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# The Bernstein operators on any finite interval revisited

DAN BĂRBOSU

ABSTRACT. One studies simultaneous approximation properties of fundamental Bernstein polynomials involved in the construction of the mentioned operators.

## 1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The classical Bernstein operator associated to a real valued function $f : [0,1] \to \mathbb{R}$ is given by

$$B_m g(x) = \sum_{k=0}^{m} p_{mk}(x) f\left(\frac{k}{m}\right) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right), \tag{1.1}$$

for any $x \in [0,1]$ and any $m \in \mathbb{N}$. It was introduced by S. N. Bernstein [6].
Note that in (1.1) the polynomials $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, $k \in \{0,1,...,m\}$ are called *Bernstein fundamental polynomials*. During the years, the operator (1.1) was intensively studied, many of its approximation properties being by now well known. In the same time, many generalizations of operator (1.1) were also considered. It is important to see which of these are real generalizations.

Suppose $a$ and $b$ are real numbers such that $a < b$. The Bernstein operator associated with any function $f : [a,b] \to \mathbb{R}$ is defined by

$$\begin{aligned} B_m^* f(x) &= \sum_{k=0}^{m} p_{m,k}^*(x) f\left(a + k - \frac{b-a}{m}\right) \\ &= \frac{1}{(b-a)^m} \sum_{k=0}^{m} \binom{m}{k} (x-a)^k (b-x)^{m-k} f\left(a + k\frac{b-a}{m}\right) \end{aligned} \tag{1.2}$$

for any $x \in [0,1]$ and any $m \in \mathbb{N}$. In (1.2) the polynomials

$$p_{m,k}^* = \frac{1}{(b-a)^m} \binom{m}{k} (x-a)^k (b-x)^{m-k}$$

are the Bernstein fundamental polynomials on $[a,b]$. The construction of operator (1.2) was described in [3] and [10]. It can be find also in the papers [1], [4], [5], [9], [11].

In the recent paper [10] were investigated qualitative and quantitative aspects regarding the operator (1.2). More exactly, were studied uniform convergence, order of approximation and asymptotic behavior It was proved that these properties are transfered from the classical operator $B_m$ to the operator $B_m^*$.

The focus of the present paper is to study properties of simultaneous approximation for $B_m^*$.

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Section 2 proves that the simultaneous approximation properties of $B_m$ remain valid for the operator $B_m^*$. Section 3 contains some applications of the results from Section 2. Here we discuss also about a Bernstein operator associated with a function $f : [0, 1[\to \mathbb{R}$, which appears in [7], [8], [4], [13].

## 2. Main results

In order to study simultaneous approximation properties of polynomials $B_m^* f$, we need the following

**Lemma 2.1.** *The fundamental Bernstein polynomials $p^*_{m,k}$ satisfy the recurrence*

$$\left(p^*_{m,k}(x)\right)' = \frac{m}{b-a}\left(p^*_{m-1,k-1}(x) - p^*_{m-1,k}(x)\right), \tag{2.3}$$

*for any $x \in ]a, b[$, where $p_{0,0}(x) := 1$, $p_{s,s-1}(x) = 0$, $s \in \mathbb{N}$.*

*Proof.* Recall that $p^*_{m,k}(x) = \frac{1}{(b-a)^m}\binom{m}{k}(x-a)^k(b-x)^{m-k}$. Then

$$\begin{aligned}
(p^*_{m,k}(x))' &= \frac{1}{(b-a)^m}\binom{m}{k}\left\{k(x-a)^{k-1}(b-x)^{m-k} - (m-k)(x-a)^k(b-x)^{m-k-1}\right\}\\
&= \frac{m}{b-a}\left\{\frac{1}{(b-a)^{m-1}}\binom{m-1}{k-1}(x-a)^{k-1}(b-x)^{m-k}\right.\\
&\quad \left. - \frac{1}{(b-a)^{m-1}}\binom{m-1}{k}(x-a)^k(b-x)^{m-k-1}\right\}\\
&= \frac{m}{b-a}\left(p^*_{m-1,k-1}(x) - p^*_{m,k-1}(x)\right).
\end{aligned}$$

□

We are now ready to compute the first order derivative of the polynomial $B_m^* f$.

