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Intuitionistic level subgroups in the Klein-4 group

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ABSTRACT. In this paper we check the status of the already known result "level subgroups of any fuzzy subgroup of a finite group forms a chain" in Intuitionistic Fuzzy environment. The tool we use for this is the Klein-4 group V, which is a non-cyclic group. We prove that V has 64 distinct Intuitionistic Fuzzy Subgroups (IFSGs) upto isomorphism. The Intuitionistic Level Subgroups (ILSGs) of only 40 among them form chains and so the result is not true in intuitionistic fuzzy case. To strengthen our findings we provide a python program to construct the geometric representations of all the 64 IFSGs and its output.

1. INTRODUCTION

The concept of fuzzy sets was introduced by Lotfi A. Zadeh [29] in 1965 as an extension of the classical notion of set. It revolutionized the concept of belongingness of an element to a set by making partial belongingness possible. According to Zadeh, a fuzzy set is a pair (X, m) where X is a set and $m : X \to [0, 1]$ is a membership function. The reference set X is called *universe* of discourse, and for each $x \in X$ the value $m(x)$ is called the *grade of membership of x in X*. The function m is called the *membership function* of the fuzzy set $A = (X, m)$. Following this in 1976 Sanchez E. [24] came up with fuzzy relations based on formal models capable of perceiving the human way of dealing with complex phenomena, which are now used throughout fuzzy mathematics and has applications in areas such as linguistics [12], decision-making [17], clustering [4], etc. Apart from these, various concepts in fuzzy mathematics have a wide range of applications in a variety of fields such as engineering [26], computer science [16], medical diagnosis [19], social behavior studies [8], decision making [20], cryptographic models and signature schemes [28], fuzzy codes [2], etc. Because of this extensive applicability of fuzzy sets and relations, researchers throughout the world were motivated to generalize most of the abstract mathematical concepts to the fuzzy context and subject them to conscientious research. As a part of this advancement, many of the abstract algebraic concepts were generalized to fuzzy setting, because of the vital role played by them in a variety of researches in computer sciences, information sciences, cryptography, coding theory, etc. The development in this direction started with the introduction of fuzzy approach to the theory of groups by Rosenfeld [22] who defined the notion of fuzzy subgroups of a group. Subsequently, numerous studies appeared in the literature on various fuzzy algebraic structures. Later, in 1983, K. T. Atanassov [3] proposed the concept of intuitionistic fuzzy sets as an extension of the theory of the fuzzy set. In 1989, Biswas [6] extended Atanassov's definition of intuitionistic fuzzy sets to group theory and fromulated the theory of intuitionistic fuzzy subgroups of a group. Even now, many new results continue to appear in this area of study.

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[1], [5], [13] and [14] are some of the very recent works in the field of intuitionistic fuzzy group theory, which contain studies about novel concepts like anti-intuitionistic fuzzy subgroups, pythagorean fuzzy subgroups, t-intuitionistic fuzzy subgroups and complex intuitionistic fuzzy subgroups.

In 1981, P. S. Das [11] studied about the level subgroups of fuzzy subgroups of a group and found out that they form a chain. Later in 2006, Ahn et.al.[27] studied some properties of level subgroups of intuitionistic fuzzy subgroups of cyclic groups. But no one has made an attempt to check whether the level subgroups of a group form a chain in the intuitionistic fuzzy context also. The intention of our work is to explore whether the finding of Das can be translated into the arena of intuitionistic fuzzy subgroups of a group. In this paper, we try to study the status of the findings of P. S. Das[11] in intuitionistic fuzzy subgroups of Klein-4 group.

2. PRELIMINARIES

Definition 2.1. [29] Let X be any non-empty set. A **Fuzzy Subset** A in X is defined to be a function $A : X \to I = [0, 1]$ which assigns a degree of membership between 0 and 1 to each element of X.

Definition 2.2. [11] If A is a fuzzy subset of a non-empty set X and $t \in I$, then t**-cut** of A (or **Level Subset** of A at t), denoted by A_t, is defined as $A_t = \{x \in X : A(x) \geq t\}$.

Definition 2.3. [22] A fuzzy subset A of a group G is said to be a **Fuzzy Subgroup** (FSG) of G if, for $x, y \in G$
(1) $A(xy) \geq \wedge[A(x), A(y)]$ ($\wedge$ being the min operator on I)
(2) $A(x^{-1}) = A(x)$.

