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Radius of exponential convexity of certain subclass of analytic functions

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ABSTRACT. Let $\mathcal{S}$ be the class of analytic normalized univalent functions defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and f, g be two functions in $\mathcal{S}$ satisfying $Re\left(\frac{zf'(z)e^{\alpha f(z)}}{g(z)}\right) > 0$ and $\left|\frac{zf'(z)e^{\alpha f(z)}}{g(z)} - 1\right| < 1$, $\alpha \in \mathbb{C} \setminus \{0\}$. We determine the radius of exponential convexity of $f \in \mathcal{S}$ whenever g satisfies (i) $Re\frac{g(z)}{z} > 0$ (ii) $Re\frac{g(z)}{z} > 1/2$.

1. INTRODUCTION

Let Δ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$ and $\mathcal{A}$ the class of normalized analytic functions on Δ. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions univalent in Δ. The class of starlike functions and convex functions are two standard subclasses of $\mathcal{S}$.

For two functions $f(z) = z + a_2z^2 + a_3z^3 + ...$ and $g(z) = z + b_2z^2 + b_3z^3 + ...$, in $\mathcal{A}$, the sharp radius of convexity of functions satisfying

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > 0$$

whenever g satisfies various geometric conditions like univalence, starlikeness, convexity, starlikeness of order α and convexity of order $\alpha, (0 \leq \alpha < 1)$ were obtained by Ratti [4]. In [5] he obtained similar results when $f, g \in \mathcal{A}$ satisfying

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$$

where again g satisfies the conditions above. Motivated by the above results, we obtain the radius of $\alpha-$ exponential convexity for functions $f \in \mathcal{S}$ whenever g satisfies certain conditions similar to above. We recall the definition of exponential convex functions.

Definition 1.1. Let $\alpha \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{S}$ is said to belong to the class $E(\alpha)$ of $\alpha-$ exponentially convex functions if $F(\Delta)$ is a convex set where $F(z) = e^{\alpha f(z)}$.

This class was introduced and studied by Arango $etal.$ [1]

Remark 1.1. For $f \in E(\alpha)$ and $x \in \bar{\Delta} \setminus \{0\}$, the function $f(xz)/x$ is not necessarily in $E(\alpha)$.

The following theorem gives an analytic characterization of exponentially convex univalent functions.

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Theorem 1.1. [1] *Let $\alpha \in \mathbb{C} \setminus \{0\}$. A function f analytic in Δ with $f(0) = 0, f^{'}(0) = 1$ is in $E(\alpha)$ if and only if*

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)} + \alpha z f^{'}(z)\right) > 0, z \in \Delta.$$

We now state few lemmas which are useful to prove our main results.

Lemma 1.1. [3] *If $h(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in $|z| < 1$ and $\Re(h(z)) > 0$ for $|z| < 1$, then*

(i) $\left|\frac{h^{'}(z)}{h(z)}\right| \leq \frac{2}{1 - |z|^2}$.

(ii) $\Re\left\{\frac{zh^{'}(z)}{h(z})\right\} \geq \frac{-2|z|}{1 - |z|^2}$.

Lemma 1.2. [4] *The function $g(z)$ is analytic for $|z| < 1$ and satisfies $g(0) = 1$ and $\Re(g(z)) > \alpha$ $(0 \leq \alpha < 1)$ for $|z| < 1$ if and only if $g(z) = \frac{1 + (2\alpha - 1)z\phi(z)}{1 + z\phi(z)}$, where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq 1$ for $|z| < 1$.*

Lemma 1.3. [4] *If $\phi(z)$ is analytic for $|z| < 1$ and $|\phi(z)| \leq 1$ for $|z| < 1$, then*

(i) $|\phi^{'}(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}$; *(ii)* $\left|\frac{z\phi^{'}(z) + \phi(z)}{1 + z\phi(z)}\right| \leq \frac{1}{1 - |z|}$.

Lemma 1.4. [4] *If $h(z) = 1 + c_1 z + c_2 z^2 + ... + ...$ is analytic in $|z| < 1$ and $\Re(h(z)) > \alpha$ $(0 \leq \alpha < 1)$ for $|z| < 1$, then $\Re(h(z)) \geq \frac{1 - (2\alpha - 1)|z|}{1 + |z|}$.*

Lemma 1.5. [2] *If $h(z) = 1 + c_1 z + c_2 z^2 + ... + ...$ is analytic in $|z| < 1$ and $Reh(z) > 0$ for $|z| < 1$, then $\Re(h(z)) \geq \frac{1 - |z|}{1 + |z|}$.*

Remark 1.2. Functions of the form $h(z) = 1 + c_1 z + ...$ defined on the unit disk satisfying $\Re(h(z)) > 0$ are called positive functions.

