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Ways to organize learning paths to discover solutions to competition problems

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ABSTRACT. In this article, we illustrate a method of organizing the process of discovering solutions to problems for mathematical contests. We describe the authors' experience in creating situations of learning individually, through cooperation and collaboration in pairs, and in small groups. The examples proposed for discussion can be considered mini-scientific works, which allow thorough research of the situation, but require detailed explanation and collaboration between students in order to refresh the supporting concepts and create generalizations.

1. INTRODUCTION

Discussions on the impact of excessive student guidance in the process of assimilating knowledge have been going on for many years. It is obvious that the dose of explanations and proofs differs from case to case. Often schools focus too much on knowledge transfer. Students usually remain passive, while the teacher strives to personally transmit the relevant material. The obsession with "passing on the material" allows for a smaller contribution from students. For this reason, we need a fundamental reorientation. The priority among teachers should not be to transfer knowledge to the students, but rather to allow students to access knowledge on their own. If students with an average level of knowledge are targeted in the teaching-learning process complete explanations are welcome. In the case of gifted student teams, the process of studying methods of solving special problems or solving complicated problems can be organized following a scheme of three steps: me $\to$ pairs $\to$ team. This scheme is designed in the following way: "me" $\to$ "me and you" $\to$ "all of us". The sequence allows for the probing of the solution or the execution of tests, particular cases. Naturally that these tests may be different for different students. In order to "shrink" the particular cases examined individually, we move on to the exploration of the problem situation in pairs. At this stage, the distribution of tasks is also accepted, if the exploration is laborious, so it is natural for students to cooperate to move faster in finding the solution. If working in pairs, the students find an elegant solution, its presentation to the whole team follows. If the pairs are far from a plausible solution, examination of the solution proposed by the authors of the problem and its presentation to the whole team follows. In this situation, students collaborate to develop the author's ideas and argue each reasoning based on their own knowledge.

We will illustrate this activity as an example of solving some problems proposed at the International Mathematical Olympiad.

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2. EXAMPLES

Problem 2.1. (**IMO 1997**) In a finite sequence of real numbers the sum of any 7 successive terms is negative and the sum of any 11 successive terms is positive. Determine the maximum number of terms in the sequence.

Solution: Let $x_1, x_2 \ldots x_n$ be this sequence.

Students are asked to work individually to discover that a case can correspond to a sequence of 77 terms. The idea of writing :

$$(x_1 + x_2 + \ldots + x_7) + (x_8 + \ldots + x_{14}) + \ldots + (x_{71} + x_{72} + \ldots + x_{77}) =$$
$$= (x_1 + x_2 + \ldots + x_{11}) + (x_{12} + \ldots + x_{22}) + \ldots + (x_{67} + \ldots + x_{77})$$

leads to a contradiction, because the first term must be negative, and the second positive. This gives us that $n < 77$.

For pairs, the idea of "shrinking" the calculations is suggested until we reach:

$$(x_1 + x_2 + \ldots + x_7) + (x_2 + x_3 + \ldots + x_8) + \ldots + (x_{11} + x_{12} + \ldots + x_{17}) =$$
$$= (x_1 + x_2 + \ldots + x_{11}) + (x_2 + x_3 + \ldots + x_{12}) + \ldots + (x_7 + x_8 + \ldots + x_{17}).$$

A contradiction is obtained which yields to the conclusion that $n < 17$.

Also in pairs, concrete examples can be built to verify the solution.

The author solution continues by an example of a sequence with the properties of the statement with 16 elements, namely $5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5$, showing that the maximum number of terms required by the problem is 16.

The main activity can be oriented towards the examination of the solution awarded with the *special prize* by the Olympic Committee :

Let $x_1, x_2 \ldots x_n$ be a finite sequence of real numbers. Let us note : $y_0 = 0$, $y_1 = x_1$, $y_k = x_1 + x_2 + \ldots + x_k$. There is a bijective correspondence between the sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ made in the opposite direction : $x_k = y_k - y_{k-1}$.

The conditions of the problem are equivalent to : $y_{k+7} - y_k < 0$ for any $0 \leq k \leq n-7$, and $y_{k+11} - y_k > 0$ for any $0 \leq k \leq n-11$. So we have that $y_{k+7} < y_k$ and $y_k < y_{k+11}$ for those k.

