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The Golden Inheritance of Renaissance

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ABSTRACT. Federico Commandino (1509–1575) was born into an influential family from Urbino, Italy. His family's political connections secured him a position as the private secretary to pope Clement VII in 1534. He dedicated most of his life translating several fundamental works from ancient Greek into Latin, including Archimedes, Ptolemy, Euclid, Aristarchus, Pappus, Apollonius, Eutocius, Heron and Serenus [Rosen, E., *Biography in Dictionary of Scientific Biography* (New York 1970-1990)]. Most notable is his 1565 work titled *De centro gravitatis*, where an interesting novel idea is presented and explored. In this work, one particular element from Euclid's *Elements* is extended to tetrahedra: the center of gravity. In our paper we explore the heritage of his work on tetrahedra and we reflect on the educational value of this cornerstone moment in the history of geometry.

Federico Commandino was a true man of the Renaissance with a life composed of completed studies in medicine at the top universities of his century, service as private secretary to the Pope and, when he felt the world of politics did not present as much interest for him as a scholarly life, he translated the most important authors of mathematics from the ancient Greek world into Latin and Italian. It may sound like a fantastic story, but the life of Fredrico Commandino is undoubtedly true as it was cultivated in a fertile era of multi-disciplinary crossroads.

FIGURE 1. Federico Commandino (1509–1575)

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Commandino's inclination to spread mathematical enlightenment resulted in his own pursuit of original mathematical research. He asked himself whether certain properties of planar figures, investigated by Euclid in his *Elements*, could be extended to three-dimensional figures.

We present below this theorem called, in some sources, (competently preserving the memory of centuries past, e.g. [3]) Commandino's Theorem. The importance of this contribution is that it brings forth a better understanding of tetrahedra, superior to any work done before.

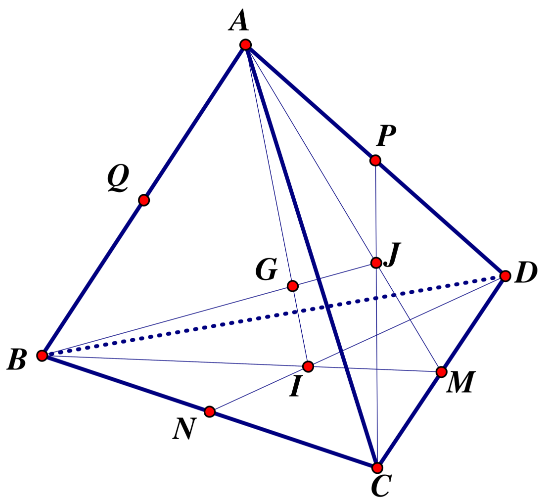
FEDERICI
COMMANDINI
VRBINATIS
LIBER DE CENTRO
GRAVITATIS
SOLIDORVM.

CVM PRIVILEGIO IN ANNOS X.
BONONIAE,
Ex Officina Alexandri Benacii.
M D LXV.

For any student pursuing the calculus sequence, a revealing moment awaits in the sections where the space geometry is investigated through methods of vector calculus. Some of the viewpoints incorporated in current textbooks present only vector solutions to three dimensional problems. However there are different methods available, pertaining to synthetic geometry in which our introduction outlined the historical period where such ideas originated. In the present note *we propose an exploration advocating for a dual approach, represented on one hand by three dimensional synthetic solutions, and on the other hand three dimensional analytic solutions*. What if our calculus textbooks presented dual perspectives like this?

These ideas belong, no doubt, in multivariable calculus sections where the geometry of lines and planes is investigated. Such a dual presentation would promote an integrated viewpoint that would make the whole idea more enjoyable and revealing to any curious mind. We start our exploration by setting forth the fact called in several sources Commandino's Theorem.

Question 1. (1835 in [3]; Problem 934 in [11].) *Prove that the straight lines joining the vertices of a tetrahedron $ABCD$ with the points of the intersection of medians on each of the faces meet in the same point G.*



Synthetic solution. Consider the tetrahedron $ABCD$ and denote the midpoints by M, N, P,and Q of edges CD, BC, AD, and AB, respectively. Let I be the center of gravity in ΔBCD and J the center of gravity in ΔACD. Medians BJ and AI of the tetrahedron meet in point G.

Due to similarity, we have the following string of proportions all coming from plane (ABM) :

$$\frac{IG}{GA} = \frac{JG}{GB} = \frac{IJ}{BA} = \frac{IM}{MB} = \frac{MJ}{MA} = \frac{1}{3}.$$

These ratios show that each pair of medians in the tetrahedron meet in a point G, namely that point dividing the median in ratio 1:3. □

This was the original way of thinking, very close to Euclid's spirit. However, it might be possible that for the calculus student raised in our century, the analytic approach looks more familiar.