Proposition 2.1. [22] *If A is FSG of a group G with identity element e, then $A(e) \geq A(x), \forall x \in G$.*

Proposition 2.2. [11] *In a group G, a fuzzy subset A will be a FSG of G if and only if A_t is a subgroup of G for $0 \leq t \leq A(e)$.*

Definition 2.4. [11] For a FSG A of a group G, the subgroup A_t is called **Level Subgroup** of A at t, for $0 \leq t \leq A(e)$.

Proposition 2.3. [11] *Let A be a FSG of a finite group G with $Im(A) = \{t_i : i = 1, 2, 3, ..., n\}$. Then the collection $\{A_{t_i} : i = 1, 2, 3, ..., n\}$ contains all level subgroups of A.
Moreover, if $t_1 > t_2 > t_3 > ... > t_n$, then all these level subgroups will form a chain $G_A = A_{t_1} \subset A_{t_2} \subset A_{t_3} \subset ... \subset A_{t_n} = G$, where $G_A = \{x \in G : A(x) = A(e)\}$.*

Definition 2.5. [3] An **Intuitionistic Fuzzy Subset** (IFS) of a set X is an object of the form $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in X\}$ where the functions $m_A, n_A : X \to I$ represent the *degree of membership* and *degree of non membership* of any element $x \in X$ and should satisfy the condition $0 \leq m_A(x) + n_A(x) \leq 1, \forall\, x \in X$.

Definition 2.6. [25] Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFS in a set X and $\alpha, \beta \in I$. Then the **Intuitionistic Level Subset** (ILS) of A at (α, β) (or (α, β)–cut of IFS A) is the crisp set $A_{\alpha,\beta} = \{x \in X : m_A(x) \geq \alpha \textit{ and } n_A(x) \leq \beta\}$.

Definition 2.7. [21] An IFS $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ of a group G is said to be an **Intuitionistic Fuzzy Subgroup** (IFSG) of G if and only if
(1) $m_A(xy) \geq \wedge[m_A(x), m_A(y)]$ ($\wedge$ being the min operator on I)
(2) $m_A(x^{-1}) = m_A(x)$

(3) $n_A(xy) \leq \vee[n_A(x), n_A(y)]$ ($\vee$ being the max operator on I) and
(4) $n_A(x^{-1}) = n_A(x)$.

Proposition 2.4. [21] *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFSG in a group G with identity element e. Then, $m_A(e) \geq m_A(x)$ and $n_A(e) \leq n_A(x)$, $\forall\, x \in G$.*

Proposition 2.5. [25] *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFS in a group G. Then,*
(1) $A_{\alpha,\beta} = \phi$, for all $\alpha > m_A(e)$ and $\beta < n_A(e)$
(2) A is an IFSG in G $\Leftrightarrow$ $A_{\alpha,\beta}$ is a subgroup of G for $0 \leq \alpha \leq m_A(e)$ and $n_A(e) \leq \beta \leq 1$ with $\alpha + \beta \leq 1$.

Definition 2.8. [9] Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFSG in a group G. Then the subgroup $A_{\alpha,\beta}$ (where $0 \leq \alpha \leq m_A(e)$ and $n_A(e) \leq \beta \leq 1$) of G is called **Intuitionistic Level Subgroup** (ILSG) of A at (α, β).

Proposition 2.6. [25] *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFSG in a group G and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I$ be such that $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$. Then $A_{\alpha_1,\beta_1} \subseteq A_{\alpha_2,\beta_2}$.*

Proposition 2.7. [9] *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in G\}$ be an IFSG in a finite group G, $Im(m_A) = \{t_i : i = 1, 2, 3, ..., n\}$ and $Im(n_A) = \{s_j : j = 1, 2, 3, ..., m\}$. Then the collection*

$$\left\{\ A_{t_i,s_j} : i = 1, 2, 3, ..., n; j = 1, 2, 3, ..., m\right\}$$

contains all ILSG's of G.

Remark 2.1. Proposition 2.7 states that the Intuitionistic Fuzzy analogue of the first part of Proposition 2.3 holds true.

