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Explicit algorithms for J-fixed points of some non linear mappings in certain Banach spaces

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ABSTRACT. Let E be a real normed linear space and E^* its dual. In a recent work, Chidume et al. [Chidume, C. E. and Idu, K. O., *Approximation of zeros of bounded maximal monotone mappings, solutions of hammerstein integral equations and convex minimizations problems*, Fixed Point Theory and Applications, **97** (2016)] introduced the new concepts of *J-fixed points* and *J-pseudocontractive mappings* and they shown that a mapping $A : E \to 2^{E^*}$ is monotone if and only if the map $T := (J - A) : E \to 2^{E^*}$ is *J-pseudocontractive*, where J is the normalized duality mapping of E. It is our purpose in this work to introduce an algorithm for approximating J-fixed points of *J-pseudocontractive* mappings. Our results are applied to approximate zeros of monotone mappings in certain Banach spaces. The results obtained here, extend and unify some recent results in this direction for the class of maximal monotone mappings in uniformly smooth and strictly convex real Banach spaces. Our proof is of independent interest.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. An operator $A : H \to H$ with domain $D(A)$ is called *monotone* if for every $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle_H \geq 0, \tag{1.1}$$

and it is called *strongly monotone* if there exists $k \in (0, 1)$ such that every $x, y \in D(A)$,

$$\langle x - y, Ax - Ay \rangle_H \geq k \|x - y\|_H^2. \tag{1.2}$$

Such operators have been studied extensively (see, e.g., Bruck Jr [7], Chidume [11], Martinet [17], Reich [19], Rockafellar [20]) because of their role in convex analysis, in certain partial differential equations, in nonlinear analysis and optimization theory.

The extension of the *monotonicity* definition to operators defined from a Banach space has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions. (see, e.g., Pascali and Sburian [18], p. 101, Rockafellar [20]).

The *first* extension involves mappings A from E to E^*. Here and in the sequel, $\langle \cdot, \cdot \rangle$ stands for the duality pairing between (a possible normed linear space) E and it dual E^*. Let E be a real normed space. A mapping $A : E \to E^*$ with domain $D(A)$ is called *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\big\langle x - y, Ax - Ay \big\rangle \geq 0, \tag{1.3}$$

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and it is called *strongly monotone* if there exists $k \in (0,1)$ such that for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq k\|x - y\|^2. \tag{1.4}$$

The *second* extension of the notion of monotonicity to real normed spaces involves mappings A from E *into itelf*. Let E be a real normed space. The map $J : E \to 2^{E^*}$ defined by:

$$Jx := \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|,\ \|x^*\| = \|x\|\right\}$$

is called the *normalized duality map* on E.

A mapping $A : E \to E$ with domain $D(A)$ is called *accretive* if for all $x, y \in D(A)$, the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \ \ \forall\, s > 0. \tag{1.5}$$

As a consequence of a result of Kato [15], it follows that A is accretive if and only if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.6}$$

Finally, A is called *strongly accretive* if there exists $k \in (0,1)$ such that for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|^2. \tag{1.7}$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *monotonicity* and *accretivity* coincide.

In several cases, solutions of $Au = 0$, coincide with the *equilibrium points* of some dynamical systems whenever the operator A is accretive (see e.g., [11], p.116). This, therefore has motivated many attempt construct algorithms for approximating zeros such operators.

Supposing that $A : E \to E$ is of accretive-type, Browder [6] defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator *pseudocontractive*. One can observe that zeros of A correspond to fixed points set of T; This observation has motivated the consideration of fixed point pseudocontractive mappings. For instance in [9], Chidume has proved in $L_p, 2 \leq p < \infty$ that the sequence $\{x_n\}$ defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n$ converges strongly to a fixed point of a Lipschitz and bounded strongly pseudocontractive mapping. As a corollary he proved that the algorithm $x_{n+1} = x_n - \lambda_n A x_n$ converges strongly to a zero of bounded and strongly accretive mapping. This result has been generalized and extended in various directions by numerous authors (see e.g., Censor and Reich [8], Chidume *et. al.* [12], Chidume and Djitte [13, 14] and references therein.