Let's write these "consecutive" inequalities, completing each time on the right with a y of minimum possible index:

$$0 = y_0 < y_{11} < y_4 < y_{15} < y_8 < y_1 < y_{12} < y_5 <$$
$$< y_{16} < y_9 < y_2 < y_{13} < y_6 < y_{17} < y_{10} < y_3 < y_{14} < y_7 < y_0 = 0.$$

In this way, it is seen that $n \geq 17$ leads to contradiction. For $n = 16$ it is observed that these inequalities contain all terms of the sequence and that it is no longer cyclic, but linear, starting with : $y_{10} < y_3 < y_{14} < \ldots$ and ending with $< y_2 < y_{13} < y_6$. By choosing $x_1, \ldots, x_{16}$ so that these inequalities are true, which is obviously possible, we can determine $x_1, \ldots, x_{16}$ by the formula at the end of the first paragraph of the solution. In this way, in addition to the conclusion that 16 is the maximum number of terms in a sequence with the properties in the statement, the general form of these sequences of 16 elements is

also obtained.
We also apply the described organizational procedure to familiarize students with the method of solving another problem proposed at the 2011 International Mathematical Olympiad (Amsterdam).

Problem 2.2. **(IMO 2011)** Let $\mathcal{A}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{A}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{A}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{A}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{A}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{A}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{A}$ as a pivot infinitely many times.

Solution:
"Me" : Students individually try to examine various situations, varying the number of points and their position in the plane. It appears that it is possible to move the line on the convex hull, and the line will not pass through other points except those on the hull.
"Me and You" : The idea that the problem be approached from the inside is suggested to the pairs, considering lines that would satisfy various conditions. Fix a line ℓ passing through one of the points of the set $\mathcal{A}$ and consider the obtained half-planes painted in orange and blue. Executing the rotation of this line allows students to observe that after the "pivot" changes its position from one point T to another point U. T will be on the same side as point U in the previous stage. That is why the number of elements in $\mathcal{A}$ in each half-plane remains the same throughout the process, except the case in which the line ℓ contains 2 points.

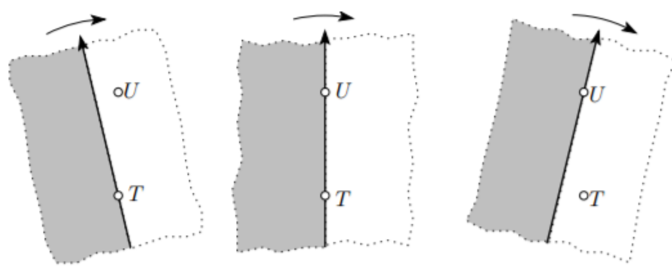

FIGURE 1. The *windmill* process

We will first consider the case $|\mathcal{A}| = 2n+1$. We claim that for any point $T \in \mathcal{A}$, there is a line that divides the set $\mathcal{A}$, so that in each half-plane there are exactly n points. To observe this, we choose a line that passes through T and does not contain other points in $\mathcal{A}$, and we assume that there are $n + r$ points in the orange part. If $r = 0$, then we have proved the statement. Suppose that $r \neq 0$. While the line rotates by $180°$ around T, the number of points on the orange side changes by ± 1 as the line passes through each point. After the rotation by $180°$, the number of points on the orange side is $n - r$. Thus, there will be an intermediate phase in which both the orange and the blue part will both contain exactly n points.
Let's choose a line that passes through an arbitrary point $P \in \mathcal{A}$ and separates n points on each side, which should be the initial state of the windmill. We will show that after a rotation by $180°$, the windmill line passes through each point in $\mathcal{A}$, as a pivot. We select

any point T in $\mathcal{A}$ and a line ℓ passing through T that separates the plane in two parts with the same number of points. The set $\mathcal{A}$ has no other point with the property that drawing through it a line parallel with ℓ separates the plane in 2 subsets with the same number of points, because, ℓ will be on the left of that line, or on the right(suppose it's on the left, there is contradiction because both lines can't have simultaneously n points on the left), which will lead us to a contradiction. Therefore, when the windmill is parallel to ℓ , it will coincide with ℓ therefore passes through T. This means that the line has returned to it's original state and the "windmill" movement continues to pass an infinite number of times through each point of $\mathcal{A}$.

Now we assume that $|\mathcal{A}| = 2n$. Analogously to the first case, for any $T \in \mathcal{A}$ there is an oriented line passing through T such that there are $n-1$ points in the orange part and n points in the blue part. Such a line passing through an arbitrary point P will be the initial state of the "windmill". We claim this line will do the job. After a rotation by $360°$ the line will return to it's original position. Now we take a fixed point T in $\mathcal{A}$, and we know there is a line such that the orange part contains $n-1$ points and the blue one n points. When the line of the windmill ℓ is parallel to this line, by the same arguments as above these lines must be the same, so after a rotation by $360°$ our windmill goes through all points of $\mathcal{A}$.