2. Main results

Theorem 2.2. *Let $f(z) = z + a_2 z^2 + ...$ and $g(z) = z + b_2 z^2 + ...$ be two functions in $\mathcal{S}$, satisfying $\Re\left(\frac{zf^{'}(z)e^{\alpha f(z)}}{g(z)}\right) > 0$. If g satisfies $\Re\left(\frac{g(z)}{z}\right) > 0, z \in \Delta$, then the radius of $\alpha-$exponential convexity of f is 0.36*

Proof. If f satisfies $\Re\left(\frac{zf^{'}(z)e^{\alpha f(z)}}{g(z)}\right) > 0$, then there is a positive function h such that

$$zf^{'}(z)e^{\alpha f(z)} = g(z)h(z).$$

Taking logarithmic derivatives on both-sides,

$$1 + \frac{zf^{''}(z)}{f^{'}(z)} + \alpha z f^{'}(z) = \frac{zg^{'}(z)}{g(z)} + \frac{zh^{'}(z)}{h(z)} \tag{2.1}$$

which implies

$$\Re\left(1 + \frac{zf^{''}(z)}{f^{'}(z)} + \alpha z f^{'}(z)\right) = \Re\left(\frac{zg^{'}(z)}{g(z)}\right) + Re\left(\frac{zh^{'}(z)}{h(z)}\right)$$

$$\geq \frac{1-2|z|-|z|^2}{1-|z|^2} - \frac{2|z|}{1-|z|^2} = \frac{1-4|z|-|z|^2}{1-|z|^2}.$$

If $\frac{1-4|z|-|z|^2}{1-|z|^2} > 0$, that is, if $|z| < \sqrt{5}-2 = 0.36$, then $\Re\left(1+\frac{zf''(z)}{f'(z)} + \alpha z f'(z)\right) > 0$. Thus, f is $\alpha-$ exponential convex in the disk $|z| < \sqrt{5}-2 = 0.36$ □

Theorem 2.3. *Let $f(z)=z+a_2z^2+\ldots$ and $g(z)=z+b_2z^2+\ldots$ be in $\mathcal{S}$, and $\Re\left(\frac{zf'(z)e^{\alpha f(z)}}{g(z)}\right) > 0$. If g satisfy $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$, $z \in \Delta$, then the radius of $\alpha-$ exponential convexity of f is* 0.28078

Proof. If f satisfies $\Re\left(\frac{zf'(z)e^{\alpha f(z)}}{g(z)}\right) > 0$, then there is a positive function h such that $zf'(z)e^{\alpha f(z)} = g(z)h(z)$. Taking logarithmic derivatives on both sides

$$1+\frac{zf''(z)}{f'(z)} + \alpha z f'(z) = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}$$

$$\Re\left(1+\frac{zf''(z)}{f'(z)} + \alpha z f'(z)\right) = \Re\left(\frac{zg'(z)}{g(z)}\right) + Re\left(\frac{zh'(z)}{h(z)}\right).$$

Since $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$ we have $\frac{g(z)}{z} = \frac{1}{1+z\phi(z)}$ for some analytic function ϕ such that $|\phi(z)| \leq 1$. Therefore,

$$\frac{zg'(z)}{g(z)} = 1 - \frac{z\phi(z)+z^2\phi'(z)}{1+z\phi(z)}$$

where $\left|\frac{z\phi(z)+z^2\phi'(z)}{1+z\phi(z)}\right| \leq \frac{|z|}{1-|z|}$.

$$\Re\left(1+\frac{zf''(z)}{f'(z)} + \alpha z f'(z)\right) \geq 1 - \frac{|z|}{1-|z|} - \frac{2|z|}{1-|z|^2} = \frac{1-3|z|-2|z|^2}{1-|z|^2}.$$

If $1-3|z|-2|z|^2 > 0$ that is if $|z| < 0.2878$ then $\Re\left(1+\frac{zf''(z)}{f'(z)} + \alpha z f'(z)\right) > 0$.

