CREAT. MATH. INFORM.
Volume **30** (2021), No. 2,
Pages 189 - 196

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2021.02.09

Special issue "30 years of publication of CMI

On first general Zagreb index of tournaments

S. PIRZADA[a], BILAL A. RATHER[b], TARIQ A. NAIKOO[c] and T. A. CHISHTI[d]

ABSTRACT. A tournament is an orientation of a complete simple graph. The score of a vertex in a tournament is the out degree of the vertex. The Zagreb index of a tournament is defined as the sum of the squares of the scores of its vertices. The *first general Zagreb index* of a tournament T is defined as $M_a(T) = \sum\limits_{i=1}^{n} s_i^a$, where a is any real number other than 0 and 1. In this paper, we obtain various lower and upper bounds for the first general Zagreb index of a tournament.

1. INTRODUCTION

A tournament is an orientation of a complete simple graph. Let T be a tournament with order n and having vertex set $\{v_1, v_2, \ldots, v_n\}$. The score of a vertex $v_i,\ 1 \leq i \leq n$, denoted by s_{v_i} (or simply by s_i), is defined as the out degree of v_i. Clearly, $0 \leq s_i \leq n-1$ for all $i,\ 1 \leq i \leq n$. The sequence $[s_1, s_2, \ldots, s_n]$ in non-increasing (or non-decreasing) order is called the score sequence of the tournament T. In a regular tournament on n (odd) vertices, each vertex has score $\frac{n-1}{2}$. Many of the important properties of tournaments were first investigated by Landau [3] (1953) in order to model dominance relations in flocks of chickens. Current applications of tournaments include the study of voting theory and social choice theory among other things. Other undefined notations and terminology can be seen in [9].

The following result [3], also called Landau's theorem, gives a necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

Theorem 1.1. (Landau [3]) *A sequence $[s_1, s_2, \ldots, s_n]$ of non-negative integers in non-decreasing order is a score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2},\ \textit{for}\ 1 \leq k \leq n \tag{1.1}$$

with equality when $k = n$.

Several results for the scores in a tournament can be seen in [5, 6].

For any two distinct vertices u and v of a tournament T, we have one of the following possibilities:

(i) An arc directed from u to v, denoted by $u(1-0)v$.
(ii) An arc directed from v to u, denoted by $u(0-1)v$.

Received: 13.01.2021. In revised form: 01.06.2021. Accepted: 08.06.2021
2010 *Mathematics Subject Classification.* 05C20.
Key words and phrases. *Tournament, score, score sequence, Zagreb index, first general Zagreb index, Landau's theorem.*
Corresponding author: S. Pirzada; pirzadasd@kashmiruniversity.ac.in

The *first general Zagreb index* [4] (also called the general zeroth-order Randić index) of a graph G, denoted by $Z_a(G)$, is defined as $Z_a(G) = \sum\limits_{i=1}^{n} d_i^a$, where a is any real number other than 0 and 1. Also, for $a = 2$, we have $Z_2(G) = \sum\limits_{i=1}^{n} d_i^2 = M_1(G)$, which is known as the *first Zagreb index*[2] of G. The first Zagreb index was used in examining the dependence of total π-electron energy of molecular structures. Analogous to this, Naikoo et al. [7] defined the first Zagreb index $M(T)$ of a tournament T as the sum of the squares of the scores of the vertices of T. That is, $M(T) = \sum\limits_{i=1}^{n} s_i^2$. Motivated by this, the *first general Zagreb index* of a tournament T is defined as $M_a(T) = \sum\limits_{i=1}^{n} s_i^a$. For $a = 0$, we have $M_0(T) = n$; and for $a = 1$ together with Inequality (1.1), $M_1(T) = \sum\limits_{i=1}^{n} s_i = \dfrac{n(n-1)}{2}$.

2. Bounds for the general Zagreb index of tournaments

For the rest of the paper, we assume the sequence to be in non-increasing order, unless otherwise stated. Let $M_k = \sum\limits_{i=1}^{k} s_i$ be the sum of largest k scores of tournament T. Then, by using the fact that $\sum\limits_{i=1}^{n} s_i = \dfrac{n(n-1)}{2}$, we have

$$\frac{M_k}{k} = \frac{\sum\limits_{i=1}^{k} s_i}{k} \geq \frac{\sum\limits_{i=k+1}^{n} s_i}{n-k} = \frac{\frac{n(n-1)}{2} - M_k}{n-k},$$

which after simplification gives

$$M_k \geq \frac{k(n-1)}{2} \tag{2.2}$$

with equality if and only if $k = n$. Now $\dfrac{k(n-1)}{2} > \dfrac{k(k-1)}{2}$ implies that $n > k$. So, for $k < n$, Inequality (2.2) is better than Inequality (1.1) and both the inequalities agree on $k = n$.