**Lemma 2.2.** *The following identity*

$$(B^*_{m,k} f)'(x) = \frac{m}{b-a}\sum_{k=0}^{m-1} p^*_{m-1,k}(x)\,\Delta^1_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) \tag{2.4}$$

*holds, where*

$$\Delta^1_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) = f\left(a + (k+1)\frac{b-a}{m}\right) - f\left(a + k\frac{b-a}{m}\right)$$

*denotes the first order finite difference of $f$ with the starting point $a + k\frac{b-a}{m}$ and step $\frac{b-a}{m}$.*

*Proof.* By applying Lemma 2.1 we get

$$(B_m^* f)'(x) = \sum_{k=0}^{m}\left(p^*_{m,k}(x)\right)' f\left(a + k\frac{b-a}{m}\right)$$

$$= \frac{m}{b-a}\sum_{k=0}^{m} p^*_{m-1,k-1}(x) f\left(a+k\frac{b-a}{m}\right) - \frac{m}{b-a}\sum_{k=0}^{m} p^*_{m-1,k}(x) f\left(a+k\frac{b-a}{m}\right).$$

If we denote

$$S_1 = \sum_{k=0}^{m} p^*_{m-1,k-1}(x) f\left(a + k\frac{b-a}{m}\right);\, S_2 = \sum_{k=0}^{m} p^*_{m-1,k}(x) f\left(a + k\frac{b-a}{m}\right),$$

then, since $p^*_{m-1,-1}(x)=0$, the sum $S_1$ can be written as

$$S_1=p^*_{m-1,-1}(x)f(a)+\sum_{k=1}^{m}p^*_{m-1,k-1}(x)f\left(a+k\frac{b-a}{m}\right)=\sum_{k=1}^{m}p^*_{m-1,k-1}(x)f\left(a+k\frac{b-a}{m}\right),$$

and by denoting $k-1:=k$, it follows that

$$S_1=\sum_{k=0}^{m-1}p^*_{m-1,k}(x)f\left(a+(k+1)\frac{b-a}{m}\right).$$

Since $p^*_{m-1,m}(x)=0$, for the sum $S_2$ we have

$$S_2=\sum_{k=0}^{m-1}p^*_{m-1,k}(x)f\left(a+k\frac{b-a}{m}\right)+p^*_{m-1,m}(x)f(b)=\sum_{k=0}^{m-1}p^*_{m-1,k}(x)f\left(a+k\frac{b-a}{m}\right),$$

and going back to the expression of $(B^*_m f)'(x)$, we get

$$(B^*_m f)'(x)=\frac{m}{b-a}(S_1-S_2)=\frac{m}{b-a}\sum_{k=0}^{m-1}p^*_{m-1,k}(x)\,\Delta^1_{\frac{b-a}{m}}f\left(a+k\frac{b-a}{m}\right).$$

□

**Remark 2.1.** Denoting the first order divided difference of $f$ on the knots $a+k\dfrac{b-a}{m}$, $a+(k+1)\dfrac{b-a}{m}$ by $\left[a+k\dfrac{b-a}{m},a+(k+1)\dfrac{b-a}{m};f\right]$ (see [12]), one obtains that (2.4) can be expressed in the form

$$(B^*_m f)'(x)=\sum_{k=0}^{m}p^*_{m-1,k}(x)\left[a+k\frac{b-a}{m},a+(k+1)\frac{b-a}{m};f\right]. \qquad (2.5)$$

Applying Lemma 2.2 and Lemma 2.1 we can prove

**Theorem 2.1.** *Let $(B^*_m f)^{(j)}$ be the $j$-th order derivative of $B^*_m f$, where $j\in\mathbb{N}$ and $j\le m$. For any $x\in]a,b[$, the following identity*

$$(B^*_m f)^{(j)}(x)=\frac{m(m-1)\dots(m-j+1)}{(b-a)^j}\sum_{k=0}^{m-j}p^*_{m-j,k}(x)\,\Delta^j_{\frac{b-a}{m}}f\left(a+k\frac{b-a}{m}\right), \qquad (2.6)$$

*holds, where $\Delta^j_{\frac{b-a}{m}}$ denotes the $j$-th order finite difference of $f$ with the step $\dfrac{b-a}{m}$.*

