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Some comparative growth properties of meromorphic function in the light of generalized relative order (α, β)

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ABSTRACT. In the paper we wish to establish some comparative growth properties of composite entire and meromorphic functions on the basis of generalized relative order (α, β) and generalized relative lower order (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1. INTRODUCTION

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [8, 11, 17]. We also use the standard notations and definitions of the theory of entire functions which are available in [16] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. A non-constant entire function f is said to have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds (see [3, 2]).When f is meromorphic, one may introduce another function $T_f(r)$, the Nevanlinna's characteristic function of f (see [8, p. 4]), playing the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a)(\overline{N}_f(r, a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left(\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a) - \overline{n}_f(0, a)}{t} dt + \overline{n}_f(0, a) \log r\right),$$

in addition we represent by $n_f(r, a)(\overline{n}_f(r, a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r, \infty)$ and $\overline{N}_f(r, \infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

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On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f (see [8, p.4]) is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where}$$
$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may employ $m_f(r,a)$ instead of $m(r, \frac{1}{f-a})$.

If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. For $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [9] defined the (p,q)-th order, denoted by $\rho^{(p,q)}(f)$, and (p,q)-th lower order, denoted by $\lambda^{(p,q)}(f)$, of an entire function, respectively, as follows:

Definition 1.1. [9] Let $p \geq q$. The (p,q)-th order $\rho^{(p,q)}(f)$ and (p,q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

For any entire function f, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [8]$\}$, one can easily verify that

$$\begin{matrix}\rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

when $p \geq 2$.

The function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Extending the notion (p,q)-th order, recently Shen et al. [13] introduced the new concept of $[p,q]$-φ order of entire and meromorphic function where $p \geq q$. Later on, combining the definition of (p,q)-order and $[p,q]$-φ order, Biswas (see, e.g., [6]) redefined the (p,q)-order of an entire and meromorphic function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [15]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$

and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$. For any $\alpha\in L$, we say that $\alpha\in L_1^0$, if $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$ and $\alpha\in L_2^0$, if $\alpha(\exp((1+o(1))x))=(1+o(1))\alpha(\exp(x))$ as $x\to+\infty$. Finally for any $\alpha\in L$, we also say that $\alpha\in L_1$, if $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$ and $\alpha\in L_2$, if $\alpha(\exp(cx))=(1+o(1))\alpha(\exp(x))$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$. Clearly, $L_1\subset L_1^0$, $L_2\subset L_2^0$ and $L_2\subset L_1$.

Considering the above, Sheremeta [14] introduced the concept of generalized order (α,β), denoted by $\rho_{(\alpha,\beta)}[f]$, of an entire function f as

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log M_f(r))}{\beta(\log r)}\ (\alpha\in L,\beta\in L).$$

For details about generalized order (α,β) one may see [14].

Now, we shall introduce the definition of the generalized order (α,β) of an entire function which considerably extend the definition of φ-order introduced by Chyzhykov et al. [7]. In order to keep accordance with Definition 1.1, have gave a minor modification to the original definition of generalized order (α,β) of an entire function (e.g. see, [12, 14]).

Definition 1.2. ($cf.$ [4]) Let $\alpha,\beta\in L$. The generalized order (α,β), denoted by $\rho_{(\alpha,\beta)}[f]$, and generalized lower order (α,β), denoted by $\lambda_{(\alpha,\beta)}[f]$, of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}.$$

If f is a meromorphic function, then

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}$$

Using the inequality $T_f(r)\leq\log M_f(r)\leq 3T_f(2r)$ $\{cf.$ [8]$\}$, for an entire function f, one may easily verify that

$$\begin{matrix}\rho_{(\alpha,\beta)}[f]\\ \lambda_{(\alpha,\beta)}[f]\end{matrix}=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(M_f(r))}{\beta(r)}=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha\in L_2$ and $\beta\in L_1$.

Definition 1.1 is a special case of Definition 1.2 for $\alpha(r)=\log^{[p]}r$ and $\beta(r)=\log^{[q]}r$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [3, 2, 10]) will come. Now in order to make some progress in the study of relative order,one may introduce the definitions of generalized relative order (α,β), denoted by $\rho_{(\alpha,\beta)}[f]_g$, and generalized relative lower order (α,β), denoted by $\lambda_{(\alpha,\beta)}[f]_g$, of a meromorphic function with respect to another entire function in the following way:

Definition 1.3. ($cf.$ [4]) Let $\alpha,\beta\in L$. The generalized relative order (α,β), denoted by $\rho_{(\alpha,\beta)}[f]_g$, and generalized relative lower order (α,β), denoted by $\lambda_{(\alpha,\beta)}[f]_g$, of a meromorphic function f with respect to an entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g=\limsup_{r\to+\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]_g=\liminf_{r\to+\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, then Definition 1.3 reduces to Definition 1.2. If $\alpha(r) = \beta(r) = \log r$, then we get the definition of relative order of meromrophic function f with respect to an entire function g introduced by Lahiri et al. [10] and if $g = \exp z$ and $\alpha(r) = \beta(r) = \log r$, then $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. And if $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ and $g = z$, then Definition 1.3 becomes the classical one given in [6].