The following result gives the upper bound for M_k.

Lemma 2.1. *If T be a tournament with n vertices, then*

$$M_k \leq \frac{nk - k + \sqrt{(nk-k)^2 - 4D}}{2}, \tag{2.3}$$

where $D = k\left(\dfrac{n(n-1)}{2}\right)^2 - (n-k)k\dfrac{n(n-1)s_n}{2}$. Equality holds if and only if $T \cong nK_1$.

Proof. Using *Cauchy-Schwartz's inequality*, Inequality (1.1) and noting that $\sum\limits_{i=1}^{n} s_i^2 \leq \dfrac{n(n-1)s_n}{2}$ [7], we have

$$\left(\frac{n(n-1)}{2} - M_k\right)^2 = \left(\sum_{i=k+1}^{n} s_i\right)^2 \leq (n-k)\left(\sum_{i=k+1}^{n} s_i^2\right) = (n-k)\left(\sum_{i=1}^{n} s_i^2 - \sum_{i=1}^{k} s_i^2\right)$$

$$\leq (n-k)\left(\frac{n(n-1)s_n}{2} - \sum_{i=1}^{k} s_i^2\right)$$
$$\leq (n-k)\left(\frac{n(n-1)s_n}{2} - \frac{M_k^2}{k}\right).$$

After making some simplifications, we obtain

$$\frac{n}{k}M_k^2 - n(n-1)M_k + \left(\frac{n(n-1)}{2}\right)^2 - (n-k)\frac{n(n-1)s_n}{2} \leq 0.$$

Hence, it follows that

$$M_k \leq \frac{nk - k + \sqrt{(nk-k)^2 - 4D}}{2},$$

where $D = k\left(\frac{n(n-1)}{2}\right)^2 - (n-k)k\frac{n(n-1)s_n}{2}$. Assume that the equality holds in (2.3). Then all the above inequalities must be equalities. So $s_1 = s_2 = \cdots = s_k = s_{k+1} = s_{k+2} = \cdots = s_n$ and T is a regular tournament. This is only possible for $T \cong nK_1$. □

Jensen's inequality. Let f be a convex function on an interval $\mathcal{I}$ and let $x_1, x_2, \ldots, x_n$ be points of $\mathcal{I}$ and let $a_1, a_2, \ldots, a_n$ be real numbers satisfying $\sum\limits_{k=1}^{n} a_k = 1$. Then

$$f\left(\sum_{k=1}^{n} a_k x_k\right) \leq \sum_{k=1}^{n} a_k f(x_k)$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 2.2. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T. Then the following hold.*

(i)

$$\sum_{i=1}^{n} s_i^a \geq n\left(\frac{(n-1)}{2}\right)^a,$$

with equality if and only if $T \cong nK_1$.

(ii)

$$\sum_{i=1}^{n} s_i^a \geq n\left(\prod_{i=1}^{n} s_i^a\right)^{\frac{1}{n}},$$

with equality if and only if $T \cong K_n$.

Proof. **(i).** Since $f(x) = x^p$ is strictly convex for $x > 0$, so by *Jensen's inequality*, we have

$$\sum_{i=1}^{n} s_i^a \geq n\left(\sum_{i=1}^{n} \frac{s_i}{n}\right)^a = n\left(\frac{n-1}{2}\right)^a,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$.
(ii). Consider the function $f(x) = \log(x)$, which is concave in $(1, \infty)$. Thus, by *Jensen's inequality*, we have

$$\log\left(\sum_{i=1}^{n} \frac{s_i^a}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n} \log s_i^a \geq \frac{1}{n}\log\prod_{i=1}^{n} s_i^a \geq \log\left(\prod_{i=1}^{n} s_i^a\right)^{\dfrac{1}{n}},$$

which is equivalent to

$$\sum_{i=1}^{n} s_i^a \geq n\left(\prod_{i=1}^{n} s_i^a\right)^{\dfrac{1}{n}} = n\Big(s_1\ s_2\ \ldots\ s_{n-1}\ s_n\Big)^{\dfrac{a}{n}},$$

and equality holds if and only if $s_1 = s_2 = \cdots = s_n$. □

Proof. If $q > p > 0$, and $x_1, x_2, \ldots, x_n$ are non negative real numbers, then

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 2.3. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T. Then*

$$\sum_{i=1}^{n} s_i^a \geq k\left(\frac{k(n-1)}{2k}\right)^a + (n-k)\left(\frac{n-1}{2}\right)^a,$$

with equality if and only if $T \cong nK_1$.