*Proof.* We proceed by mathematical induction with respect to $j$. For $j=1$, (2.6) holds by virtue of Lemma 2.2. Suppose

$$(B^*_m f)^{(j-1)}(x)=\frac{m(m-1)\dots(m-j+2)}{(b-a)^{j-1}}\sum_{k=0}^{m-j+1}p^*_{m-j+1,k}(x)\Delta^{j-1}_{\frac{b-a}{m}}f\left(a+k\frac{b-a}{m}\right).$$

Then

$$(B^*_m f)^{(j)}(x)=\frac{m(m-1)\dots(m-j+2)}{(b-a)^{j-1}}\sum_{k=0}^{m-j+1}\left(p^*_{m-j+1}(x)\right)'\Delta^{j-1}_{\frac{b-a}{m}}f\left(a+k\frac{b-a}{m}\right).$$

Now, by applying Lemma 2.1 one obtains: $(B^*_m f)^{(j)}(x)=$

$$=\frac{m(m-1)\dots(m-j+1)}{(b-a)^j}\cdot\sum_{k=0}^{m-j}\left\{p^*_{m-j,k-1}(x)-p^*_{m-j,k}(x)\right\}\Delta^{j-1}_{\frac{b-a}{m}}f\left(a+k\frac{b-a}{m}\right).$$

By denoting

$$S_1 = \sum_{k=0}^{m-j+1} p^*_{m-j,k-1}(x)\Delta^{j-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right), S_2 = \sum_{k=0}^{m-j+1} p^*_{m-j,k}(x)\Delta^{j-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right),$$

it follows that

$$(B^*_m f)^{(j)}(x) = \frac{m(m-1)\dots(m-j+1)}{(b-a)^j}(S_1 - S_2). \tag{2.7}$$

Using the fact that $p^*_{m-j+1,-1}(x) = 0$, the sum $S_1$ we can written under the form

$$\begin{aligned} S_1 &= p^*_{m-j,-1}(x)\Delta^{j-1}_{\frac{b-a}{m}} f(a) + \sum_{k=1}^{m-j+1} p^*_{m-j,k-1}(x)\Delta^{j-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) \\ &= \sum_{k=1}^{m-j+1} p^*_{m-j,k-1}(x)\Delta^{j-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right). \end{aligned}$$

By making the change $k - 1 := k$, it follows that $S_1$ can be expressed as

$$S_1 = \sum_{k=0}^{m-j} p^*_{m-j,k}(x)\, \Delta^{j-1}_{\frac{b-a}{m}}\, f\left(a + (k+1)\frac{b-a}{m}\right).$$

Since $p^*_{m-j,m-j+1}(x) = 0$, for the sum $S_2$ we have

$$\begin{aligned} S_2 &= \sum_{k=1}^{m-j} p^*_{m-j,k}(x)\Delta^{j-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) + p^*_{m-j,m-j+1}(x) \\ &= \sum_{k=1}^{m-j} p^*_{m-j,k}(x)\, \Delta^{j-1}_{\frac{b-a}{m}}\, f\left(a + k\frac{b-a}{m}\right), \end{aligned}$$

Going back to (2.7), we get:

$$\begin{aligned} (B^*_m f)^{(j)}(x) &= \frac{m(m-1)\dots(m-j+1)}{(b-a)^j} \sum_{k=0}^{m-j} p^*_{m-j,k}(x) \Big\{ \Delta^{j-1}_{\frac{b-a}{m}}\left(a + (k+1)\frac{b-a}{m}\right) \\ &\quad - \Delta^{k-1}_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right)\Big\} \\ &= \frac{m(m-1)\dots(m-j+1)}{(b-a)^j} \sum_{k=0}^{m-j} p^*_{m-j,k}(x)\, \Delta^{j}_{\frac{b-a}{m}}\, f\left(a + k\frac{b-a}{m}\right). \end{aligned}$$

□

**Corollary 2.1.** *The polynomial $B^*_m f$ can be represented under the form*

$$B^*_m f(x) = \sum_{j=0}^{m} \binom{m}{j} \left(\frac{x-a}{b-a}\right)^j \Delta^j_{\frac{b-a}{m}} f(a). \tag{2.8}$$