Definition 2.9. [10] Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in X\}$ be an IFS of a non-empty finite set X with $Im(m_A) = \{t_i : i = 1, 2, 3, ..., n\}$ and $Im(n_A) = \{s_j : j = 1, 2, 3, ..., m\}$ where $1 \geq t_1 > t_2 > ... > t_n \geq 0$ and $0 \leq s_1 < s_2 < ... < s_m \leq 1$. The *finite sequence* consisting of all intuitionistic level subsets of A, given by $\tilde{Ł}(A) = \{A_{t_1,s_1}, A_{t_1,s_2}, ..., A_{t_1,s_m}, A_{t_2,s_1}, A_{t_2,s_2}, ..., A_{t_2,s_m}, ..., A_{t_n,s_1}, A_{t_n,s_2}, ..., A_{t_n,s_m}\}$, is called the **Intuitionistic Level Representation (ILR)** of A.

Definition 2.10. [Isomorphic Intuitionistic Fuzzy Subsets][10] Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in X\}$ and $B = \{\langle x, m_B(x), n_B(x)\rangle : x \in X\}$ be two IFS's of a non-empty set X. We say that A and B are isomorphic, denoted by $A \cong B$, if for all $x, y \in X$
(I_1) $m_A(x) < m_A(y) \Leftrightarrow m_B(x) < m_B(y)$
(I_2) $m_A(x) = m_A(y) \Leftrightarrow m_B(x) = m_B(y)$
(I_3) $n_A(x) < n_A(y) \Leftrightarrow n_B(x) < n_B(y)$
(I_4) $n_A(x) = n_A(y) \Leftrightarrow n_B(x) = n_B(y)$

If two IFS's A and B of a non-empty set X are isomorphic, then the degrees of membership and non-membership of various elements of X w.r.t. A and B will have the same hierarchical ordering, but may differ in values.

Definition 2.11. [15] The **Klein 4-group** is a group with four elements, in which (i) each element is self-inverse and (ii) composing any two of the three non-identity elements produces the third one. It is usually denoted by $V = \{e, a, b, c\}$, where e is the identity element.

It may be observed that V is a non-cyclic group as none of its elements can be a generator for it.

3. ILSGs in Klein-4 Group

Throughout this section we will denote the Klein 4-group by $V = \{e, a_1, a_2, a_3\}$, where $a_1 = a, a_2 = b, a_3 = c$.

Proposition 3.8. *Let* $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ *be an IFSG in* V. *Then* $|Im(m_A)| \leq 3$ *and* $|Im(n_A)| \leq 3$.

Proof. Suppose $|Im(m_A)| > 3$. Then $|Im(m_A)| = 4$ (since, $m_A : V \to I$ and $|V|$ = 4). Let $Im(m_A) = \{t_1, t_2, t_3, t_4\}$ where $1 \geq t_1 > t_2 > t_3 > t_4 \geq 0$. By proposition 2.4 $m_A(e) = t_1$. Let $m_A(a_i) = t_2, m_A(a_j) = t_3, m_A(a_k) = t_4$ for any $i, j, k = 1, 2, 3$ where $i \neq j \neq k$. Then $m_A(a_k) < m_A(a_j)$, which is a contradiction to first axiom of IFSG in definition 2.7. This rules out the possibility of $|Im(m_A)| > 3$. Hence $|Im(m_A)| \leq 3$.
It can be similarly proved that $|Im(n_A)| \leq 3$. □

Proposition 3.9. *Let* $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ *be an IFSG in* V. *If* $m_A(a_i) > m_A(a_j)$, *then* $m_A(a_k) = m_A(a_j)$ *for any* $i, j, k = 1, 2, 3$ *where* $i \neq j \neq k$.

Proof. Suppose this is not true. That is, $m_A(a_i) > m_A(a_j)$ but $m_A(a_k) \neq m_A(a_j)$. Also by first axiom of IFSG in definition 2.7, $m_A(a_k) \geq \wedge[m_A(a_i), m_A(a_j)] = m_A(a_j)$. Hence $m_A(a_k) > m_A(a_j)$. Then the remaining possibilities are

$$\begin{aligned} m_A(a_k) &> m_A(a_i) > m_A(a_j) \\ m_A(a_k) &= m_A(a_i) > m_A(a_j) \\ m_A(a_i) &> m_A(a_k) > m_A(a_j) \end{aligned} \tag{3.1}$$