Proof. By *power mean inequality*, we have

$$\left(\frac{\sum\limits_{i=1}^{k} s_i^a}{k}\right)^{\dfrac{1}{a}} \geq \frac{M_k}{k},$$

that is,

$$\sum_{i=1}^{k} s_i^a \geq \frac{M_k^a}{k^{a-1}},$$

with equality if and only if $s_1 = s_2 = \cdots = s_k$.

Similarly,

$$\sum_{i=k+1}^{n} s_i^a \geq \frac{\left(\dfrac{n(n-1)}{2} - M_k\right)^a}{(n-k)^{a-1}},$$

equality holds if and only $s_{k+1} = s_{k+2} = \cdots = s_n$.

Thus,

$$\sum_{i=1}^{n} s_i^a = \sum_{i=1}^{k} s_i^a + \sum_{i=k+1}^{n} s_i^a \geq \frac{M_k^a}{k^{a-1}} + \frac{\left(\dfrac{n(n-1)}{2} - M_k\right)^a}{(n-k)^{a-1}}.$$

Let $x = M_k$ and consider the function

$$f(x) = \frac{x^a}{k^{a-1}} + \frac{\left(\dfrac{n(n-1)}{2} - x\right)^a}{(n-k)^{a-1}}.$$

Now, solving $f^{'}(x) \geq 0$, we see that $f(x)$ is increasing for $x \geq \frac{k(n-1)}{2}$ and by Inequality (2.2), $M_k \geq \frac{k(n-1)}{2}$. Therefore, we get

$$\sum_{i=1}^{n} s_i^a = f(M_k) \geq f\left(\frac{k(n-1)}{2}\right) = k\left(\frac{k(n-1)}{2k}\right)^a + (n-k)\left(\frac{n-1}{2}\right)^a.$$

Assume equality occurs, then all the above inequalities are equalities, so that $s_1 = s_2 = \cdots = s_k = s_{k+1} = \cdots = s_n$ and $M_k = \frac{k(n-1)}{2}$, which is true for $T \cong K_1$. $\square$

Diaz-Metcalf inequality [1]. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers satisfying $ra_i \leq b_i \leq Ra_i$, for $1 \leq i \leq n$. Then,

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\sum_{i=1}^{n} a_i b_i,$$

equality holds if and only if $b_i = Ra_i$ or $b_i = ra_i$ for $1 \leq i \leq n$.

Theorem 2.4. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T. Then*

$$ns_n^a \leq \sum_{i=1}^{n} s_i^a \leq ns_1^a,$$

with equality holds if and only if T is a regular tournament.

Proof. Choosing $b_i = s_i^{\frac{a}{2}},\ a_i = 1,\ r = s_n^{\frac{a}{2}}$ and $R = s_1^{\frac{a}{2}}$, then we have

$$s_n^{\frac{a}{2}} \leq s_i^{\frac{a}{2}} \leq s_1^{\frac{a}{2}}.$$

By Diaz-Metcalf Inequality, we have

$$\begin{aligned}\sum_{i=1}^{n} s_i^a + (s_1 s_n)^{\frac{a}{2}}\sum_{i=1}^{n} 1 &\leq (s_n^{\frac{a}{2}} + s_1^{\frac{a}{2}})\sum_{i=1}^{n} s_i^{\frac{a}{2}}\\ &\leq (s_n^{\frac{a}{2}} + s_1^{\frac{a}{2}})\sum_{i=1}^{n} s_1^{\frac{a}{2}},\end{aligned}$$

which implies that

$$\sum_{i=1}^{n} s_i^a + (s_1 s_n)^{\frac{a}{2}} n \leq (s_n^{\frac{a}{2}} + s_1^{\frac{a}{2}}) n s_1^{\frac{a}{2}},$$

which further implies that

$$\sum_{i=1}^{n} s_i^a \leq ns_1^a,$$

with equality if and only if $s_1 = s_2 = \cdots = s_n$.

