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Solving a system of integral equations by using some tripled fixed point theorems

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ABSTRACT. A tripled fixed point theorems in ordered metric spaces is used in order to prove the existence and uniqueness of a solution for a class of integral equations. The conditions of the theorem are much weaker than those existing in literature and the theorem is useful for solving some general problems. An example to illustrate our theoretical results is also given.

1. INTRODUCTION

Fixed point theory in partial ordered metric spaces, that is, in metric spaces endowed with a partial ordering, has developed rapidly in recent years. The study of coupled fixed point theory has been considered in 2004 by Ran and Reurings [16] and in 2006 by Bhaskar and Lakshmikantham [10]. A rich literature on the existence of coupled fixed points of mixed monotone, monotone and non-monotone mappings, has been developed ever since publication of that paper (see [7], [8], [14]). Berinde and Borcut [6], introduced the concept of tripled fixed point and proved some related fixed point theorems. After that various results on the existence of tripled fixed points for various classes of mappings have been obtained, see [4], [11], [12], [13], [15]), for a selection of them.
In this paper we obtain the existence of a solution for a class of tripled integral equations in the framework of tripled fixed point theorem on partially ordered metric spaces.

2. PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let (X, d) be a partially ordered set and d be a metric on X, with precondition that (X, d) is complete metric space. In addition, we define such a partial order in the product space $X \times X \times X$, for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \iff x \geq u, y \leq v, z \geq w.$$

Definition 2.1. ([11]) Let (X, d) be a partially ordered set, $F : X \times X \times X \to X$ be a mapping. If $F(x, y, z)$ is monotone non-decreasing in x and z, and non-increasing in y, that is, for any $x, y, z \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \implies F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \implies F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1, z_2 \in X, z_1 \leq z_2 \implies F(x, y, z_1) \leq F(x, y, z_2),$$

then we say that F has the *mixed-monotone property*.

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Definition 2.2. ([6]) Let $F : X \times X \times X \to X$ be a mapping. If

$$F(x,y,z) = x,\ F(y,x,y) = y,\ F(z,y,x) = z,$$

for any $(x,y,z) \in X \times X \times X$ then we say that (x,y,z) is a *tripled fixed point of* F.

Definition 2.3. ([6]) Let (X,d) be a metric space. On the product space X^3, we consider the metric $\bar{d} : X \times X \times X \to X$, defined by

$$\bar{d}((x,y,z),(u,v,w)) = d(x,u) + d(y,v) + d(z,w), \tag{2.1}$$

for all $(x,y,z),(u,v,w) \in X^3$.

Theorem 2.1. ([6]) *Let $F : X \times X \times X \to X$ be a continuous mapping satisfy the above mixed monotone property on X. Suppose that there exists a $k \in [0,1)$ with*

$$d(F(x,y,z),F(u,v,w)) \leq \frac{k}{3}\left[d(x,u) + d(y,v) + d(z,w)\right],$$

for any $(u,v,w) \leq (x,y,z)$. If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0),\ y_0 \geq F(y_0,x_0,y_0),\ z_0 \leq F(z_0,y_0,x_0),$$

then, there exists $x,y,z \in X$, such that

$$x = F(x,y,z), y = F(y,x,y), z = F(z,y,x).$$

Theorem 2.2. ([13]) *Let (X,d) be a complete metric space, $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X.*