The main aim of this paper is to establish some newly developed results related to the growth rates of composition of entire and meromorphic functions on the basis of generalized relative order (α, β) and generalized relative lower order (α, β) of meromorphic function with respect to another entire function which extend some earlier results (see, e.g., [5]). Henceforth we assume that $\alpha, \beta \in L_1$.

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [1] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f(\exp(r^{\mu})).$$

3. Main results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a meromorphic function and g, h be any two entire functions such that*

$$\liminf_{r\to+\infty} \frac{\alpha(T_h^{-1}(r))}{(\beta(r))^{\gamma}} = A, \text{ a real number } > 0 \tag{3.1}$$

and

$$\liminf_{r\to+\infty} \frac{\alpha(T_h^{-1}(T_f(\exp r^{\mu})))}{(\alpha(T_h^{-1}(r)))^{\eta+1}} = B, \text{ a real number } > 0 \tag{3.2}$$

for any γ, η, μ satisfying $0 < \gamma < 1,\ \eta > 0, \gamma(\eta+1) > 1$ and $0 < \mu < \rho_g \leq \infty$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h = \infty.$$

Proof. From (3.1) we have for all sufficiently large values of r that

$$\alpha(T_h^{-1}(r)) \geq (A-\varepsilon)(\beta(r))^{\gamma} \tag{3.3}$$

and from (3.2) we obtain for all sufficiently large values of r that

$$\alpha(T_h^{-1}(T_f(\exp r^{\mu}))) \geq (B-\varepsilon)(\alpha(T_h^{-1}(r)))^{\eta+1}. \tag{3.4}$$

Also $T_h^{-1}(r)$ is an increasing function of r, it follows from (3.3), (3.4) and Lemma 2.1 for a sequence of values of r tending to infinity that

$$\begin{aligned}
\alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq \alpha(T_h^{-1}(T_f(\exp(r^{\mu}))))\\
i.e.,\ \alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq (B-\varepsilon)(\alpha(T_h^{-1}(r)))^{\eta+1}\\
i.e.,\ \alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq (B-\varepsilon)[(A-\varepsilon)(\beta(r))^{\gamma}]^{\eta+1}\\
i.e.,\ \alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq (B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}\\
i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} &\geq \frac{(B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}}{\beta(r)}
\end{aligned}$$

$$i.e.,\ \limsup_{r\to+\infty} \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} \geq \liminf_{r\to+\infty} \frac{(B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}}{\beta(r)}.$$

Since $\varepsilon(>0)$ is arbitrary and $\gamma(\eta+1)>1$, it follows from above that

$$\rho_{(\alpha,\beta)}[f\circ g]_h=\infty.$$

This proves the theorem. □

Example 3.1. Let all the conditions of Theorem 3.1 hold for a meromorphic function f and for entire functions g,h and if we take $\alpha(r)=\log^{[p]}(r)$ and $\beta(r)=\log^{[q]}(r)$ with $p\geq q$ & $p\geq 2$, then one can easily show $\rho^{(p,q)}[f\circ g]_h=\infty$.

Theorem 3.2. *Let f be a meromorphic function and g,h be any two entire functions such that*

$$\liminf_{r\to+\infty}\frac{\alpha(T_h^{-1}(\exp(r^{\mu})))}{(\beta(r))^{\gamma}}=A,\ \text{a real number }>0 \tag{3.5}$$

and

$$\liminf_{r\to+\infty}\frac{\log(\frac{\alpha(T_h^{-1}(T_f(\exp r^{\mu})))}{\alpha(T_h^{-1}(\exp r^{\mu}))})}{(\alpha(T_h^{-1}(\exp r^{\mu})))^{\eta}}=B,\ \text{a real number }>0 \tag{3.6}$$

for any γ,η satisfying $\gamma>1,\ 0<\eta<1$, $\gamma\eta>1$ and $0<\mu<\rho_g\leq\infty$. Then

$$\rho_{(\alpha,\beta)}[f\circ g]_h=\infty.$$

Proof. From (3.5) we have for all sufficiently large values of r that

$$\alpha(T_h^{-1}(\exp(r^{\mu})))\geq(A-\varepsilon)(\beta(r))^{\gamma} \tag{3.7}$$

and from (3.6) we obtain for all sufficiently large values of r that

$$\begin{aligned}\log(\frac{\alpha(T_h^{-1}(T_f(\exp r^{\mu})))}{\alpha(T_h^{-1}(\exp r^{\mu}))}) &\geq (B-\varepsilon)(\alpha(T_h^{-1}(\exp r^{\mu})))^{\eta}\\ i.e.,\ \frac{\alpha(T_h^{-1}(T_f(\exp r^{\mu})))}{\alpha(T_h^{-1}(\exp r^{\mu}))} &\geq \exp[(B-\varepsilon)(\alpha(T_h^{-1}(\exp r^{\mu})))^{\eta}]. \end{aligned}\tag{3.8}$$