Unfortunately, the success achieved in using geometric properties developed from the mid 1980s to early 1990s in approximating zeros of *accretive-type mappings* has not carried over to approximating zeros of *monotone-type operators* in general Banach spaces. Part of the difficulties is due to the fact that in the case of a montone mapping $A : E \to E^*$, the mapping $I - A$ used in the accretive case not defined. This, has motivated some research efforts to introduce some analogeous concepts to pseudocontractive type mappings. In this direction, Shashad and Zegeye [22] introduced the notion of *mono-pseudocontractive* mappings. Then, they proved that for a closed, convex and nonempty subset C of a smooth, reflexive and strictly convex real Banach space E, a mapping $A : C \to E^*$ is monotone if and only if $T = J^{-1}(J - A) : E \to E$ is mono-pseudocontractive. Then they used this result to approximate fixed points of mono-pseudocontractive mappings.

Recently Chidume et al. [10] introduced the class of *J-pseudocontractive* mappings.

Let E be real normed linear space with dual space E^*. A mapping $T : E \to 2^{E^*}$ is called *J-pseudocontractive* if for every $x, y \in E$, the following inequality holds :

$$\langle \tau - \zeta, x - y\rangle \le \langle \eta - \mu, x - y\rangle \quad \text{for all} \quad \tau \in Tx, \zeta \in Ty, \eta \in Jx, \nu \in Jy.$$

In the same way as accretive mappings and pseudocontractive mappings, a connection is made between monotone mappings and J-pseudocontractive mappings. In fact, as a result of Chidume *et al.* [10], a multivalued mapping $A : E \to 2^{E^*}$ is monotone if and only if $T := (J - A) : E \to 2^{E^*}$ is J-pseudocontractive, where J is the normalized duality mapping of E. A point $x \in E$ is called a *J-fixed point* of T if there exists $\eta \in Jx$ such that $\eta \in Tx$. We denote by $F^J(T)$ the set of J-fixed points of T. In particular, if J is single valued, then $F^J(T) = \{x \in E : Jx \in Tx\}$. Finally, for $u \in E$, $0 \in Au$ if and only if u is a J-fixed point of T.

It is our purpose in this work to introduce an algorithm for approximating J-fixed points of *J-pseudocontractive* mappings. Our results are applied to approximate zeros for monotone mappings in certain Banach spaces. The results obtained in this work extend some recent results in this direction for the class of maximal monotone mappins defined in uniformly smooth and strictly convex real Banach spaces.

2. Preliminaries

Let E be a normed linear space. E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.8}$$

exist for each $x, y \in S_E$ (Here $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \le c\tau^q$, then E is said to be *q-uniformly smooth*.

A normed linear space E is said to be strictly convex if:

$$\|x\| = \|y\| = 1,\ x \ne y \ \Rightarrow\ \left\|\frac{x + y}{2}\right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x + y\| \ : \|x\| = \|y\| = 1,\ \|x - y\| \ge \epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \ge c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex.

Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth and } p - \text{uniformly convex} & \text{if } 2 \le p < \infty; \\ 2 - \text{uniformly convex and } p - \text{uniformly smooth} & \text{if } 1 < p < 2. \end{cases}$$

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive smooth and strictly convex Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E.

Let E be a smooth real Banach space with dual E^*. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$, was introduced by Alber (see e.g [2]) as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \text{ for } x, y \in E, \tag{2.9}$$

where J is the normalized duality mapping from E into 2^{E^*}.

The Lyapunov functional has been studied by Alber [2], Alber and Guerre-Delabriere [4], Kamimura and Takahashi [3], Reich [19] and a host of other authors. It follows from (2.9) that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{2.10}$$

Definition 2.1. Let C be a nonempty, closed and convex subset of a smooth real Banach E. The generalized projection operator is a mapping $\Pi_C : E \to C$ that assigns to each $x \in E$ the corresponding unique element $\hat{x} \in E$ such that $\phi(\hat{x}, x) = \inf\{\phi(y, x) : y \in C\}$. That is

$$\Pi_C x = \hat{x}; \quad \hat{x} : \phi(\hat{x}, x) = \inf\{\phi(y, x) : y \in C\}.$$

If $E = H$ is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. Therefore Π_C coincides with the metric projection operator in Hilbert spaces.