Analytic solution. Consider the tetrahedron $ABCD$ such that the coordinates are as follows. $B(0,0,0)$, $A(m,n,p)$, $C(c,0,0)$, and $D(a,b,d)$. Then the center of gravity of the face BCD is $I\left(\dfrac{a+c}{3}, \dfrac{b}{3}, \dfrac{d}{3}\right)$, and hence the direction of the median AI in the tetrahedron is

$$\overrightarrow{AI} = \left(\frac{a+c}{3} - m, \frac{b}{3} - n, \frac{d}{3} - p\right).$$

The symmetric equations of line (AG) are

$$\frac{X-m}{\frac{a+c}{3} - m} = \frac{Y-n}{\frac{b}{3} - n} = \frac{Z-p}{\frac{d}{3} - p}.$$

To complete the proof we need to show that point G, with the property that the position vector of $G = \dfrac{1}{4}(A+B+C+D)$, satisfies the equation of line (AG). Since $G\left(\dfrac{m+a+c}{4}, \dfrac{n+b}{4}, \dfrac{p+d}{4}\right)$, a direct substitution yields immediately

$$\frac{\frac{m+a+c}{4} - m}{\frac{a+c}{3} - m} = \frac{\frac{n+b}{4} - n}{\frac{b}{3} - n} = \frac{\frac{p+d}{4} - p}{\frac{d}{3} - p} = \frac{3}{4}.$$

Similar computations can show that point G lies on the other three medians of the tetrahedron. □

The following result was stated and proved first in the same reference.

Question 2. *With the notations established in the previous problem, the straight line MG passes through the midpoint of segment AB. Or, otherwise stated, in any tetrahedron the segments joining midpoints of opposite edges pass through point G.*

Synthetic argument It is just a direct consequence of the previous proof. MG lies on the straight line MQ, which is median in triangle ABM.

Consistent to the dual approached spirit of our exploration, we present a second proof using the analytic method.

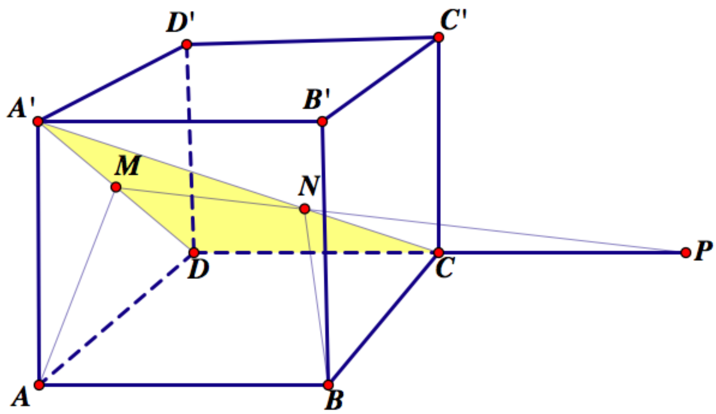
Analytic argument. In the tetrahedron $ABCD$, denote by M, N, P, and Q the midpoints of edges CD, BC, AD, and AB, respectively. Then $P\left(\dfrac{m+a}{2}, \dfrac{n+b}{2}, \dfrac{p+d}{2}\right)$, $M\left(\dfrac{a+c}{2}, \dfrac{b}{2}, \dfrac{d}{2}\right)$, $Q\left(\dfrac{m}{2}, \dfrac{n}{2}, \dfrac{p}{2}\right)$, and, finally, $N\left(\dfrac{c}{2}, 0, 0\right)$. A direct computation yields $\overrightarrow{PN} = \left(\dfrac{m+a-c}{2}, \dfrac{n+b}{2}, \dfrac{p+d}{2}\right)$, and thus

$$(PN): \quad \frac{X - \frac{c}{2}}{\frac{m+a-c}{2}} = \frac{Y}{\frac{n+b}{2}} = \frac{Z}{\frac{p+d}{2}}.$$

Point $G\left(\dfrac{m+a+c}{4}, \dfrac{n+b}{4}, \dfrac{p+d}{4}\right)$ lies on (PN) since

$$\frac{\frac{m+a+c}{4} - \frac{c}{2}}{\frac{m+a-c}{2}} = \frac{\frac{n+b}{4}}{\frac{n+b}{2}} = \frac{\frac{p+d}{4}}{\frac{p+d}{2}} = \frac{1}{2}.$$

Since the outcome of the ratio is $\frac{1}{2}$ we can immediately conclude that G is the midpoint of segment PN. Similarly, one can see that G lies on MQ, and that it is the midpoint of MQ. $\square$



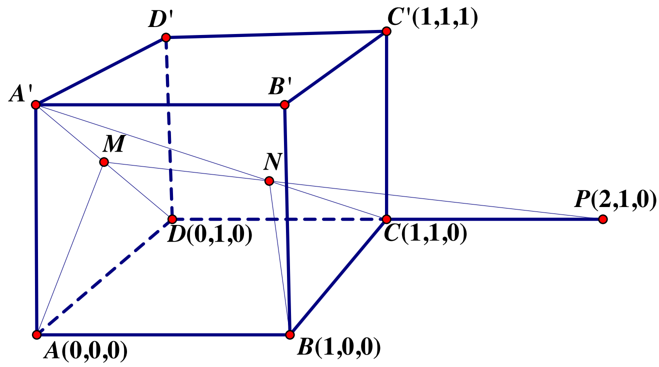
The following problem was authored by Ion Tiotioi and was used as Problem 3 for the Second round of the 2008 Romanian Olympiads ([5], p.47). Our presentation includes a lovely synthetical argument (published in the mentioned source) and an approach by a straightforward analytic idea.

