CREAT. MATH. INFORM.
Volume **30** (2021), No. 2,
Pages 197 - 202

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2021.02.10

Special issue "30 years of publication of CMI

Generalized (α, β)-derivations and left Ideals in Prime Rings

MD HAMIDUR RAHAMAN

ABSTRACT. Let R be a prime ring with center $Z(R)$, λ a nonzero left ideal, α, β are automorphisms of R and R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. In the present paper, we prove that if any one of the following holds:
(i) $F([x, y]) - b\alpha(x \circ y) \in Z(R)$
(ii) $F([x, y]) + b\alpha(x \circ y) \in Z(R)$
(iii) $F(x \circ y) - b\alpha([x, y]) \in Z(R)$
(iv) $F(x \circ y) + b\alpha([x, y]) \in Z(R)$
for all $x, y \in \lambda$ and for some $b \in R$ then R is commutative. Also some related results have been obtained.

1. INTRODUCTION

In all that follows, unless stated otherwise, R will be an associative ring with the center $Z(R)$, α, β be the automorphisms of R, λ a left ideal of R. For any $x, y \in R$, the symbols $[x, y]$ and $x \circ y$ stand for the Lie commutator $xy - yx$ and Jordan commutator $xy + yx$, respectively. A ring R is called 2-torsion free, if whenever $2a = 0$, with $a \in R$, then $a = 0$. If $S \subseteq R$, then we can define the left (resp. right) annihilator of S as $l(S) = \{x \in R \mid xs = 0 \ \ for\ all \ \ s \in S\}$ (resp. $r(S) = \{x \in R \mid sx = 0 \ \ for\ all \ \ s \in S\}$).

Recall that a ring R is prime if for any $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$, and is semiprime if for any $x \in R$, $xRx = \{0\}$ implies $x = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \in L$ for all $u \in L$ and $r \in R$, and a Lie ideal L is called square-closed if $u^2 \in L$ for all $u \in L$. By a derivation, we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let α and β be automorphisms of R, an additive mapping $d : R \longrightarrow R$ is said to be an (α, β)-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Following [1], an additive mapping $F : R \longrightarrow R$ is called a generalized (α, β)-derivation on R if there exists an (α, β)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Note that for I_R the identity map on R, this notion includes those of (α, β)-derivation when $F = d$, derivation when $F = d$ and $\alpha = \beta = I_R$ and generalized derivation, which is the case when $\alpha = \beta = I_R$.

Many results indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result of Posner [10] states that if R is a prime ring and d a non-zero derivation of R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R must be commutative. Over the last few decades, several authors

Received: 30.05.2020. In revised form: 03.04.2021. Accepted: 10.04.2021
2010 *Mathematics Subject Classification.* 16W25, 16R50, 16N60.
Key words and phrases. *Prime and semiprime rings, generalized (α, β)-derivations, (α, β)-derivations, automorphisms and ideals.*

have investigated the relationship between the commutativity of the ring R and certain specific types of derivations of R (see [3], [4], [5] and [7] where further references can be found).

Daif and Bell [6] showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x,y])-[x,y]=0$ or $d([x,y])+[x,y]=0$ for all $x,y\in I$, then $I\subseteq Z(R)$. In particular, if $I=R$ then R is commutative. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [11], Quadri et al., proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that any one of the following holds: (i) $F([x,y])-[x,y]=0$ (ii) $F([x,y])+[x,y]=0$ (iii) $F(x\circ y)-x\circ y=0$ (iv) $F(x\circ y)+x\circ y=0$ for all $x,y\in I$, then R is commutative. Following this line of investigation, Asma Ali, D. Kumar and P. Miyan [1], explored the commutativity of a prime ring R admitting a generalized derivation F satisfying any one of the following conditions: (i) $F([x,y])-[x,y]\in Z(R)$ (ii) $F([x,y])+[x,y]\in Z(R)$ (iii) $F(x\circ y)-x\circ y\in Z(R)$ (iv) $F(x\circ y)+x\circ y\in Z(R)$ for all $x,y\in I$, a nonzero right ideal of R. On the other hand, Marubayashi et al.[8], established that if a 2-torsion free prime ring R admits a nonzero generalized (α,β)-derivation F associated with an (α,β)-derivation d such that either $F([u,v])=0$ or $F(u\circ v)=0$ for all $u,v\in U$, where U is a nonzero square-closed Lie ideal of R, then $U\subseteq Z(R)$. In the present paper, our purpose is to prove the similar results for the case when the generalized (α,β)-derivation F acts on one sided ideal of R.