Suppose either,

1. F is continuous or;

2. X has the following properties:

(a) *if a non decreasing sequence $x_n \to x$, then $x_n \leq x$, for all n;*

(b) *if a non increasing sequence $y_n \to y$, then $y_n \geq y$, for all n.*

If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0),\ y_0 \geq F(y_0,x_0,y_0),\ z_0 \leq F(z_0,y_0,x_0),$$

then, there exists $x,y,z \in X$, such that

$$x = F(x,y,z), y = F(y,x,y), z = F(z,y,x).$$

3. Main results

In this section, inspired by the works [6], [13], we present an existence theorem for a nonlinear tripled system and we give an example to support our theoretical result. Consider the following system of integral equations

$$\begin{cases} x(t) = f(t) + \int\limits_a^b K(t,s,x(s),y(s),z(s))ds \\ y(t) = f(t) + \int\limits_a^b K(t,s,y(s),x(s),y(s))ds \\ z(t) = f(t) + \int\limits_a^b K(t,s,z(s),y(s),x(s))ds \end{cases} \tag{3.2}$$

where $a,b \in \mathbb{R},\ a < b,\ x,y,z \in C[a,b],\ f : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ are given mapping.

In this context, we consider on $X = C[a,b]$ the metric

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

and, correspondingly, on X^3 we shall have the metric $\bar{d}$ defined by (2.1)

$$\bar{d}((x,y,z),(u,v,w)) = d(x,u) + d(y,v) + d(z,w).$$

The main result of this note is the following theorem.

Theorem 3.3. *Consider the nonlinear system (3.2). Suppose that the following conditions hold:*

i) $f : [a,b] \to \mathbb{R}$ *and* $K : [a,b] \times [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *are continuous;*

ii) $K(t,s,\cdot,\cdot,\cdot)$ *has the generalized mixed monotone property with respect to the last three variables, for all* $t, s \in [a,b]$;

iii) There exists $\alpha, \beta, \gamma : [a,b] \times [a,b] \to \mathbb{R}_+$ *such that, for each* $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$ *with* $x_1 \le y_1$, $x_2 \ge y_2$, $x_3 \le y_3$ *we have*

$$|K(t,s,x_1(s),x_2(s),x_3(s)) - K(t,s,y_1(s),y_2(s),y_3(s))| \le \alpha(t,s)\cdot|x_1-y_1| + \beta(t,s)\cdot|x_2-y_2| + \gamma(t,s)\,|x_3-y_3|\,;$$

iv) $\max\limits_{t\in[a,b]}\left(\int\limits_a^b \alpha(t,s)ds\right) < \frac{1}{3},\ \max\limits_{t\in[a,b]}\left(\int\limits_a^b \beta(t,s)ds\right) < \frac{1}{3},\ \max\limits_{t\in[a,b]}\left(\int\limits_a^b \gamma(t,s)ds\right) < \frac{1}{3};$

v) There exists $x_0, y_0, z_0 \in C[a,b]$ *such that*

$$\begin{cases} x_0(t) \le \varphi(t) + \int\limits_a^b K(t,s,x_0(s),y_0(s),z_0(s))ds \\ y_0(t) \ge \varphi(t) + \int\limits_a^b K(t,s,y_0(s),x_0(s),y_0(s))ds \\ z_0(t) \le \varphi(t) + \int\limits_a^b K(t,s,z_0(s),y_0(s),x_0(s))ds \end{cases} \quad (3.3)$$

or

$$\begin{cases} x_0(t) \ge \varphi(t) + \int\limits_a^b K(t,s,x_0(s),y_0(s),z_0(s))ds \\ y_0(t) \le \varphi(t) + \int\limits_a^b K(t,s,y_0(s),x_0(s),y_0(s))ds \\ z_0(t) \ge \varphi(t) + \int\limits_a^b K(t,s,z_0(s),y_0(s),x_0(s))ds \end{cases} \quad (3.4)$$

for all $t \in [a,b]$.
Then, there exists a solution $(\bar{x},\bar{y},\bar{z})$ *for the system (3.2).*

Proof. We prove that all assumptions of Theorem 2.1 are satisfied.

Define $F : X \times X \times X \to X$ by

$$F(x,y,z) = \phi(t) + \int\limits_a^b K(t,s,x(s),y(s),z(s))ds,$$

for each $t \in [a,b]$.