A more general solution, proposed by an ex-Olympiad student can be examined next. This solution is a mini-scientific work, which allows thorough research of the situation, but requires detailed explanation and collaboration between students in order to update the supporting concepts.

Initial considerations: We will denote by $\mathcal{A}$ the set of corresponding points, and by $\mathcal{M}$ our mills. Let $\mathcal{S}$ be the convex hull of $\mathcal{A}$. For a certain line, we choose it's direction, then the rotation of the line will correspond to the rotation of it's "direction", so each line in the mill will have a direction and so we can say that a point is on line's left side or on line's right side. We notice that the rotation of the mill can be reversed, the reverse having the same principle of rotation, but in a counterclockwise direction. Furthermore we notice that the mill is periodical because there is a finite number of pairs of points in $\mathcal{M}$. So if the mill passes through a point once, it will pass through it an infinite number of times. Suppose that the mill $\mathcal{M}$ does not pass through a point P from $\mathcal{A}$. We choose a straight line from the mill. Then P will be either on it's left or on it's right. By rotating this line, P will not be able to change from it's from left to right, because then by continuity, P would have to pass through the line, which is impossible by assumption. Thus the point P is either always on the left side of all the lines from our windmill, or on the right side. In the first case, we call $\mathcal{M}$ the mill a left-mill, and in the second, a right-mill. We now assume that each mill avoids some point in $\mathcal{A}$, so each is a left or right mill.

Claim:
a) A windmill can't be a left-mill and a right-mill at the same time.
b) Let X be a point on the perimeter of $\mathcal{S}$. Then every mill passes through X.

Proof. a) Suppose that P is to the left of the lines in $\mathcal{M}$ and Q is to the right of the lines in $\mathcal{M}$ (obviously $P \neq Q$). $\mathcal{M}$ will contain a line d parallel to PQ. then P and Q will be on the same side of the line d, but one of them should be on the left and one of them on the right. Contradiction.

b) For each line d (the direction), we denote : A, B first and second point of intersection of the straight line d with $\mathcal{S}$. When d rotates clockwise, A, B move on the perimeter of $\mathcal{S}$ continuously in a clockwise direction (or remain fixed if d rotates around them). Since the mill is periodic, so it returns to the initial position, both A and B must travel, in clockwise order, the entire perimeter of $\mathcal{S}$ which implies the conclusion. □

For a straight line ℓ with direction, we denote $d(\ell)$ the half-plane to the right of ℓ and $s(\ell)$ the half-plane to the left of ℓ.

Reparametrization: For a mill $\mathcal{M}$ generated by a line (with direction) we denote this "initial" line by $\mathcal{M}_0$. Then we denote $\mathcal{M}_t$ the line obtained when $\mathcal{M}_0$ rotates in total with the angle t (around several points, in a row), so :

$$\mathcal{M}_{t+2\pi} = \mathcal{M}_t \text{ and } \mathcal{M} = \cap_{t=0}^{\infty} \mathcal{M}_t$$

Now we can speak about the "reparametrization" of a mill: if $\gamma : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is an infinite continuous increasing function, we denote $\mathcal{M}'$ by $\mathcal{M}' = \mathcal{M}_{\gamma(t)}$. Obviously the set of points in $\mathcal{M}'$ is the same as the set of points in $\mathcal{M}$, in fact $\mathcal{M}'$ is the same $\mathcal{M}$ rotated with different "speed". This definition is important because of the following sentence:

Lemma 2.1. *Let X be a vertex on the convex hull. And, let Y, and Z be two points on the perimeter of $\mathcal{S}$, so that Y is between X and Z on the perimeter. In other words : $d(XY) \cap Int(\mathcal{S}) \subseteq d(XZ) \cap Int(\mathcal{S})$. Then the mill $\mathcal{M}_1$ generated by XY (rotated around X) will be to the "right" of the mill $\mathcal{M}_2$ generated by XZ (rotated around X) in the following sense: there is a reparametrization $\mathcal{M'}_1$ of $\mathcal{M}_1$ with the property that $d((\mathcal{M'}_1)_t) \cap Int(\mathcal{S}) \subseteq d((\mathcal{M'}_2)_t) \cap Int(\mathcal{S})$.*