If $m_A(a_k) > m_A(a_i) > m_A(a_j)$, then $m_A(a_j) < m_A(a_i) = \wedge[m_A(a_i), m_A(a_k)]$ which is a contradiction to first axiom of IFSG in definition 2.7. The other three possibilities in (3.1) will also give rise to contradictions similar to this. Thus all possibilities listed in (3.1) are eliminated. Hence $m_A(a_k) = m_A(a_j)$. □

Proposition 3.10. *Let* $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ *be an IFSG in* V. *If* $n_A(a_i) < n_A(a_j)$, *then* $n_A(a_k) = n_A(a_j)$ *for any* $i, j, k = 1, 2, 3$ *where* $i \neq j \neq k$.

Proof. Similar to proof of proposition 3.9. □

Proposition 3.11. *Let* $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ *be an IFSG in* V. *If* $m_A(a_i) = m_A(a_j)$, *then* $m_A(a_k) \geq m_A(a_i) = m_A(a_j)$ *for any* $i, j, k = 1, 2, 3$ *where* $i \neq j \neq k$.

Proof. If $m_A(a_k) < m_A(a_i) = m_A(a_j)$, then $m_A(a_k) < \wedge[m_A(a_i), m_A(a_j)]$ which is a contradiction to first axiom of IFSG in definition 2.7. This completes the proof of the proposition. □

Proposition 3.12. *Let* $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ *be an IFSG in* V. *If* $n_A(a_i) = n_A(a_j)$, *then* $n_A(a_k) \leq n_A(a_i) = n_A(a_j)$ *for any* $i, j, k = 1, 2, 3$ *where* $i \neq j \neq k$.

Proof. Similar to proof of proposition 3.11. □

Remark 3.2. Propositions 3.9 and 3.11 imply that the hierarchy of membership degrees in any IFSG in V should be as follows:

$$\begin{aligned} m_A(e) &\geq m_A(a_i) > m_A(a_j) = m_A(a_k) \\ m_A(e) &\geq m_A(a_k) = m_A(a_i) = m_A(a_j) \\ m_A(e) &\geq m_A(a_k) > m_A(a_i) = m_A(a_j) \end{aligned}$$

for any $i, j, k = 1, 2, 3$ where $i \neq j \neq k$. It may be noted that, the third one is equivalent to the first and hence can be omitted.

Similarly, by propositions 3.10 and 3.12 the hierarchy of non-membership degrees in any IFSG in V should be:

$$n_A(e) \leq n_A(a_i) < n_A(a_j) = n_A(a_k)$$
$$n_A(e) \leq n_A(a_i) = n_A(a_j) = n_A(a_k)$$

for any $i, j, k = 1, 2, 3$ where $i \neq j \neq k$.
All the possible hierarchies of membership and non-membership degrees in any IFSG A in V are listed in the table 1.

TABLE 1. The possible hierarchies of membership and non-membership degrees in any IFSG A in V

	Membership degrees	Non-membership degrees
1	$m_A(e) > m_A(a_1) > m_A(a_2) = m_A(a_3)$	$n_A(e) < n_A(a_1) < n_A(a_2) = n_A(a_3)$
2	$m_A(e) > m_A(a_2) > m_A(a_1) = m_A(a_3)$	$n_A(e) < n_A(a_2) < n_A(a_1) = n_A(a_3)$
3	$m_A(e) > m_A(a_3) > m_A(a_1) = m_A(a_2)$	$n_A(e) < n_A(a_3) < n_A(a_1) = n_A(a_2)$
4	$m_A(e) > m_A(a_1) = m_A(a_2) = m_A(a_3)$	$n_A(e) < n_A(a_1) = n_A(a_2) = n_A(a_3)$
5	$m_A(e) = m_A(a_1) > m_A(a_2) = m_A(a_3)$	$n_A(e) = n_A(a_1) < n_A(a_2) = n_A(a_3)$
6	$m_A(e) = m_A(a_2) > m_A(a_1) = m_A(a_3)$	$n_A(e) = n_A(a_2) < n_A(a_1) = n_A(a_3)$
7	$m_A(e) = m_A(a_3) > m_A(a_1) = m_A(a_2)$	$n_A(e) = n_A(a_3) < n_A(a_1) = n_A(a_2)$
8	$m_A(e) = m_A(a_1) = m_A(a_2) = m_A(a_3)$	$n_A(e) = n_A(a_1) = n_A(a_2) = n_A(a_3)$