Also $T_h^{-1}(r)$ is an increasing function of r, it follows from(3.7), (3.8) and Lemma 2.1 for a sequence of values of r tending to infinity that

$$\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)}\geq\frac{\alpha(T_h^{-1}(T_f(\exp(r^{\mu}))))}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)}\geq\frac{\alpha(T_h^{-1}(T_f(\exp(r^{\mu}))))}{\alpha(T_h^{-1}(\exp(r^{\mu})))}\cdot\frac{\alpha(T_h^{-1}(\exp(r^{\mu})))}{\beta(r)}$$

$$\begin{aligned} i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} &\geq \exp[(B-\varepsilon)(\alpha(T_h^{-1}(\exp r^{\mu})))^{\eta}]\cdot\frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}\\ i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} &\geq \exp[(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\gamma\eta}]\cdot\frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}\end{aligned}$$

$$i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)}\geq\exp[(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\gamma\eta-1}\beta(r)]\cdot\frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)}\geq(\exp(\beta(r)))^{(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\gamma\eta-1}}\cdot\frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$\begin{aligned} &i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)}\\ &\geq\ \liminf_{r\to+\infty}\left((\exp(\beta(r)))^{(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\gamma\eta-1}}\cdot\frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}\right).\end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary and $\gamma > 1$, $\gamma\eta > 1$, $\liminf_{r\to+\infty}(\exp(\beta(r)))^{(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\gamma\eta-1}}$ exits. Therefore theorem follows from above. □

Example 3.2. Let all the conditions of Theorem 3.2 hold for a meromorphic function f and for entire functions g, h and if we take $\alpha(r) = \log^{[m]}(r)$ and $\beta(r) = \log r$ with $m \geq 2$, then one can easily show $\rho^{[m]}[f \circ g]_h = \infty$.

Theorem 3.3. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \rho_g \leq \infty$ and $\lambda_{(\alpha,\beta)}[f]_h > 0$. Then*

$$\rho_{(\alpha,\beta)}[f \circ g]_h = \infty.$$

Proof. Suppose $0 < \mu < \rho_g \leq \infty$. As $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2.1 for a sequence of values of r tending to infinity that

$$\begin{aligned} \alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq \alpha(T_h^{-1}(T_f(\exp(r^{\mu})))) \\ i.e.,\ \alpha(T_h^{-1}(T_{f\circ g}(r))) &\geq (\lambda_{(\alpha,\beta)}[f]_h - \varepsilon)\beta(\exp(r^{\mu})) \qquad (3.9)\\ i.e.,\ \frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} &\geq \frac{(\lambda_{(\alpha,\beta)}[f]_h - \varepsilon)\beta(\exp(r^{\mu}))}{\beta(r)} \\ i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} &\geq \liminf_{r\to+\infty}\frac{(\lambda_{(\alpha,\beta)}[f]_h - \varepsilon)\beta(\exp(r^{\mu}))}{\beta(r)} \\ i.e.,\ \rho_{(\alpha,\beta)}[f \circ g]_h &= \infty. \end{aligned}$$

Thus the theorem follows. □

Example 3.3. Let $f(z) = \exp z$ and $g(z) = h(z) = \exp z$, then $T_f(r) = \frac{r}{\pi}$, $T_{f\circ g}(r) \sim \frac{e^r}{(2\pi^3 r)^{\frac{1}{2}}}$. Here $\rho_g = 1$. If we take $\alpha(r) = \log(r^{\frac{1}{2}})$ and $\beta(r) = \log r$, then $\lambda_{(\alpha,\beta)}[f]_h = \frac{1}{2}$. Now

$$\begin{aligned} \rho_{(\alpha,\beta)}[f \circ g]_h &= \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} \sim \limsup_{r\to+\infty}\frac{\alpha\left(\frac{e^r}{(2\pi r)^{\frac{1}{2}}}\right)}{\beta(r)} \\ &= \limsup_{r\to+\infty}\frac{\frac{1}{2}(r - \frac{1}{2}\log r + O(1))}{\log r} \\ &= \infty.'' \end{aligned}$$