2. Main results

Throughout the present paper we shall make use of the following identities without any specific mention: For all $x,y,z\in R$
(i) $[xy,z]=x[y,z]+[x,z]y$,
(ii) $[x,yz]=y[x,z]+[x,y]z$,
(iii) $xo(yz)=(xoy)z-y[x,z]=y(xoz)+[x,y]z$,
(iv) $(xy)oz=x(yoz)-[x,z]y=(xoz)y+x[y,z]$.

Theorem 2.1. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α,β)-derivation F associated with a nonzero (α,β)-derivation d such that $d(Z(R))\neq(0)$. If $F([x,y])-b\alpha(x\circ y)\in Z(R)$ for all $x,y\in\lambda$ and for some $b\in R$, then R is commutative.

Proof. It is easy to check that $d(Z(R))\subseteq Z(R)$. Since $d(Z(R))\neq(0)$, there exists $0\neq a\in Z(R)$ such that $0\neq d(a)\in Z(R)$. By assumption, we have

$$F([x,y])-b\alpha(x\circ y)\in Z(R) \text{ for all } x,y\in\lambda. \tag{2.1}$$

Replacing y by ay in (2.1), we get

$$(F([x,y])-b\alpha(x\circ y))\alpha(a)+\beta([x,y])d(a)\in Z(R) \text{ for all } x,y\in\lambda. \tag{2.2}$$

Combining equation (2.1) and (2.2) and using the fact $\alpha(a)\in Z(R)$, we get $\beta([x,y])d(a)\in Z(R)$, which implies that $[\beta([x,y])d(a),r]=0=[\beta([x,y]),r]d(a)$ for all $x,y\in\lambda$ and $r\in R$. Since R is prime and $0\neq d(a)\in Z(R)$, we have

$$[\beta([x,y]),r]=0 \text{ for all } x,y\in\lambda; r\in R. \tag{2.3}$$

Replacing y by yx in (2.3) and using (2.3), we get

$$\beta([x,y])[\beta(x),r]=0 \text{ for all } x,y\in\lambda; r\in R. \tag{2.4}$$

Replacing r by $r\beta(s)$ in (2.4) and using (2.4), we arrive at $\beta([x,y])r[\beta(x),\beta(s)] = 0$ for all $x, y \in \lambda$ and $r, s \in R$. The primeness of R yields that for each $x \in \lambda$, either $\beta([x,y]) = 0$ or $[\beta(x), \beta(s)] = 0$. Equivalently, either $[x, \lambda] = 0$ or $[x, R] = 0$. Set $\lambda_1 = \{x \in \lambda \mid [x, \lambda] = 0\}$ and $\lambda_2 = \{x \in \lambda \mid [x, R] = 0\}$. Then, λ_1 and λ_2 are both additive subgroups of λ such that $\lambda = \lambda_1 \cup \lambda_2$. Thus, by Brauer's trick, we have either $\lambda = \lambda_1$ or $\lambda = \lambda_2$. If $\lambda = \lambda_1$, then $[\lambda, \lambda] = 0$, and if $\lambda = \lambda_2$, then $[\lambda, R] = 0$. In both cases, we conclude that λ is commutative and so, by a result of [9], R is commutative. □

Using similar arguments as used in the above theorem, we can prove the following:

Theorem 2.2. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F([x,y]) - b\alpha([x,y]) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Corollary 2.1. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(xy) - b\alpha(xy) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Proof. For any $x, y \in \lambda$, we have $F(xy) - b\alpha(xy) \in Z(R)$. Interchanging the role of x and y, we have $F(yx) - b\alpha(yx) \in Z(R)$. Therefore we have that $(F(xy) - \alpha(xy)) - (F(yx) - \alpha(yx)) \in Z(R)$ that is $F([x,y]) - b\alpha([x,y] \in Z(R)$ and hence the result follows. □

Theorem 2.3. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F([x,y]) + b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Proof. If $F([x,y]) + b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then the generalized (α, β)-derivation $-F$ satisfies the condition $F([x,y]) - b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$. It follows from Theorem 2.1 that R is commutative. □

Theorem 2.4. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) - b\alpha([x,y]) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Proof. By assumption We have that

$$F(x \circ y) - b\alpha([x,y]) \in Z(R) \text{ for all } x, y \in \lambda. \tag{2.5}$$

Since $d(Z(R)) \neq (0)$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$.