Proposition 3.13. *Given $t_1, t_2, t_3, s_1, s_2, s_3 \in I$ with $1 \geq t_1 \geq t_2 \geq t_3 \geq 0$ and $0 \leq s_1 \leq s_2 \leq s_3 \leq 1$, there exist exactly 64 non-isomorphic IFSG's in V with membership degrees t_1, t_2, t_3 and non-membership degrees s_1, s_2, s_3.*

Proof. It is clear from table 1 that, there are 8 possible hierarchies of membership degrees and 8 possible hierarchies of non-membership degrees in any IFSG in V. This means, membership degrees can be assigned in 8 different ways, following which non-membership degrees can also be assigned in 8 different ways. Hence, by the fundamental principle of counting, exactly $8 \times 8 = 64$ distinct (non-isomorphic) IFSG's can be defined in V. □

Proposition 3.14. *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ be an IFSG in V. If $m_A(a_i) \geq m_A(a_j)$ and $n_A(a_i) \leq n_A(a_j)$ for any $i, j = 1, 2, 3$ where $i \neq j$, then the ILSG's of A form a chain.*

Proof. Suppose $m_A(a_i) \geq m_A(a_j)$ and $n_A(a_i) \leq n_A(a_j)$. Then from the previous discussions we get: $m_A(e) \geq m_A(a_i) \geq m_A(a_j) = m_A(a_k)$ and $n_A(e) \leq n_A(a_i) \leq n_A(a_j) = n_A(a_k)$ where $i \neq j \neq k$. Hence there exist non-negative real numbers $t_1 \geq t_2 \geq t_3$ and $s_1 \leq s_2 \leq s_3$ in I, such that $m_A(e) = t_1, m_A(a_i) = t_2, m_A(a_j) = m_A(a_k) = t_3$ and $n_A(e) = s_1, n_A(a_i) = s_2, n_A(a_j) = n_A(a_k) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\} \\ &= \{\{e\}, \{e\}, \{e\}, \{e\}, \{e, a_i\}, \{e, a_i\}, \{e\}, \{e, a_i\}, V\}\end{aligned}$$

Hence, the distinct ILSG's of A are: $\{e\}, \{e, a_i\}, V$ which form the chain $A_{t_1,s_1} \subseteq A_{t_2,s_2} \subseteq A_{t_3,s_3}$. □

Proposition 3.15. *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ be an IFSG in V. If $m_A(a_i) > m_A(a_j)$ and $n_A(a_i) > n_A(a_j)$ for any $i, j = 1, 2, 3$ where $i \neq j$, then the ILSG's of A does not form a chain.*

Proof. Suppose $m_A(a_i) > m_A(a_j)$ and $n_A(a_i) > n_A(a_j)$. Then from the previous discussions we get: $m_A(e) \geq m_A(a_i) > m_A(a_j) = m_A(a_k)$ and $n_A(e) \leq n_A(a_j) < n_A(a_i) = n_A(a_k)$ where $i \neq j \neq k$. Hence there exist non-negative real numbers $t_1 \geq t_2 > t_3$ and $s_1 \leq s_2 < s_3$ in I, such that $m_A(e) = t_1, m_A(a_i) = t_2, m_A(a_j) = m_A(a_k) = t_3$ and $n_A(e) = s_1, n_A(a_j) = s_2, n_A(a_i) = n_A(a_k) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\} \\ &= \{\{e\}, \{e\}, \{e\}, \{e\}, \{e\}, \{e, a_j\}, \{e\}, \{e, a_i\}, V\}\end{aligned}$$

Hence, the distinct ILSG's of A are: $\{e\}, \{e, a_i\}, \{e, a_j\}, V$ which does not form a chain. □

Remark 3.3. Proposition 3.15 says that, the only possibilities where the ILSG's do not form a chain are when the membership and non-membership degrees of any two elements follow the same hierarchical ordering (other than equality).

Combining the above two propositions we get the following result.

