}(y)) = y - p_a(E_x\cup L) + p_a(E_x \cup L) + h^1(\mathcal{I}_{E_x}(y)) -$

$h^1(\mathcal{I}_{E_x\cup L}(y))$. Since $\deg(L\cap E_x)=e$, $p_a(E_x\cup L)-p_a(E_x)=e-1$. Since $y\ge d+k-e+1-1$, the restriction map $H^0(\mathcal{O}_L(y))\to H^0(\mathcal{O}_{L\cap E_x}(y))$ is surjective.Thus Mayer-Vietoris exact sequence

$$0\to\mathcal{O}_{E_x\cup L}(y)\to\mathcal{O}_{E_x}(y)\oplus\mathcal{O}_L(y)\to\mathcal{O}_{L\cap E_x}(y)\to 0$$

gives $h^1(\mathcal{I}_{E_x\cup L}(y))=h^1(\mathcal{I}_{E_x}(y))$. Thus $h^0(\mathcal{I}_{E_x\cup L}(y))=h^0(\mathcal{I}_{E_x}(y))-y+e-1$. $\square$

Remark 2.2. In the set-up of Theorem 1.1 take $x\equiv 0 \pmod d$, say $x=dm$ for some $m\in\{1,\dots,d\}$. The definition of E_x as a complete intersection gives $h^0(\mathcal{I}_{E_{dm}}(m-1))=0$. However, if $m\ne d$ $h^0(\mathcal{I}_{Y_{dm}}(m))=1$ and the unique element S of $|\mathcal{I}_{E_{dm}}(m)|$ is a union of planes and hence $E_{dm}\ne Y_S$.

Proof of Theorem 1.3: The family of all degree d surfaces with a quadric as an irreducible component has dimension $\dim|\mathcal{O}_{\mathbb{P}^3}(d-1)|+\dim|\mathcal{O}_{\mathbb{P}^3}(2)|=\binom{d+3}{3}-(d+1)^2+8=\binom{d+1}{3}+8$. Since each irreducible quadric surface contains ∞^1 lines, we get the lower bound.

Fix a general $X\in\mathcal{J}$. If X is irreducible, then it is sufficient to use part (c). Assume that X is reducible and call z the minimal degree of an irreducible component of X. The minimality of z implies $z\le d/2$. By assumption $z\ge 3$. Thus we may assume $d\ge 6$. Since not all surfaces of degree z or of degree $d-z$ contain infinitely many lines, we get $\dim\mathcal{J}\le\binom{d-z+3}{3}+\binom{z+3}{3}-4$. The function $f_d(z)=\binom{d-z+3}{3}+\binom{z+3}{3}-4$ has a strict minimum at 3 and $d-3$ and these equal minima give the inequality $\dim\mathcal{J}\le\binom{d}{3}+16$. $\square$

3. Conclusions and suggestions for further works

There are several important papers on linear subspaces contained in complete intersections ([4, 5, 11, 14]). Although slice rank is defined only for hypersurfaces, complete intersections briefly occurred in the proof of at least one theorem on the strength of hypersurfaces ([2, §2]). We did not tried to extend the results proved in this paper to the case of complete intersections (not an easy task for the interested reader), because we are unable to find a clear link between the results of the present paper (not just the old papers [4, 5, 10, 11, 14, 21]). All the quoted papers use algebraic or complex analytic tools. Hypersurfaces may be defined over $\mathbb{R}$ and they may contain linear subspaces defined over $\mathbb{R}$ (the interested ones) or pairs of complex conjugate linear subspaces defined over $\mathbb{C}$ (the ones to avoid). Certainly tools of Real Algebraic Geometry and Real Semialgebraic Geometry may be used to study them. One of the referees suggested to try also Hard Real Analysis, e.g. [13].

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