The functional ϕ and the generalized projection operator enjoy the following properties.

Lemma 2.1. [1] *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then for all $y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x).$$

Lemma 2.2. [1] *Let C be a convex subset of a real smooth Banach space E. Let $x \in C$, then $x_0 = \Pi_C x$ if and only if*

$$\langle y - x_0, Jx - Jx_0\rangle \leq 0, \ \forall\, y \in C.$$

Lemma 2.3 (Kamimura and Takahashi, [3]). *Let E be a smooth and uniformly convex real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Define the functional $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2.$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}x^*) \ \forall\, x \in E, \ x^* \in E^*.$$

Lemma 2.4 (Alber, [2]). *Let E be a reflexive strictly convex and smooth Banach space with E^* as it dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.11}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

In the sequel we shall use the next results.

Lemma 2.5 (Xu [23]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following inequality*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, n \geq n_0, \tag{2.12}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ are real sequences satisfying: $\sum \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, the sequence (a_n) converges to zero as $n \to \infty$.

Lemma 2.6. [16] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. There exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all sufficiently large numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_{k+1}} \quad \text{and} \quad a_k \leq a_{m_{k+1}}.$$

Precisely, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.7. [5] *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \;\forall\; y \in C.$$

Lemma 2.8. [22] *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping. For $r > 0$ define the mapping $F_r : E \to C$ as follows:*

$$F_r x := \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \;\forall\; y \in C\}$$

for all $x \in E$. Then the following hold:
(1) F_r is single-valued;
(2) $F(F_r)$ is closed and convex;
(3) $\phi(p, F_r x) + \phi(F_r x, x) \leq \phi(p, x)$ for all $p \in F(F_r)$.

3. Main results

We begin with the following results:

Lemma 3.9. *Let E be a smooth, strictly convex, and reflexive real Banach space. Let $T : E \to E^*$ be a continuous J-pseudocontractive mapping. Then, for $r > 0$ and $x \in E$, there exists $z \in E$ such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)Jz - Jx\rangle \leq 0 \;\forall\; y \in E.$$

Proof. Since T is J-pseudocontractive and continuous then $A := (J - T) : E \to E^*$ is a continuous and monotone mapping. Moreover, for $z \in E$

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \;\forall\; y \in E$$

is equivalent to

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)Jz - Jx\rangle \leq 0 \;\forall\; y \in E.$$

Therefore the result follows from Lemma 2.7. □

Lemma 3.10. *Let E be a smooth, strictly convex, and reflexive real Banach space. Let $T : E \to E^*$ be a continuous J-pseudocontractive mapping. For $r > 0$ define the mapping $T_r : E \to E$ as follows:*

$$T_r x := \{z \in E : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)Jz - Jx\rangle \leq 0 \; \forall \; y \in E\}$$

for all $x \in E$. Then the following hold :
(1) T_r is single-valued;
(2) $F(T_r)$ is closed and convex;
(3) $\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x)$ for all $p \in F(T_r)$;
(4) $F(T_r) = F^J(T)$.

Proof. Since T is J-pseudocontractive and continuous, then $A := (J - T) : E \to E^*$ is a continuous and monotone. Moreover

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \; \forall \; y \in C$$

is equivalent to

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)Jz - Jx\rangle \leq 0 \; \forall \; y \in E.$$

Therefore (1)-(3) follow by taking $A := J - T$ in Lemma 2.8.
Now let $x \in E$ such that $x = T_r x$. We have

$$\langle y - x, Tx - Jx\rangle \leq 0 \; \forall \, y \in E.$$

This implies that $Jx = Tx$. Therefore $F(T_r) \subset F^J(T)$.
On the other hand suppose that $Tx = Jx$. Let $y \in E$, we have

$$\langle y - x, Tx\rangle - \frac{1}{r}\langle y - x, (1 + r)Jx - Jx\rangle = \langle y - x, Tx - Jx\rangle - \frac{1}{r}\langle y - x, Jx - Jx\rangle = 0.$$

This implies that $x \in T_r x$. Since T_r is single-valued then $T_r x = x$.
Therefore, $F^J(T) \subset F(T_r)$. So $F(T_r) = F^J(T)$.