Theorem 3.1. *Let $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in V\}$ be an IFSG in V. Then the ILSGs of A form a chain if, and only if, $m_A(a_i) \geq m_A(a_j)$ and $n_A(a_i) \leq n_A(a_j)$ for all $i, j = 1, 2, 3$ with $i \neq j$.*

Remark 3.4. Throughout the rest of this paper we may denote the IFSGs in V by $A(i; j)$; $i; j = 1; 2; 3; \ldots 8$, where the membership degrees in $A(i; j)$ are chosen as per the i^{th} row and the non-membership degrees as per the j^{th} row of table 1.

Remark 3.5. According to theorem 3.1, corresponding to each of the hierarchies of membership levels in rows 1, 2, 3, 5, 6 and 7 of table 1, exactly four hierarchies of non-membership levels will form IFSGs in V whose ILSGs does not form a chain and corresponding to each of the hierarchies of membership levels in rows 4 and 8 of table 1, all hierarchies of non-membership levels will form IFSGs in V whose ILSGs form chains.
For example, corresponding to the hierarchy of membership levels in row 1 of table 1 ($m_A(a_1) > m_A(a_2)$ and $m_A(a_1) > m_A(a_3)$), the ILSGs of all the IFSGs except $A(1, 2)$ [$n_A(a_1) > n_A(a_2)$], $A(1, 3)$ [$n_A(a_1) > n_A(a_3)$], $A(1, 6)$ [$n_A(a_1) > n_A(a_2)$] and $A(1, 7)$ [$n_A(a_1) > n_A(a_3)$] form chains.

Proposition 3.16. *The probability that the ILSG's corresponding to a randomly defined IFSG of V forms a chain is 13/16.*

Proof. As stated in proposition 3.13, 64 distinct IFSG's can be defined on V (upto isomorphism). By theorem 3.1, ILSG's corresponding to exactly 24 among them will not form a chain. Hence the proportion of IFSG's in which the ILSG's form a chain is $\frac{40}{64} = \frac{5}{8}$. □

4. A Python program to Construct IFSGs in Klein-4 Group

In this section we reinforce our findings in the previous sections by actually constructing all the 64 IFSGs of V for a given set of membership and non-membership levels. Since it is difficult to do it manually, we have done it using a python program[7, 23, 18]. Here we present that python program which gives as output the diagrammatic representations of all the 64 IFSGs with the membership and non-membership levels input by the user, and the outputs obtained.
In this section we will use the usual notation of Klein-4 group, which is $V = \{e, a, b, c\}$, in order to avoid complications in the figures.

Remark 4.6. We can give a **geometric representation to an IFS** $A = \{\langle x, m_A(x), n_A(x)\rangle : x \in X\}$ in a non-empty finite set X. For this, m_A is taken along x-axis and n_A along y-axis. Then an element x of X is represented as an element of A by the point $(m_A(x), n_A(x))$ in the coordinate plane. In this representation all elements of A will lie inside the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$.

Example 4.1. Consider the IFSG $A(1,1)$ in V. Then according to table 1, membership function will be given as: $m_{A(1,1)}(e) = t_1, m_{A(1,1)}(a) = t_2, m_{A(1,1)}(b) = m_{A(1,1)}(c) = t_3$ and non-membership function will be given as: $n_{A(1,1)}(e) = s_1, n_{A(1,1)}(a) = s_2, n_{A(1,1)}(b) = n_{A(1,1)}(c) = s_3$, where $1 \geq t_1 \geq t_2 \geq t_3 \geq 0$ and $0 \leq s_1 \leq s_2 \leq s_3 \leq 1$. The geometric representation of this IFSG is given in figure 1.Also from figure 1 it is evident that the ILSGs of $A(1,1)$ are $A(1,1)_{t_1,s_1} = \{e\}$ [the region marked as (I)], $A(1,1)_{t_2,s_2} = \{e, a\}$ [the region marked as (II)] and $A(1,1)_{t_3,s_3} = V$ [the region marked as (III)].