Theorem 3.4. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \rho_g \leq \infty$ and $\lambda_{(\alpha,\beta)}[f]_h > 0$. Then*

$$\limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\alpha(T_h^{-1}(T_f(r)))} = \infty.$$

Proof. In view of Theorem 3.3, we obtain that

$$\begin{aligned} \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\alpha(T_h^{-1}(T_f(r)))} &\geq \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\beta(r)} \cdot \\ &\quad \liminf_{r\to+\infty}\frac{\beta(r)}{\alpha(T_h^{-1}(T_f(r)))} \\ i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\alpha(T_h^{-1}(T_f(r)))} &\geq \rho_{(\alpha,\beta)}[f \circ g]_h \times \frac{1}{\rho_{(\alpha,\beta)}[f]_h} \\ i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\alpha(T_h^{-1}(T_f(r)))} &= \infty. \end{aligned}$$

Thus the theorem follows. □

Example 3.4. Let $f(z) = \exp z$ and $g(z) = h(z) = \exp z$, then $T_f(r) = \frac{r}{\pi}$, $T_{f\circ g}(r) \sim \frac{e^r}{(2\pi^3 r)^{\frac{1}{2}}}$. Here $\rho_g = 1$. If we take $\alpha(r) = \log(r^{\frac{1}{2}})$ and $\beta(r) = \log r$, then $\lambda_{(\alpha,\beta)}[f]_h = \frac{1}{2}$. Now

$$\begin{aligned}\limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\alpha(T_h^{-1}(T_f(r)))} &\sim \limsup_{r\to+\infty}\frac{\alpha\left(\frac{e^r}{(2\pi r)^{\frac{1}{2}}}\right)}{\alpha(r)}\\ &= \limsup_{r\to+\infty}\frac{\frac{1}{2}(r-\frac{1}{2}\log r+O(1))}{\frac{1}{2}\log r}\\ &= \infty.\end{aligned}$$

Theorem 3.5. *Let f be a meromorphic function and h be an entire function such that $0 < \lambda_{(\alpha,\beta)}[f]_h \leq \rho_{(\alpha,\beta)}[f]_h < \infty$. Also let g be an entire function with non-zero order. Then for every positive constant A and every real number γ,*

$$\limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\{\alpha(T_h^{-1}(T_f(r^A)))\}^{1+\gamma}} = \infty,$$

when $\lim_{r\to+\infty}\frac{\log\beta(\exp(r^\mu))}{\log\beta(r^A)} = +\infty$ *for any* $\mu > 0$.

Proof. If γ be such that $1+\gamma \leq 0$ then the theorem is trivial. So we suppose that $1+\gamma > 0$.

From the definition of $\rho_{(\alpha,\beta)}[f]_h$, it follows for all sufficiently large values of r that

$$\begin{aligned}\alpha(T_h^{-1}(T_f(r^A))) &\leq (\rho_{(\alpha,\beta)}[f]_h+\varepsilon)\beta(r^A)\\ i.e.,\ \{\alpha(T_h^{-1}(T_f(r^A)))\}^{1+\gamma} &\leq (\rho_{(\alpha,\beta)}[f]_h+\varepsilon)^{1+\gamma}(\beta(r^A))^{1+\gamma}.\end{aligned}\tag{3.10}$$

Now from (3.9) and (3.10), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\{\alpha(T_h^{-1}(T_f(r^A)))\}^{1+\gamma}} \geqslant \frac{(\lambda_{(\alpha,\beta)}[f]_h-\varepsilon)\beta(\exp(r^\mu))}{(\rho_{(\alpha,\beta)}[f]_h+\varepsilon)^{1+\gamma}(\beta(r^A))^{1+\gamma}}.$$

Since $\frac{\beta(\exp(r^\mu))}{(\beta(r^A))^{1+\gamma}} \to +\infty$ as $r \to +\infty$, the theorem follows from above. □

Example 3.5. Let all the conditions of Theorem 3.5 hold for a meromorphic function f and for entire functions g, h. If we take $\alpha(r) = \log^{[m]}(r)$ and $\beta(r) = \log r$ with $m \geq 2$, then one can easily show $\limsup_{r\to+\infty}\frac{\alpha(T_h^{-1}(T_{f\circ g}(r)))}{\{\alpha(T_h^{-1}(T_f(r^A)))\}^{1+\gamma}} = \infty$.

4. Conclusion

The present paper deals with the extension of the works on the growth properties of composite entire and meromorphic functions on basis of their generalized relative order (α,β) where α and β are continuous non-negative functions on $(-\infty,+\infty)$. The concept of generalized relative order (α,β) should have a broad range of applications in complex dynamics, factorization theory of entire functions of single complex variable, the solution of complex differential equations etc. which may be an ample scope of further research.

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