Proof. We build the reparametrization in the following way : we already know that the first intersection of the line (with direction) with $\mathcal{S}$ moves continuously in a clockwise direction around the perimeter of $\mathcal{S}$. Then we can uniquely define reparametrization of $\mathcal{M'}_1$, so that the first intersection of the straight line $\mathcal{M'}_1(t)$ with the perimeter of $\mathcal{S}$ coincides with the intersection of the line $\mathcal{M}_2(t)$ with the perimeter of $\mathcal{S}$. This reparametrization is unique unless this intersection is a vertex P of $\mathcal{S}$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ both rotate around this point. On this time interval, we make the fallowing reparametrization : $\mathcal{M}_1$ moves at a constant speed so that $\mathcal{M'}_1$ and $\mathcal{M}_2$ meet another point at the same time, i.e. γ to be linear in interval.

$$\mathcal{M'}_1(t) \cap Int(P) = (U_t V_t), \mathcal{M'}_2(t) \cap Int(P) = (U_t W_t).$$

It is enough to show that V_t is between U_t and V_t on the perimeter of $\mathcal{S}$, on the clockwise direction. This is true for t very small because $U_0 = X$, $V_0 = Y$, $W_0 = Z$. Next we use continuity, the limit case would be when $V_t = W_t$. In this case, the mills will be either the same (in which case the problem is obvious) or we will be in the situation when $U_t V_t = AB$, where A, B are vertices, and $\mathcal{M}_1$ rotated around A until it changed into B, and $\mathcal{M}_2$ rotated around B until it changed into A. However, it is easy to see that in this case V_t does not pass over W_t but moves backwards towards U_t: for U to the right of AB, the order of the intersection of UA with the perimeter of $\mathcal{S}$ and the intersection of UB with the perimeter of $\mathcal{S}$ is the same as order intersection of UB perimeter of $\mathcal{S}$ and the intersection and UA with the perimeter of $\mathcal{S}$. If U is to the left of the line, for $U_t = U$, these intersections describe exactly V_t and W_t. The lemma is proved. □

This lemma is important in the following corollary, which results immediately from it, without the need for demonstration.

Corollary 2.1. *Let X be a vertex of $\mathcal{S}$ and Y, Z on the perimeter of $\mathcal{S}$ so that Y is between X and Z on the perimeter of $\mathcal{S}$, in clockwise order. Let $\mathcal{M}_1$ be the mill generated by XY (around X) and $\mathcal{M}_2$ the mill generated by XZ. If $\mathcal{M}_1$ is a left-mill then $\mathcal{M}_2$ is a left-mill, moreover if $\mathcal{M}_1$ does not pass through a certain vertex U then $\mathcal{M}_2$ does not pass through U either. Conversely, if $\mathcal{M}_2$ is a right-mill then $\mathcal{M}_1$ is a right-mill, and if $\mathcal{M}_2$ does not pass through V then $\mathcal{M}_1$ does not pass through V either.*

Note: The concept of reparametrization was introduced to facilitate the demonstration of this corollary, which is essential in solving the problem.

The solution to the problem: With the help of this corollary, we can now finish the problem.

Let be X a vertex of $\mathcal{S}$. Draw all the lines between X and other vertices of the set $\mathcal{A}$. They divide the perimeter of $\mathcal{S}$ into several "arcs" (which are actually segments). We know that each arc determines a mill: if Y, Y' are on the same arc then XY and XY' generate the same mill because XY will eventually rotate to become XY' or vice versa. We order these arcs in clockwise order. According to the corollary, the first few arcs will give right-mills, and the last will give left mills. Let's take the last arc that gives us a right-mill. Then it will not go through a vertex P of $\mathcal{A}$. According to the corollary, the other mills on the right will be "further to the right" (after parameterization) of it. So they will have P on the right too. Similarly, all left-mills will not pass through any point Q. Thus we established that left-mills passing through X are not passing through P and the right-mills are not passing through Q. We now consider the mill generated by PQ. It will pass through X because X is on convex hull, but it can't be a left-mill or a right-mill as it passes through P and Q. Contradiction.

3. Conclusion

The peer learning process contributes to student's personal development and self-affirmation. In this way, tolerance towards different ways of thinking are being educated, communication skills and competitive spirit are developed. These strategies are effective, if the ways of grouping individuals are carefully selected, to ensure a positive interdependence between students, they stimulate involvement in the task, leading to an interactive learning process. In collaborative learning, students work together to solve the same problem, to study solutions to complicated problems, to research authentic innovations. Serious problems require collaboration between students.

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