Then the system (3.2) can be written as a tripled fixed point problem of F :

$$\begin{cases} F(x,y,z) = x \\ F(y,x,y) = y \\ F(z,y,x) = z \end{cases}$$

First, we will prove that F has the mixed monotone property. For $x_1 \leq x_2$ and $t \in [a,b]$, we have:

$$F(x_1,y,z) - F(x_2,y,z) = \int_a^b [K(t,s,x_1(s),y(s),z(s)) - K(t,s,x_2(s),y(s),z(s))]\, ds \leq 0,$$

due to condition (ii).

Thus, $F(x_1,y,z) \leq F(x_2,y,z)$ for every element $x_1 \leq x_2,\ t \in [a,b]$. Similary, we show that $F(x,y_1,z) \geq F(x,y_2,z)$ for all $y_1 \geq y_2$ and $F(x,y,z_1) \leq F(x,y,z_2)$ for all $z_1 \leq z_2$. Then, for all $x_1 \leq y_1,\ x_2 \geq y_2,\ x_3 \leq y_3$, we have

$$d(F(x_1,x_2,x_3),F(y_1,y_2,y_3)) =$$

$$\max_{t\in[a,b]} \left| \int_a^b [K(t,s,x_1(s),x_2(s),x_3(s)) - K(t,s,y_1(s),y_2(s),y_3(s))]ds \right| \leq$$

$$\leq \max_{t\in[a,b]} \int_a^b |K(t,s,x_1(s),x_2(s),x_3(s)) - K(t,s,y_1(s),y_2(s),y_3(s))|\, ds \leq$$

$$\leq \max_{t\in[a,b]} \int_a^b \alpha(t,s)ds \cdot d(x_1,y_1) + \max_{t\in[a,b]} \int_a^b \beta(t,s)ds \cdot d(x_2,y_2) + \max_{t\in[a,b]} \int_a^b \gamma(t,s)ds \cdot d(x_3,y_3).$$

Using (iv), we find that all the assumptions of Theorem 3.3 are satisfied and hence the system (3.2) has a unique solution. □

Example 3.1. For the system (3.2), let $a = 0,\ b = 1$ and

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{t}{2}, \\ \alpha(t,s) &= ts^3, \\ \beta(t,s) &= ts^4, \\ \gamma(t,s) &= ts^5, \end{aligned}$$

and

$$K(t,s,x(s),y(s),z(s)) = \alpha(t,s)\cdot x(s) - \beta(t,s)\cdot y(s) + \gamma(t,s)\cdot z(s).$$

Then (3.2) becomes

$$\begin{cases} x(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3x(s) - ts^4y(s) + ts^5z(s)]ds \\ y(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3y(s) - ts^4x(s) + ts^5y(s)]ds \\ z(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3z(s) - ts^4y(s) + ts^5x(s)]ds \end{cases}$$

$$F(x,y,z) = \frac{1}{2} + \frac{t}{2} + \int_0^1 \alpha(t,s)\cdot x(s) - \beta(t,s)\cdot y(s) + \gamma(t,s)\cdot z(s)$$

Obviously,

$$\max_{t\in[0,1]}\int_0^1 ts^3ds = \max_{t\in[0,1]} t\cdot \frac{s^4}{4}\Big|_0^1 = \max_{t\in[0,1]}\frac{t}{4}\le\frac{1}{4}<\frac{1}{3},$$

$$\max_{t\in[0,1]}\int_0^1 ts^4ds = \max_{t\in[0,1]} t\cdot \frac{s^5}{5}\Big|_0^1 = \max_{t\in[0,1]}\frac{t}{5}\le\frac{1}{5}<\frac{1}{3},$$

$$\max_{t\in[0,1]}\int_0^1 ts^5ds = \max_{t\in[0,1]} t\cdot \frac{s^6}{6}\Big|_0^1 = \max_{t\in[0,1]}\frac{t}{6}\le\frac{1}{6}<\frac{1}{3}.$$