Similarly, choosing $b_i = s_i^a,\ a_i = 1,\ r = s_n^a$ and $R = s_1^a$ in Diaz-Metcalf inequality, we have

$$\sum_{i=1}^{n} s_i^{2a} + (s_1 s_n)^a n \leq (s_n^a + s_1^a)\sum_{i=1}^{n} s_i^a,$$

which implies that

$$\sum_{i=1}^{n} s_i^a \geq \frac{1}{s_n^a + s_1^a}\left(\sum_{i=1}^{n} s_i^{2a} + n(s_1 s_n)^a\right)$$
$$\geq \frac{1}{s_n^a + s_1^a}\left(n s_n^{2a} + n(s_1 s_n)^a\right) = n s_n^a.$$

Equality holds if and only if T is a regular tournament. □

Ozeki's Inequality [8]. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2,$$

where $M_1 = \max a_i,\ M_2 = \max b_i,\ m_1 = \min a_i$ and $m_2 = \min b_i,\ i = 1, 2, \ldots, n$.

Theorem 2.5. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T with $s_i \geq 1$ for all i. Then*

$$\sqrt{n^2 s_n^{2a} - \frac{n^2}{4}(s_1^a - s_n^a)^2} \leq \sum_{i=1}^{n} s_i^a \leq \frac{n^2}{4}(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}})^2 + n^2 s_1^a,$$

with equality if and only if T is a regular tournament.

Proof. Choosing $a_i = s_i^{\frac{a}{2}},\ b_i = 1,\ M_1 = s_1^{\frac{a}{2}},\ m_1 = s_n^{\frac{a}{2}}$ and $M_2 = m_2 = 1$ in Ozeki's inequality, we have

$$n\sum_{i=1}^{n} s_i^a - \left(\sum_{i=1}^{n} s_i^{\frac{a}{2}}\right)^2 \leq \frac{n^2}{4}\left(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}}\right)^2,$$

which implies that

$$n\sum_{i=1}^{n} s_i^a \leq \frac{n^2}{4}\left(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}}\right)^2 + \left(\sum_{i=1}^{n} s_i^{\frac{a}{2}}\right)^2,$$
$$\leq \frac{n^2}{4}\left(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}}\right)^2 + \left(\sum_{i=1}^{n} s_1^{\frac{a}{2}}\right)^2$$
$$= \frac{n^2}{4}\left(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}}\right)^2 + n^2 s_1^a.$$

Thus,

$$\sum_{i=1}^{n} s_i^a \leq \frac{n}{4}\left(s_1^{\frac{a}{2}} - s_n^{\frac{a}{2}}\right)^2 + n s_1^a.$$

Equality occurs if and only if T is a regular tournament.

Similarly, choosing $a_i = s_i^a,\ b_i = 1,\ M_1 = s_1^a, m_1 = s_n^a$ and $m_2 = M_2 = 1$ in Ozeki's inequality, we have

$$n\sum_{i=1}^{n} s_i^{2a} \leq \frac{n^2}{4}(s_1^a - s_n^a)^2 + \left(\sum_{i=1}^{n} s_i^a\right)^2,$$

which implies that

$$\left(\sum_{i=1}^{n} s_i^a\right)^2 \geq n\sum_{i=1}^{n} s_n^{2a} - \frac{n^2}{4}(s_1^a - s_n^a)^2.$$

Finally, we get

$$\sum_{i=1}^{n} s_i^a \geq \left(n^2 s_n^{2a} - \frac{n^2}{4}(s_1^a - s_n^a)^2\right)^{\frac{1}{2}},$$

with equality if and only if T is a regular tournament. □

Pólya-Szegö inequality [10]. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be non negative real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_ib_i\right)^2,$$

where $M_1 = \max a_i,\ M_2 = \max b_i,\ m_1 = \min a_i$ and $m_2 = \min b_i,\ i = 1, 2, \ldots, n.$

Theorem 2.6. *Let $[s_1, s_2, \ldots, s_n]$ be the score sequence of a tournament T, with $s_i \geq 1$ for all i. Then*

$$\frac{ns_n^a}{\frac{1}{2}\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}} \leq \sum_{i=1}^{n} s_i^a \leq \frac{ns_1^{\frac{a}{2}}}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2.$$

Proof. Let $a_i = s_i^{\frac{a}{2}},\ b_i = 1$, so that $M_1 = s_1,\ m_n = s_n,\ M_2 = m_2 = 1$. Now, from Pólya-Szegö inequality, we have

$$\begin{aligned} n\sum_{i=1}^{n} s_i^a &\leq \frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2 \left(\sum_{i=1}^{n} s_i^{\frac{a}{2}}\right)^2 \\ &\leq \frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2 s_1^{\frac{a}{2}} n^2. \end{aligned}$$

Thus,

$$\sum_{i=1}^{n} s_i^a \leq \frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2 s_1^{\frac{a}{2}} n.$$

Suppose equality holds. Then $s_1 = s_i = s_n$, which is possible for regular tournaments. Conversely, if T is a regular tournament, then equality holds.