Thus f is $\alpha-$ exponential convex in $|z| \leq 0.28078$ □

Theorem 2.4. *Let $f(z)= z+a_2z^2+\ldots$ and $g(z)= z+b_2z^2+\ldots$ be in $\mathcal{S}$, and $\left|\frac{zf'(z)e^{\alpha f(z)}}{g(z)}-1\right| < 1$, where $\alpha \in \mathbb{C}\setminus\{0\}$ for $|z| < 1$. If $\Re\left(\frac{g(z)}{z}\right) > 0$ then f is of $\alpha-$ exponential convex in $|z| < \frac{1}{3}$.*

Proof. Let $h(z) = \frac{zf'(z)e^{\alpha f(z)}}{g(z)} - 1$. Then $h(z)$ is analytic in Δ with $h(0) = 0, |h(z)|<1$ for $|z|<1$. By Schwarz's lemma, $h(z)= z\phi(z)$, with $|\phi(z)|\leq 1$. Therefore,

$$\frac{zf'(z)e^{\alpha f(z)}}{g(z)} - 1 = z\phi(z) i.e$$

$zf'(z)e^{\alpha f(z)} = g(z)(1+z\phi(z))$. Taking logarithmic derivative on both sides

$$\frac{1}{z} + \frac{f''(z)}{f'(z)} + \alpha f'(z) = \frac{g'(z)}{g(z)} + \frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}$$

which gives

$$1+\frac{zf^{''}(z)}{f^{'}(z)}+z\alpha f^{'}(z)=\frac{zg^{'}(z)}{g(z)}+\frac{z(z\phi^{'}(z)+\phi(z))}{1+z\phi(z)}.$$

Therefore,

$$\Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}+z\alpha f^{'}(z)\right)=\Re\left(\frac{zg^{'}(z)}{g(z)}\right)+\Re\left(\frac{z(z\phi^{'}(z)+\phi(z))}{1+z\phi(z)}\right)$$
$$\geq\frac{1-2|z|-|z|^2}{1-|z|^2}-\frac{|z|}{1-|z|}=\frac{1-3|z|}{1-|z|^2}.$$

If $1-3|z|>0$ then $\Re\left(1+\dfrac{zf^{''}(z)}{f^{'}(z)}+z\alpha f^{'}(z)\right)>0.$

Hence f satisfying the given condition is $\alpha-$ exponential convex in the disk $|z|<\dfrac{1}{3}.$ □

Theorem 2.5. *Let* $f(z)=z+a_2z^2+\ldots$ *and* $g(z)=z+b_2z^2+\ldots$ *be in* $\mathcal{S}$, *and* $\left|\dfrac{zf^{'}(z)e^{\alpha f(z)}}{g(z)}-1\right|<1$ *where* $\alpha\in\mathbb{C}\setminus\{0\}$ *for* $|z|<1$. *If* $\Re\left(\dfrac{g(z)}{z}\right)>\dfrac{1}{2}$ *then* f *is of* $\alpha-$ *exponential convex in* $|z|<\dfrac{1}{3}$.

Proof. In lines similar to the proof of the previous Theorem

$$\Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}+z\alpha f^{'}(z)\right)\geq\Re\left(\frac{zg^{'}(z)}{g(z)}\right)-\frac{|z|}{1-|z|}\geq 1-\frac{|z|}{1-|z|}-\frac{|z|}{1-|z|}=\frac{1-3|z|}{1-|z|}.$$

If $1-3|z|>0$, that is, $|z|<\dfrac{1}{3}$, it is true that $\Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}+z\alpha f^{'}(z)\right)>0.$ □

References

[1] Arango, J. H., Mejia, D. and Ruscheweyh, S., *Exponentially Convex Univalent Functions*, Comp. Vari. Ellip. Eqns, **33** (1997), No. 1, 33–50

[2] Caratheodory, C., *Conformal Representations*, Cambridge University Press, Cambridge, 1952

[3] MacGregor, T. H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104** (1962), 532–537

[4] Ratti, J. S., *The Radius of Convexity of Certain Analytic Functions*, J. Pure and Appl. Math., **1** (1970), 30–36

[5] Ratti, J. S., *The Radius of Convexity of Certain Analytic Functions - II*, Int. J. Math and Math Sci., **3** (1980), 483–489

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