Let $x_0 = 0,\ y_0 = 2,\ z_0 = 0$, we have

$$\begin{cases} x_0 \le F(x_0, y_0, z_0) = \dfrac{1}{2}, \\ y_0 \ge F(x_0, y_0, z_0) = \dfrac{1}{2} + \dfrac{t}{10}, \quad \text{for any } t\in[0,1]. \\ z_0 \le F(x_0, y_0, z_0) = \dfrac{1}{2} + \dfrac{t}{10}, \end{cases}$$

As a results, all the conditions of Theorem 3.3 are satisfied. This means that (3.2) has a triplet unique solution

$$\begin{cases} x^* = F(x^*, y^*, z^*), \\ y^* = F(y^*, x^*, y^*), \quad (x^*, y^*, z^*)\in C[0,1]\times C[0,1]\times C[0,1]. \\ z^* = F(z^*, y^*, x^*). \end{cases}$$

4. Conclusions

In this paper, we obtained a tripled fixed point theorem which is suitable for a class of systems of integral equations. It is based on a fixed point theorem on partially ordered metric spaces. Moreover, we find that all the conditions of the theorem become weaker than those existing in literature. It is also shown that we can obtain much weaker conditions for some types of integral equations.

A concrete example to illustrate the theoretical results is also considered in all computational details.

References

[1] Abbas, M., Khan. M. A. and Radenovi, S., *Common coupled fixed point theorems in cone metric space for w-compatible mappings*, Appl. Math. Comput., **217** (2010), No. 1, 195–202

[2] Amini-Harandi, A., *Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem*, Mathematical and Computer Modelling, **57** (2013), No. 9-10, 2343–2348

[3] Aydi, H. and Karapnar, E., *Triple fixed point in ordered metric spaces*, Bull. Math. Anal. Appl., **4** (2012), No. 1, 197–207

[4] Agarwal, R. P., El-Gebeily, M. A. and Oregano, D., *Generalized contractions in partially otdered metric spaces*, Appl. Anal., **87** (2008), No. 1, 109–116

[5] Bartoszewski, Z., *Solving boundary value problems for delay differential equations by a fixed-point method*, J. Comput. Appl. Math., **236** (2011), No. 6, 1576–1590

[6] Berinde, V. and Borcut, M., *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 4889–4897

[7] Berinde, V., Khan, A. R. and Păcurar, M., *Coupled solutions for a bivariate weakly nonexpansive operator by iterations*, Fixed Point Theory Appl., 2014, 2014:149, 12 pp.
[8] Berinde, V., *Coupled coincidence point theorems for mixed monotone nonlinear operators*, Comput. Math. Appl., **64** (2012), No. 6, 1770–1777
[9] Berzig, M. and Samet, B., *An extension of coupled fixed points concept in higher di-mension and applications*, Comput. Math. Appl., **63** (2012), No. 8, 1319–1334
[10] Bhaskar, T. G. and Lakshmikantam, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), No. 7, 1379–1393
[11] Borcut, M., *Tripled fixed point theorems mappings in partially ordered metric spaces*, Hacet. J. Math. Stat. (Submited)
[12] Borcut, M., *Tripled coincidente point theorems for monotone contractive type mappings in partially ordered metric spaces*, Creative Math. Inform., **21** (2012), 135–142
[13] Borcut, M., *Puncte triple fixe pentru operatori definiţi pe spaţii metrice parţial ordonate*, Risoprint, Cluj-Napoca, 2016
[14] Gu, F. and Yin, Y., *A new common coupled fixed point theorem in generalized metric space and applications to integral equations*, Fixed Point Theory Appl. 2013, 2013:266, 16 pp.
[15] Gupta, A. and Manro, S., *A new type of tripled fixed point theorem in partally ordered complete metric space*, Advances in Analysis, **2** (2017), 63–70
[16] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc., **132** (2004), 1435–1443

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