*Proof.* Applying the Taylor's formula to the $m$-th degree polynomial $B^*_m f$, we have

$$B^*_m f(x) = \sum_{j=0}^{m} \frac{(B^*_m f)^{(j)}(a)}{j!}(x-a)^j.$$

By virtue of Theorem 2.1

$$(B^*_m f)^{(j)}(a) = \frac{m(m-1)\dots(m-j+1)}{(b-a)^j} \sum_{k=0}^{m-j} p^*_{m-j,k}(a)\Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right).$$

But $p^*_{m-j,0}(a) = 1$ and $p^*_{m-j,k}(a) = 0$ for any $k \in \{1, 2, \dots, m\}$ which imply

$$(B^*_m f)^{(j)}(a) = \frac{m(m-1)\dots(m-j+1)}{(b-a)^j}\, \Delta^j_{\frac{b-a}{m}} f(a),$$

and hence

$$B^*_m f(x) = \sum_{j=0}^{m} \binom{m}{j} \left(\frac{x-a}{b-a}\right)^j \Delta^j_{\frac{b-a}{m}} f(a).$$

□

**Remark 2.2.** $(i)$ In [10] it was proved that the sequence $\{B^*_m f\}_{m\in\mathbb{N}}$ converges to $f$ uniformly on $[a, b]$, for any $f \in C[a, b]$.

$(ii)$ We shall prove that the sequence $\{(B^k_m f)^{(j)}\}_{m\in\mathbb{N}}$ converges to $f^{(j)}$ uniformly on $[a, b]$, for any $f \in C^j[a, b]$ where $j \in \mathbb{N}$, $j \le m$.

$(iii)$ We will use the Landau's symbol $o$, which we recall here

$$f(x) = o(g(x))(x \to x_0) \Leftrightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

Using the symbol $o$, the equality $\lim\limits_{m\to\infty} u\left(\dfrac{1}{m}\right)= 0$ can be written in the form $u\left(\dfrac{1}{m}\right)=o(1)$.

**Lemma 2.3.** *For any $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, $j \le m$, $k \le m-j$, the following equality*

$$\frac{m(m-1)\dots(m-j+1)}{(b-a)^j}\, \Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) = (1 + o(1)) f^{(j)}(\xi_k) \tag{2.9}$$

*holds, where $\xi_k \in \left]a + k\frac{b-a}{m}, a + (k+1)\dfrac{b-a}{m}\right[$, $k \in \{0, 1, \dots, m-1\}$ and $f \in C^j[a, b]$.*

*Proof.* Using Landau's symbol $o$, we can write

$$\begin{aligned}
&\frac{m(m-1)\dots(m-j+1)}{(b-a)^j}\, \Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right)\\
&= \frac{1}{(b-a)^j}\, m^j \left(1 + \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\dots\left(1 - \frac{j-1}{m}\right) \Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right)\\
&= \frac{1}{(b-a)^j} m^j (1 + o(1)) \Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right).
\end{aligned} \tag{2.10}$$

Using the relationship between finite and divided differences [2], we have

$$\frac{1}{(b-a)^j}\, m^j \Delta^j_{\frac{b-a}{m}} f\left(a + k\frac{b-a}{m}\right) = j!\left[a + k\frac{b-a}{m}, \dots, a + (k+j)\frac{b-a}{m}; f\right]. \tag{2.11}$$

Because $f \in C^j[a, b]$, the mean value theorem for divided differences [12] leads to

$$\left[a + k\frac{b-a}{m}, \dots, a + (k+j)\frac{b-a}{m}; f\right] = \frac{f^{(j)}(\xi_k)}{j!}, \tag{2.12}$$

where $\xi_k \in \left]a + k\frac{b-a}{m}, a + (k+1)\frac{b-a}{m}\right[$, $k \in \{0, 1, \dots, m-1\}$.

From (2.10), (2.11) and (2.12) we get (2.9). □

**Theorem 2.2.** *If $j \in \mathbb{N}_0$, $j \le m$ and $f \in C^j[a, b]$, the sequence $\{(B^*_m f)^{(j)}\}_{m\in\mathbb{N}}$, converges to $f$, uniformly on $[a, b]$.*

*Proof.* For $j = 0$ the assertion was proved in [10]. Suppose $j \in \mathbb{N}$. From Theorem 2.1 and Lemma 2.3 it follows

$$(B_m^* f)^{(j)}(x) = \sum_{k=0}^{m-j} p_{m-j,k}^*(x)(1 + o(1))f^{(j)}(\xi_k) \tag{2.13}$$