□

Algorithm. Let us now present our algorithm : for E a smooth, stricly convex reflexive real Banach space and $u, x_1 \in E$ arbitrarily chosen in E, let $\{x_n\}$ be the sequence generated as follows :

$$\begin{cases} u_n &= T_{r_n} x_n \\ x_{n+1} &= J^{-1}\Big(\alpha_n Ju + (1 - \alpha_n)Ju_n\Big), \; n \geq 1 \end{cases} \tag{3.13}$$

where $\{\alpha_n\}$ is real seqience in $(0, 1)$ satisfying : $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$; and $\{r_n\}$ is a real sequence in $[c, \infty)$ for some constant $c > 0$.

We now prove the following theorem.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex real Banach space and let E^* be it dual. Let $T : E \to E^*$ be a J-pseudocontractive and continuous mapping. Suppose that $F = F^J(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by* (3.13) *converges strongly to some $x^* = \Pi_F u$.*

Proof. **step 1:** *we show that $\{x_n\}$ is bounded.*

We proceed by induction. Let $x^* = \Pi_F u$. There exists $r > 0$ sufficiently large such that

$$\max\{\phi(x^*, u), \phi(x^*, x_1)\} \leq r.$$

Suppose that $\phi(x^*, x_n) \leq r$. From (3.13), Using Lemma 3.10, we have

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Ju_n)) \\
&= \|x^*\|^2 - 2\langle x^*, \alpha_n Ju + (1-\alpha_n)Ju_n\rangle + \|\alpha_n Ju + (1-\alpha_n)Ju_n\|^2 \\
&\leq \|x^*\|^2 - 2\alpha_n\langle x^*, Ju\rangle - 2(1-\alpha_n)\langle x^*, Ju_n\rangle + \alpha_n\|Ju\|^2 + (1-\alpha_n)\|Ju_n\|^2 \\
&= \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, u_n) \\
&\leq \alpha_n\phi(x^*, u) + (1-\alpha_n)\phi(x^*, x_n).
\end{aligned}$$

Using the induction assumption , it follows that $\phi(x^*, x_{n+1}) \leq r$. Therefore, $\{x_n\}$ is bounded. Since $\phi(x^*, u_n) \leq \phi(x^*, x_n) \leq r$,, then the sequence $\{u_n\}$, also bounded.

Step 2: *we prove that $x_n \to x^*$ as $n \to \infty$.*

Using Lemma 2.4 and Lemma 3.10, we have

$$\begin{aligned}
&\phi(x^*, x_{n+1}) = V(x^*, Jx_{n+1}) \leq V(x^*, Jx_{n+1} - \alpha_n(Ju - Jx^*)) + 2\langle x_{n+1} - x^*, \alpha_n(Ju - Jx^*)\rangle \\
&= \phi(x^*, \alpha_n Jx^* + (1-\alpha_n)Ju_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \leq (1-\alpha_n)\phi(x^*, u_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_n)\Big(\phi(x^*, x_n) - \phi(u_n, x_n)\Big) + 2\alpha_n\langle x_n - x^*, Ju - Jx^*\rangle + 2\alpha_n\|x_{n+1} - x_n\| \cdot \|Ju - Jx^*\|.
\end{aligned} \tag{3.14}$$

Therefore,

$$\phi(x^*, x_{n+1}) \leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle x_n - x^*, Ju - Jx^*\rangle + 2\alpha_n\|x_{n+1} - x_n\| \cdot \|Ju - Jx^*\|. \tag{3.15}$$

We observe that

$$\phi(u_n, x_{n+1}) = \phi(u_n, J^{-1}(\alpha_n Ju + (1-\alpha_n)Ju_n) \leq \alpha_n\phi(u_n, u).$$

and since $\{u_n\}$ is bounded and $\alpha_n \to 0$, it follows that $\phi(u_n, x_{n+1}) \to 0$ as $n \to \infty$.