Again, choosing $a_i = s_i^a,\ b_i = 1$ in Pólya-Szegö inequality, we obtain

$$n\sum_{i=1}^{n} s_i^{2a} \leq \frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2 \left(\sum_{i=1}^{n} s_i^a\right)^2,$$

Therefore,

$$\left(\sum_{i=1}^{n} s_i^a\right)^2 \geq \frac{n\sum_{i=1}^{n} s_i^{2a}}{\frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2} \geq \frac{n\sum_{i=1}^{n} s_n^{2a}}{\frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2} = \frac{n^2 s_n^{2a}}{\frac{1}{4}\left(\sqrt{\frac{s_1}{s_n}} + \sqrt{\frac{s_n}{s_1}}\right)^2},$$

and hence

$$\sum_{i=1}^{n} s_i^a \geq \frac{ns_n^a}{\frac{1}{2}\left(\sqrt{\frac{s_1}{s_n}}+\sqrt{\frac{s_n}{s_1}}\right)}.$$

It can be easily verified that equality occurs if and only if T is a regular tournament. $\square$

Example Consider the transitive tournament of order 5 with score sequence $[4,3,2,1,0]$. Lat $a=2$. By simple calculations, we have $\sum_{i=1}^{n} s_i^a = 30$ and $n(\frac{n-1}{2})^a = 20$ so that Theorem 2.2 is true. The bounds in Theorems 2.3, 2.4, 2.5 and 2.6 can be similarly verified.

Conclusions. As the first Zagreb index and the first generalized Zagreb index have been extensively studied for graphs, the investigation of the later in tournaments has been initiated in this paper. We obtained some upper and lower bounds for first generalized Zagreb index of tournaments mostly in terms of the order of the tournament. These bounds can be improved in future using more invariants of a tournament, for example, the number of arcs.

REFERENCES

[1] Diaz, J. B.; Metcalf, F. T. Stronger forms of a class of inequalities of G. Pólya, G. Szegö and L. K. Kantorovich, *Bull. Am. Math. Soc.* **69** (1963), 415–418.
[2] Gutman, I.; Trinajstić, N. Graph theory and molecular orbitals, Total $\pi-$electron energy of alternate hydrocarbons, *Chem. Phys. Lett.* **17** (1972), 535–538.
[3] Landau, H. G. On dominance relations and the structure of animal societies, III, the conditions for a score structure, *Bull. Math. Biophys.* **15** (1953), 143–148.
[4] Li, X.; Zheng, J. A verified approach to the extremal trees for the different indices, *MATCH Commun. Math. Comput. Chem.* **54** (2005), 195–208.
[5] Moon, J. W. *Topics on tournaments*, Holt, Rinehart and Winston, New York, 1968.
[6] Naikoo, T. A. On scores in tournaments, *Acta Univ. Sapientiae Informatica* **10** (2018), no. 2, 257–267.
[7] Naikoo, T. A.; Rather, B. A.; Pirzada, S., On Zagreb index of tournaments, *Kragujevac J. Maths.* to appear.
[8] Ozeki, N. On the estimation of the inequality by the maximum and minimum values, *J. College Arts Sci. Chiba Univ.* **5** (1968), 199–203.
[9] Pirzada, S. *An Introduction to Graph Theory*, Universities Press, Orient BlackSwan, Hyderabad 2012.
[10] Pólya, G.; Szego, G. *Problems and theorems in analysis*, Vol. 1, New York, Springer 1992.

[a,b,d]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KASHMIR
HAZRATBAL, SRINAGAR, KASHMIR, INDIA
Email address: pirzadasd@kashmiruniversity.ac.in
Email address: bilalahmadrr@gmail.com
Email address: tachishti@uok.edu.in

[c]DEPARTMENT OF MATHEMATICS
ISLAMIA COLLEGE OF SCIENCE AND COMMERCE
SRINAGAR, KASHIMIR, INDIA
Email address: tariqnaikoo@rediffmail.com