But

$$f^{(j)}(\xi_k) = \left(f^{(j)}(\xi_k) - f^{(j)}\left(a + k\frac{b-a}{m}\right)\right) + f^{(j)}\left(a + k\frac{b-a}{m}\right)$$

and because $\left|\xi_k - \left(a + k\frac{b-a}{m}\right)\right| < \frac{b-a}{m}$ and $f \in C^{(j)}[a,b]$, we get

$$f^{(j)}(\xi_k) - f^{(j)}\left(a + k\frac{b-a}{m}\right) = o(1).$$

It follows that we can write

$$f^{(j)}(\xi_k) = o(1) + f^{(j)}\left(a + k\frac{b-a}{m}\right). \tag{2.14}$$

Using (2.13) and (2.14), we obtain

$$\begin{aligned} (B_m^* f)^{(j)}(x) &= \sum_{k=0}^{m-j} p_{m-j,k}^*(x)(1 + (1))\left((1) + f^{(j)}\left(a + k\frac{b-a}{m}\right)\right) \\ &= \sum_{k=0}^{m-j} p_{m-j,k}^*(x) f\left(a + k\frac{b-a}{m}\right) + o(1) = B_{m-j}^* f^{(j)}(x) + o(1). \end{aligned} \tag{2.15}$$

The sequence $\{B_{m-j}^* g\}_{m\in\mathbb{N}}$ converges to $g$ uniformly on $[a,b]$ for any $g \in C[a,b]$.

Choosing $g = f^{(j)} \in C[a,b]$ from (2.15) one obtains that

$$\lim_{m\to\infty} (B_m^* f)^{(j)}(x) = f^{(j)}(x),$$

uniformly on $[a,b]$. □

## 3. Applications

Below we present some particular cases of the Bernstein operator $B_m^*$ obtained from (1.2) for specific values of $a, b$.

**Case 1**. For $a := 0$, $b := 1$ one obtains the Bernstein operator [11].
As consequences of the results from Section 2 one recover well known properties of Bernstein operator (1.1). The most important of them are the following corollaries.

**Corollary 3.2.** *If $j \in \mathbb{N}$, $j \le m$ the $j$-th order derivative of the Bernstein polynomial is given by*

$$(B_m f)^{(j)}(x) = m(m-1)\dots(m-j+1)\sum_{k=0}^{m-j} p_{m-j,k}(x)\Delta_{\frac{1}{m}}^{j} f\left(\frac{k}{m}\right). \tag{3.16}$$

**Corollary 3.3.** *The Bernstein polynomial* (3.16) *can be represented under the form*

$$B_m f(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta_{\frac{1}{m}}^{j} f(0), \text{ for any } x \in [0,1]. \tag{3.17}$$

**Corollary 3.4.** *If $j \in \mathbb{N}_0)$, $j \le m$ and $f \in C^j[0,1]$, the sequence $\{(B_m f)^{(j)}\}_{m\in\mathbb{N}}$ converges to $f^{(j)}$ uniformly on $[0,1]$, for any $f \in C^j[0,1]$.*

**Case 2**. For $a := 0$, $b := \dfrac{m}{m+1}$ one obtains the Bernstein operator associated to any function $f : \left[0, \dfrac{m}{m+1}\right] \to \mathbb{R}$ given by

$$B^*_m f(x) = \left(\frac{m+1}{m}\right)^m \sum_{k=0}^{m} \binom{m}{k} x^k \left(\frac{m}{m+1} - x\right)^{m-k} f\left(\frac{k}{m+1}\right) \tag{3.18}$$

for any $x \in \left[0, \dfrac{m}{m+1}\right]$ and any $m \in \mathbb{N}$. Relation (3.18) appears in the papers [7], [8] with a small modification, namely the knots $\dfrac{k}{m+1}$ are replaced with the knots $\dfrac{k}{m}$. The authors of the cited papers call modified relation (3.18) "new Bernstein operator". Bărbosu and Deo in [4] remarked that the definition of "new Bernstein operator" is wrong and clarified that relation (3.18) represents definition of the Bernstein operator. Recently Siddiqui, Agrawal and Gupta [13] used and modified the definition of "new Bernstein type operator", in the sense of (3.18) and claimed that they have obtained "modified new Bernstein operator". In fact, they recovered the Bernstein operator $B^*_m$. Coming back to the operators (3.18) introduced in [4], as a consequence of results from Section 2, we can state