This fact and Lemma 2.3 imply that

$$u_n - x_{n+1} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}$$

For the remaining of the proof, we split it into two cases:

Case 1: suppose that $\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n)\ \forall\, n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Since $\{\phi(x^*, x_n)\}$ is bounded from below, then the sequence $\{\phi(x^*, x_n)\}$ converges. Using (3.14) we have $\phi(u_n, x_n) \to 0$. Therefore,

$$u_n - x_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

From (3.16) and (3.17), it follows that

$$x_n - x_{n+1} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.18}$$

Let us now prove that

$$\limsup\langle x_n - x^*, Ju - Jx^*\rangle \leq 0.$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \rightharpoonup z \in E \tag{3.19}$$

and

$$\limsup\langle x_n - x^*, Ju - Jx^*\rangle = \lim_k\langle x_{n_k} - x^*, Ju - Jx^*\rangle.$$

It follows from (3.17) and (3.19) that

$$u_{n_k} \rightharpoonup z.$$

On the other hand, from (3.13), we have

$$\langle y-u_{n_k}, Ju_{n_k}-Tu_{n_k}\rangle + \langle y-u_{n_k}, \frac{Ju_{n_k}-Jx_{n_k}}{r_{n_k}}\rangle \geq 0, \tag{3.20}$$

for all $y \in E$.

Now for $y \in E$, let $v_t := ty + (1-t)z$ for $t \in (0,1]$. From (3.20) and the fact T is J-pseudocontractive, it follows that

$$\begin{aligned}\langle v_t-u_{n_k}, Jv_t-Tv_t\rangle &\geq \langle v_t-u_{n_k}, Jv_t-Tv_t\rangle - \langle v_t-u_{n_k}, Ju_{n_k}-Tu_{n_k}\rangle \\ - \langle v_t-u_{n_k}, \frac{Ju_{n_k}-Jx_{n_k}}{r_{n_k}}\rangle &= \langle v_t-u_{n_k}, Jv_t-Ju_{n_k}\rangle - \langle v_t-u_{n_k}, Tv_t-Tu_{n_k}\rangle \\ &+ \langle u_{n_k}-v_t, \frac{Ju_{n_k}-Jx_{n_k}}{r_{n_k}}\rangle \geq \langle u_{n_k}-v_t, \frac{Ju_{n_k}-Jx_{n_k}}{r_{n_k}}\rangle.\end{aligned}$$

Since J is uniformly continuous on bounded sets and $u_{n_k}-x_{n_k}\to 0$ as $k\to\infty$, it follows

$$\lim_{k\to\infty}\langle u_{n_k}-v_t, \frac{Ju_{n_k}-Jx_{n_k}}{r_{n_k}}\rangle = 0.$$

Therefore, by taking limit as $k\to\infty$ in both sides in (3) we obtain

$$\langle v_t-z, Jv_t-Tv_t\rangle \geq 0, \ \forall\, t\in(0,1].$$

This implies that for all $y\in E$ we have

$$\langle y-z, Jv_t-Tv_t\rangle \geq 0, \ \forall\, t\in(0,1].$$

By letting $t\to 0$ we have

$$\langle y-z, Jz-Tz\rangle \geq 0, \ \forall\, y\in E.$$

So, $Jz-Tz=0$; that is $z\in F^J(T)$.

From Lemma 2.2, it follows that

$$\lim_k\langle x_{n_k}-x^*, Ju-Jx^*\rangle = \langle z-x^*, Ju-Jx^*\rangle \leq 0.$$

That is

$$\limsup\langle x_n-x^*, Ju-Jx^*\rangle \leq 0.$$

Therefore, by Lemma 2.5 and (3.15) we have $\phi(x^*,x_n)\to 0$. Hence, Lemma 2.3 implies that $x_n\to x^*$.