Replacing y by cy in (2.5), we get

$$(F(x \circ y) - \alpha([x,y])\alpha(c) + \beta(x \circ y)d(c) \in Z(R) \text{ for all } x, y \in \lambda. \tag{2.6}$$

Combining (2.5) and (2.6), we find that $\beta(x \circ y)d(c) \in Z(R)$ and hence $\beta(x \circ y) \in Z(R)$. This implies that

$$[\beta(x \circ y), r] = 0 \text{ for all } x, y \in \lambda; r \in R. \tag{2.7}$$

Replacing yx for y in (2.7) and using (2.7), we have

$$\beta(x \circ y)[\beta(x), r] = 0 \text{ for all } x, y \in \lambda; r \in R. \tag{2.8}$$

Replacing r by $r\beta(s)$ in (2.8) and using (2.8), we have $\beta(x \circ y)r[\beta(x), \beta(s)] = 0$ for all $x, y \in \lambda$ and $r, s \in R$. The primeness of R yields that for each $x \in \lambda$, either $\beta(x \circ y) = 0$ or $[\beta(x), \beta(s)] = 0$. Now applying similar arguments as used in the proof of Theorem 2.1, we have either $x \circ y = 0$ for all $x, y \in \lambda$; or $[\lambda, R] = 0$. In the former case, replacing x by

xz and using the fact $x \circ y = 0$ we find $[x, y]z = 0$ for all $x, y, z \in \lambda$. This implies that $[x, y]\lambda = 0$ and hence $[x, y]R\lambda = 0$. Since λ is a nonzero left ideal and R is prime, we get $[\lambda, \lambda] = 0$. Thus, λ is commutative and so R. In the latter case, we have $[\lambda, R] = 0$, in particular $[\lambda, \lambda] = 0$ and hence we get the required result. □

Using similar arguments as above with necessary variations, we can prove the following:

Theorem 2.5. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) - b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Theorem 2.6. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) + b\alpha([x, y]) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Proof. If $F(x \circ y) + b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$, then the generalized (α, β)-derivation $-F$ satisfies the condition $F(x \circ y) - b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$. It follows from Theorem 2.4 that R is commutative. □

Theorem 2.7. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(x \circ y) + b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, then R is commutative.

Corollary 2.2. Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$. If $F(xy) + b\alpha(xy) \in Z(R)$ for all $x, y \in \lambda$ and $b \in \{0, 1, -1\}$, then R is commutative.

Proof. For any $x, y \in \lambda$, we have $F(xy) + b\alpha(xy) \in Z(R)$. Interchanging the role of x and y, we have $F(yx) + b\alpha(yx) \in Z(R)$. Therefore we have that $(F(xy) + b\alpha(xy)) + (F(yx) + b\alpha(yx)) \in Z(R)$ that is $F(x \circ y) + b\alpha(x \circ y) \in Z(R)$ and hence the result follows. □

Theorem 2.8. Let R be a prime ring and λ a nonzero left ideal of R such that $r(\lambda) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F(\alpha([x, y])) = 0$ for all $x, y \in \lambda$, then R is commutative.

Proof. By assumption, we have

$$F(\alpha([x, y])) = 0 \text{ for all } x, y \in \lambda. \tag{2.9}$$

Replacing y by yx in (2.9) and using (2.9) , we get $\beta\alpha([x, y])d(\alpha(x) = 0$, which implies

$$[x, y]\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0 \text{ for all } x, y \in \lambda. \tag{2.10}$$

Now substituting ry for y in (2.10) and using (2.10) , we obtain $[x, r]y\alpha^{-1}\beta^{-1}(d(\alpha(x)) = 0$ for all $x, y \in \lambda$ and $r \in R$. In particular, $[x, R]R\lambda\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0$ for all $x \in \lambda$. The primeness of R yields that for each $x \in \lambda$, either $[x, R] = 0$ or $\lambda\alpha^{-1}\beta^{-1}(d(\alpha(x))) = 0$, in this case $d(\alpha(x)) = 0$. In view of similar arguments as used in the proof of Theorem 2.1, we have either $[\lambda, R] = 0$ or $d(\alpha(\lambda)) = 0$. If $[\lambda, R] = 0$, then λ is commutative and we are done. If $d(\alpha(\lambda)) = 0$, then $0 = d(\alpha(R\lambda)) = d(\alpha(R))\alpha^2(\lambda) + \beta(\alpha(R))d(\alpha(\lambda))$, which reduces to $d(\alpha(R))\alpha^2(\lambda) = 0$. And hence $d(\alpha(R))\alpha^2(R\lambda) = 0 = d(\alpha(R))\alpha^2(R)\alpha^2(\lambda) = d(\alpha(R))R\alpha^2(\lambda)$. Since λ is nonzero and the last relation forces that $d(\alpha(R)) = 0$ i.e $d = 0$, contradiction. □