**Corollary 3.5.** *If $j \in \mathbb{N}$, $j \le m$ the $j$-th order derivative of operator* (3.18) *is expressed by*

$$(B^*_m f)^{(j)}(x) = \left(\frac{m+1}{m}\right)^j m(m-1)\dots(m-j+1) \sum_{k=0}^{m-j} p^*_{m-j,k}(x) \Delta^j_{\frac{1}{m+1}} f\left(\frac{k}{m+1}\right), \tag{3.19}$$

*where the fundamental Bernstein polynomials are given by*

$$p^*_{m,k}(x) = \left(\frac{m+1}{m}\right)^m \binom{m}{k} x^k \left(\frac{m}{m+1} - x\right)^{m-k}. \tag{3.20}$$

**Corollary 3.6.** *The polynomial* (3.18) *can be expressed under the form*

$$B^*_m f(x) = \sum_{j=0}^{m} \binom{m}{j} \left(\frac{m+1}{m}\right)^j \Delta^j_{\frac{1}{m+1}} f(0). \tag{3.21}$$

Regarding the approximation properties of the operator (3.18), first we prove

**Lemma 3.4.** *The sequence of polynomials* (3.18) *converges to $f$, uniformly on any compact $[0, a] \subset [0, 1[$.*

*Proof.* Applying the results from [10] (or by direct computation), we get

$$B^*_m((t-x)^2; x) = \frac{x(m-(m+1)x)}{m(m+1)}, \ \forall\, x \in \left[0, \frac{m}{m+1}\right].$$

For each $a > 0$, it follows $\lim\limits_{m\to\infty} B^*_m((t-x)^2; x) = 0$, uniformly on $[0, a]$.

Consequently, applying the Bohman-Korovkin theorem [2], we get that $\{B^*_m f\}_{m\in\mathbb{N}}$ converges to $f$, uniformly on $[0, a]$, for any $f \in C[0, 1[$. □

The properties of simultaneous approximation are described in the following

**Theorem 3.3.** *Let $j$ be a non-negative integer and denote by $(B^*_m f)^{(j)}$ the $j$-th order derivative of the polynomial* (3.18). *The sequence $\{(B^*_m f)^{(j)}\}_{m\in\mathbb{N})}$ converges to $f^{(j)}$, uniformly on $[0, a] \subset [0, 1[$, for any $C^j\left[0, \dfrac{n}{n+1}\right]$.*

*Proof.* For $j = 0$ the assertion was proved in Lemma 3.1. Suppose $j \in \mathbb{N}$. Proceeding as in Lemma 2.3, we can write

$$m(m-1)\dots(m-j+1)\Delta^j_{\frac{1}{m+1}} f\left(\frac{k}{m+1}\right) = (1+o(1))f^{(j)}(\xi_k) \tag{3.22}$$

where $\xi_k \in \left]\frac{k}{m+1}, \frac{k+1}{m+1}\right[$, $k \in \{0, 1, \dots, m-1\}$.
Using (3.22) and Corollary 3.4, it follows

$$(B_m^* f)^{(j)}(x) = \left(\frac{m+1}{m}\right)^m \sum_{k=0}^{m-j} p^*_{m-j,k}(x)(1+o(1))f^{(j)}(\xi_k). \tag{3.23}$$

From (3.23), as in the proof of Theorem 2.2, one arrives to

$$(B_m^* f)^{(j)}(x) = \left(\frac{m+1}{m}\right)^j B^*_{m-j} f^{(j)}(x) + o(1) \tag{3.24}$$

The sequence $\{B^*_{m-j} g\}_{m\in\mathbb{N}}$ converges to $g$, uniformly on $[0, a] \subset [0, 1[$, for any $g \in C\left[0, \frac{m}{m+1}\right]$. Choosing $g = f^{(j)} \in C\left[0, \frac{m}{m+1}\right]$, from (3.24) one obtains that $\{(B_m^* f)^{(j)}\}_{m\in\mathbb{N}}$ converges to $f^{(j)}$, for any $f \in C^j\left[0, \frac{m}{m+1}\right]$. □

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
NORTH UNIVERSITY CENTRE AT BAIA MARE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
*E-mail address*: barbosudan@yahoo.com