Case 2: suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(x^*,x_{n_i}) \leq \phi(x^*,x_{n_{i+1}}) \ \forall\, i\in\mathbb{N}.$$

Then by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\}$ such that $m_k\to\infty$ and

$$\phi(x^*,x_{m_k}) \leq \phi(x^*,x_{m_{k+1}}) \quad \text{and} \quad \phi(x^*,x_k) \leq \phi(x^*,x_{m_{k+1}}).$$

Since $\{\phi(x^*,x_{m_k})\}_k$ is increasing and bounded from above, then it converges as $k\to\infty$. Therefore, using the same arguments as in **case 1**, we have $\phi(x^*,x_{m_k})\to 0$. Since

$\phi(x^*, x_k) \leq \phi(x^*, x_{m_{k+1}})$, it follows that $\phi(x^*, x_k) \to 0$. Therefore, by Lemma 2.3, we have $x_k \to x^*$. This completes the proof. □

4. Application to zeros of monotone maps

Let E be a uniformly smooth and strictly convex real Banach space and let E^* be it dual. Let $A : E \to E^*$ be a monotone and continuous mapping. Let $r > 0$, define the mapping $A_r : E \to 2^E$ as follows:

$$A_r x := \{z \in C : \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \leq 0 \;\forall\, y \in E\} \quad \text{for all } \; x \in E.$$

Let $T = J - A$. Since A is monotone and continuous then T is J-pseudocontractive and continuous. Moreover, we have

$$F^J(T) = A^{-1}(0) \text{ and, for } x \in E, \quad T_r x = A_r x.$$

It follows from this analysis that A_r is well defined and is single valued. Let us consider the following algorithm. Let $u, x_1 \in E$ chosen arbitrarily. Given x_n, the next iterate is obtained as follows:

$$x_{n+1} = J^{-1}\Big(\alpha_n Ju + (1 - \alpha_n) JA_{r_n} x_n\Big),\ n \geq 1, \tag{4.21}$$

where $\alpha_n \in (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$ and $\{r_n\} \in [c, \infty)$ for some constant $c > 0$.

Theorem 4.2. *Let E be a uniformly smooth and strictly convex real Banach space and let E^* be it dual. Let $A : E \to E^*$ be a monotone and continuous mapping. Suppose that $A^{-1}(0) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by* (4.21) *converges strongly to some $x^* = \Pi_{A^{-1}(0)} u$.*

Proof. By Theorem 3.1 and the above analysis we have x_n converges strongly to some $x^* = \Pi_{F^J(T)} u$. That is $x_n \to x^* = \Pi_{A^{-1}(0)} u$. □

5. Application to convex minimization problems

Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semi-continuous function. It is well known from a result that it sub differential ∂f is maximal monotone. Further if f is bounded then ∂f is bounded on bounded subset. Henceforth the above result is applicable in minimization problems in the following sense.

Theorem 5.3. *Let E be a uniformly smooth and strictly convex real Banach space. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and lower semi-continuous function. Assume that $A := \partial f$ is continuous and that f has a minimum in E. Let u, x_1 be arbitrarily in E. The sequence $\{x_n\}$ defined iteratively from x_1 by*

$$x_{n+1} = J^{-1}\Big(\alpha_n Ju + (1 - \alpha_n) JA_{r_n} x_n\Big),\ n \geq 1$$

converges strongly to a minimum point of f.

Proof. Let $A := \partial f$ we have A is bounded, monotone and continuous. Moreover, $0 \in Ax^*$ if and only if $f(x^*) = \min_{x\in E} f(x)$, hence the result follows from Theorem 4.2. □

Corollary 5.1. *Let E be a uniformly smooth and strictly convex real Banach space. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex, proper and C^1-function. Suppose that f has a minimum in E. Let $u, x_1 \in E$ then the sequence $\{x_n\}$ defined iteratively from x_1 by*

$$x_{n+1} = J^{-1}\Big(\alpha_n Ju + (1 - \alpha_n) J\nabla f_{r_n}(x_n)\Big),\ n \geq 1,$$

converges strongly to a minimum point of f.

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