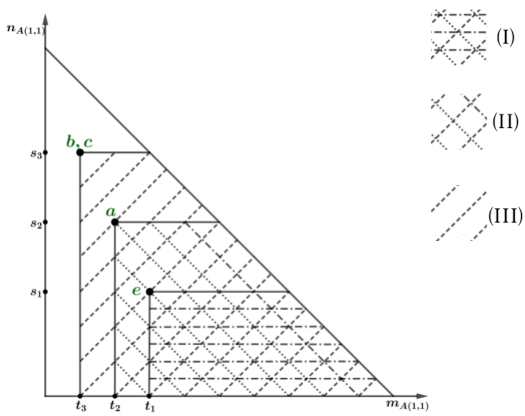

FIGURE 1. Geometric representation of $A(1,1)$ as in example 4.1

Example 4.2. Consider the IFSG $A(4,8)$ in V. Then according to table 1, membership function will be given as: $m_{A(4,8)}(e) = t_1, m_{A(4,8)}(a) = m_{A(4,8)}(b) = m_{A(4,8)}(c) = t_2$ and non-membership function will be given as: $n_{A(4,8)}(e) = m_{A(4,8)}(a) = m_{A(4,8)}(b) = m_{A(4,8)}(c) = s_1$, where $1 \geq t_1 \geq t_2 \geq 0$ and $0 \leq s_1 \leq 1$. The geometric representation of this IFSG is given in figure 2. Also from figure 2 it is evident that the ILSGs of $A(4,8)$ are $A(4,8)_{t_1,s_1} = \{e\}$ and $A(4,8)_{t_2,s_1} = V$.

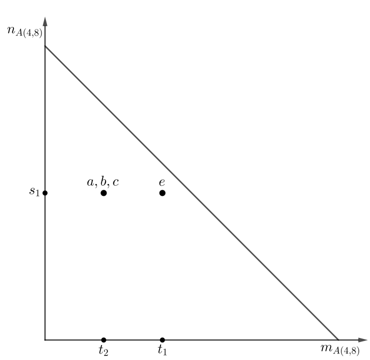

FIGURE 2. Geometric representation of $A(4,8)$ as in example 4.2

The python program for constructing diagrammatic representations of all the 64 IFSGs with the membership and non-membership levels input by the user is as follows:

LISTING 1. Python program for pictorising ILSGs in V

```
import matplotlib.pyplot as plt
#Swap Function
def swapEntries(list, entry1, entry2):
    list[entry1], list[entry2] = list[entry2], list[entry1]
    return list
#IFSG plotting function
def plotIFSG(M,N,B,A):
    plt.figure(figsize=(2,2))
    plt.xlim(0, 1)
    plt.ylim(0, 1)
    plt.title("A(%s, %s)"%(B,A))
    plt.scatter(M, N)
    plt.annotate("e", (M[0], N[0]))
    plt.annotate("a", (M[1], N[1]))
    plt.annotate("b", (M[2], N[2]))
    plt.annotate("c", (M[3], N[3]))
    plt.show()
#To plot IFSGs w.r.t different hierarchies of non membership levels
def nIFSGs(s1,s2,s3,a,b):
    n = [s1, s2, s3, s3]
    plotIFSG(m,n,b,a)
    for i in range (1,3):
        n = swapEntries(n, i, i+1)
        plotIFSG(m,n,b,a+i)
    n = [s1, s2, s2, s2]
    plotIFSG(m,n,b,a+3)
#Main program
print("Enter the membership degrees in decreasing order \n")
t1 = float(input())
t2 = float(input())
t3 = float(input())
print("Enter the non membership degrees in increasing order \n
(The sum of highest membership and non membership degrees should be
less than or equal to 1) \n")
s1 = float(input())
s2 = float(input())
s3 = float(input())
#To plot IFSGs w.r.t different hierarchies of membership levels
for k in range (1,6,4):
    m = [t1, t2, t3, t3]
    nIFSGs(s1,s2,s3,1,k)
    s1=s2
    nIFSGs(s1,s2,s3,5,k)
    for j in range (1,3):
        m = swapEntries(m, j, j+1)
        nIFSGs(s1,s2,s3,1,k+j)
        s1=s2
        nIFSGs(s1,s2,s3,5,k+j)
    m = [t1, t2, t2, t2]
    nIFSGs(s1,s2,s3,1,k+3)
    s1=s2
    nIFSGs(s1,s2,s3,5,k+3)
    t1=t2
```

The diagrammatic representations of the 64 distinct IFSGs corresponding to the membership degrees $t_1 = 0.6, t_2 = 0.4, t_3 = 0.2$ and non-membership degrees $s_1 = 0.1, s_2 = 0.3, s_3 = 0.4$, obtained by execution of the above Python program are shown in figure 3.