Question 3. *Consider the cube $ABCDA'B'C'D'$ and the points M, N, P such that M is the foot of the perpendicular from A to plane $(A'CD)$, N is the foot of the perpendicular from B to the diagonal $A'C$, and P is the symmetric of D with respect to C. Show that the points M, N, P are collinear.*

Synthetic argument ([5], p. 47). Denoting by a the length of the cube edge. Point M is the midpoint of $A'D$. By the leg theorem, we express

$$CN = \frac{BC^2}{A'C} = \frac{a^2}{a\sqrt{3}} = \frac{a}{\sqrt{3}} = \frac{A'C}{3}.$$

Note that the three points M, N, P belong to plane $(A'CD)$, which means that one planar argument in $(A'CD)$ suffices. By applying Menelaus Theorem in $\triangle A'CD$, the collinearity of points M, N, P is insured by the relation $\frac{DP}{PC} \cdot \frac{CN}{NA'} \cdot \frac{A'M}{MD} = 1$. Since $\frac{DP}{PC} = 2$, $\frac{CN}{NA'} = \frac{1}{2}$, $\frac{A'M}{MD} = 1$, the relation needed in Menelaus Theorem is satisfies, hence the points M, N, P are collinear.



Analytic argument. Choose the system of coordinates such that $A(0{,}0{,}0)$, $B(1{,}0{,}0)$, $C(1,1{,}0)$, $D(0,1{,}0)$, and $A'(0,0,1)$. Then $M\left(0,\frac{1}{2},\frac{1}{2}\right)$, and $P(2,1,0)$, which yields immediately $\overrightarrow{MP} =< 2,\frac{1}{2},-\frac{1}{2} >$, so the straight line (MP) satisfies $x = 2+2s$, $y = 1+\frac{1}{2}s$, and $z = -\frac{1}{2}s$, for some $s \in \mathbb{R}$.

On the other hand, $\overrightarrow{A'C} =< 1,1,-1 >$. The straight line $(A'C)$ satisfies the parametric equations $x = t$, $y = t$, $z = 1-t$, for some $t \in \mathbb{R}$. The plane through B perpendicular to $A'C$ is

$$1(x-1)+1(y-0)-1(z-0)=0,$$

which reduces to $x-1+y-z=0$.

By intersecting this plane with the straight line $(A'C)$ we obtain $t = \frac{2}{3}$, which means that point N lies at $\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$.

By checking these coordinates in the parametric equations of line (MP), we see that the parameter $s = -\frac{2}{3}$ corresponds to point N. This proves that $N \in (MP)$. $\square$

These examples were well-suited to illustrate the educational benefits of developing a multifaceted approach to problems of this nature. For this reason, we would hope to present two different arguments in our classrooms today. More examples along the same lines could be found to better point out the bivalent substance of methods available in the three dimensional geometry.

This dual approach for three-dimensional geometry problems (analytic vs. synthetic, see also the interesting discussion in [7]) intertwines several layers of methods developed in various centuries. We have seen that Commandino's original vision was motivated by a synthetic quest. The fundamental ideas of analytic geometry originate in René Descartes' (1596–1650) third appendix to *Discourse de la méthode*. At the end of the 19th century, in a moment of synthesis [6], various authors thought about combining both methods in their monographs and problem books. We referred above to [11], a work still of interest today (e.g. [8]). Gheorghe Ţiţeica (1873-1939), a former Ph.D. student of Gaston Darboux (1842-1917), remains widely remembered as the founder of the affine differential geometry [1], and revered as an exquisite educator. Besides his stellar research, he was interested in promoting the education of gifted students, and this is how his collection of problems was written (with a first edition in 1906, a major revisitation in 1929, and an editorial upgrade done at over two decades after the author's passing [11]).

When Élie Cartan visited Bucharest in 1931 [2], he discussed extensively with his former colleague Ţiţeica how such ideas nurture mathematical research training. Ţiţeica himself was taking the rôle of mentor very seriously. His guidance meant a lot for many of the students raised in the tradition of the *Gazette*, e.g. Dan Barbilian or Sebastian Kaufmann [10]. Building experiences such as these in mathematical education leaves us with the firm belief that a dual presentation of some parts of three-dimensional geometry would be particularly useful and possibly influential in our calculus classes today.

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