Using the same techniques with necessary variations, we can prove the following:

Theorem 2.9. Let R be a prime ring and λ a nonzero left ideal of R such that $r(\lambda) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F(\alpha(x \circ y)) = 0$ for all $x, y \in \lambda$, then R is commutative.

Corollary 2.3. Let R be a prime ring and λ a nonzero left ideal of R such that $r(\lambda) = 0$. If R admits a generalized (α, β)-derivation F associated with a nonzero (α, β)-derivation d such that $F(\alpha(x^2)) = 0$ for all $x, y \in \lambda$, then R is commutative.

Proof. By our assumption, we have

$$F(\alpha(x^2)) = 0 \text{ for all } x \in \lambda. \tag{2.11}$$

Linearization of the above equation (2.11) and using equation (2.11), we have

$$F(\alpha(x \circ y)) = 0 \text{ for all } x, y \in \lambda. \tag{2.12}$$

By the Theorem 2.9, we get the result. □

The following example illustrates that R to be prime is essential in the hypothesis of Theorem 2.2, Theorem 2.3, Theorem 2.6 and Theorem 2.8.

Example 2.1. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$ and $\lambda = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | a, b \in S \right\}$, a nonzero left ideal of R, where S is any ring. Define maps $F,\ d\ , \alpha, \beta : R \longrightarrow R$ as follows:

$$F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is straightforward to check that F is a generalized (α, β)-derivation associated with a nonzero (α, β)-derivation d such that $d(Z(R)) \neq (0)$.

It is easy to see that

(i) $F([x, y]) - b\alpha([x, y]) \in Z(R)$,
(ii) $F([x, y]) + b\alpha([x, y]) \in Z(R)$,
(iii) $F(x \circ y) - b\alpha(x \circ y) \in Z(R)$ (iv) $F(x \circ y) + b\alpha(x \circ y) \in Z(R)$ for all $x, y \in \lambda$ and for some $b \in R$, however R is not commutative.

References

[1] Ali, A.; Kumar, D.; Miyan, P. On generalized derivations and commutativity of prime ang semiprime rings. *Hacettepe J. Math. Statistics* **40** (2011), 367–374.

[2] Ashraf, M.; Ali, A.; Ali, S. On Lie ideals and generalized (θ, ϕ)-derivations in prime rings. *Comm. Algebra* **32** (2004), 2977–2985.

[3] Ashraf, M.; Rehman, N. On commutativity of rings with derivations. *Results Math.* **42** (2002), 3–8.

[4] Bell, H. E.; Daif, M. N. On commutativity and strong commutativity-preserving maps *Canad. Math. Bull.* **37** (1994), 443–447.

[5] Bell, H. E.; Daif, M. N. On derivations and commutativity in prime rings. *Acta. Math. Hungerica* **66** (1995), 337–343.

[6] Daif, M. N.; Bell, H. E. Remarks on derivations on semiprime rings. *Int. J. Math. & Math. Sci.* **15** (1992), 205–206.
[7] Golbasi, O. Commutativity of semiprime rings with genearlized derivations. *Indian J. Pure appl. Math.* **40** (2009), 191–199.
[8] Marubayashi, H.; Ashraf, M.; Rehman, N.; Ali, S. On generalized (α, β)-derivations in prime rings. *Algebra Colloquium* **17** (2010), 865–874
[9] Mayne, J. H. Centralizing mappings of prime rings. *Canad. Math. Bull.* **27** (1984), 122–126.
[10] Posner, E. C. Derivations in prime rings. *Proc. Amer. Math. Soc.* **8** (1957), 1093–1100.
[11] Quadri, M. A.; Khan, M. S.; Rehman, N. Generalized derivations and commutativity of prime rings. *Indian J. Pure Appl. Math.* **34** (2003), 1393–1396.

Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email address: rahamanhamidmath@gmail.com