Remark 4.7. While running the program the 64 outputs are obtained one below the other. We have arranged and compiled them into a single figure for the sake of better understanding.

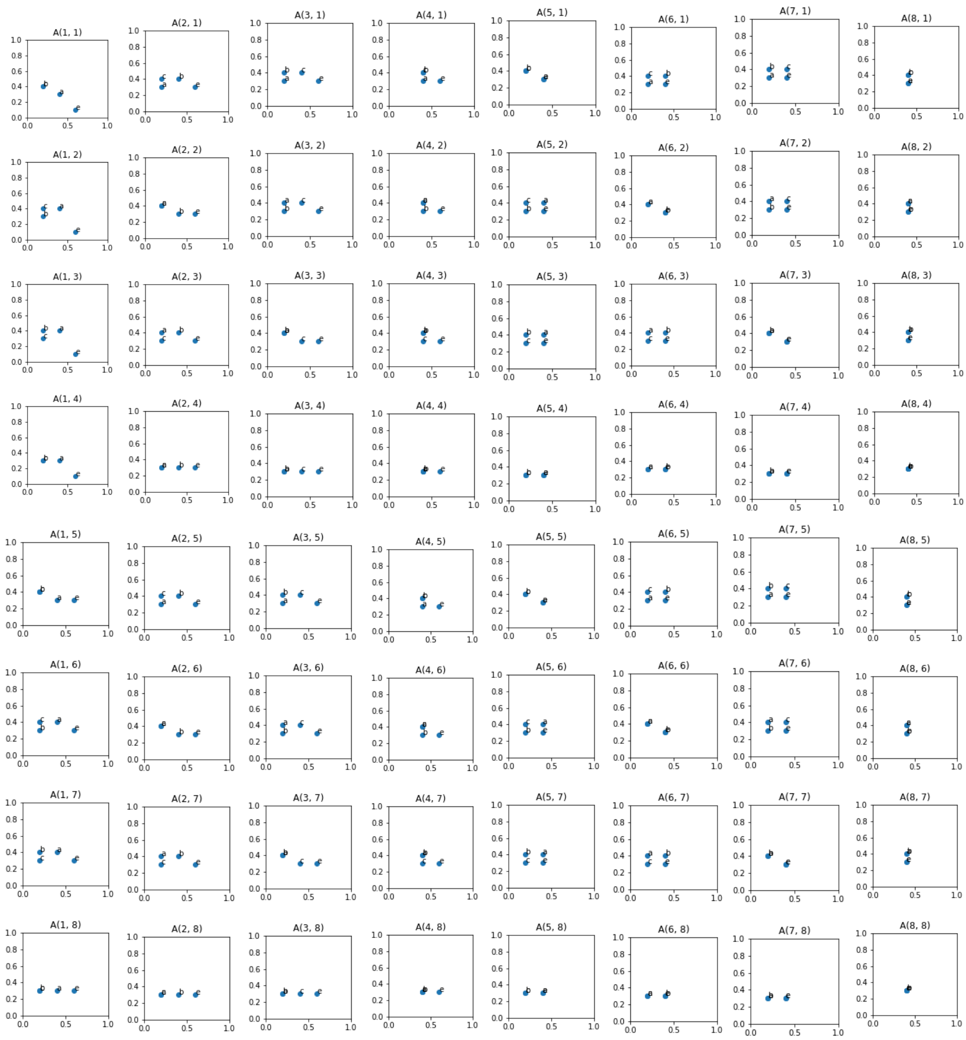

FIGURE 3. Output obtained from the Python Program

5. CONCLUSIONS

It was verified in the process of fuzzifying the abstract algebraic principles and theories that a chain is formed by the level subgroups of a fuzzy subgroup of any group. The aim of our research is to investigate whether this result can be extended to intuitionistic fuzzy subgroups. In this paper, we carry out this inquisition in Klein-4 group V. We have proved that, any IFSG of the Klein-4 group V can have at most three membership and non-membership levels. We have also proved that, V has 64 distinct IFSGs upto isomorphism. For only 40 among them the ILSGs form chains. We then support our findings by constructing all the 64 IFSGs of V for a given set of membership and non-membership levels using a python program. The python program and the geometric representations of all the 64 IFSGs of V obtained as its output are also provided.

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