CREAT. MATH. INFORM. Online version at http://creative-mathematics.ubm.ro/
29 (2020), No. 1, 57 - 64 Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

n-Commuting skew higher derivation on semiprime rings

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ABSTRACT. In this article we study skew higher derivation $(d_i)_{i\in\mathbb{N}}$ on semiprime ring R with suitable torsion restriction and we prove that every n-centralizing skew higher derivation is n-commuting. Further, we show that if a ring R has n-centralizing skew higher derivation then either R is commutative or some linear combination of $(d_i)_{i\in\mathbb{N}}$ maps center of R to zero.

1. INTRODUCTION

Throughout the article, R is a semiprime ring with center $Z(R)$. A ring R is said to be a prime ring if for any $x, y \in R$, $xRy = 0$ implies either $x = 0$ or $y = 0$ and R is said to be a semiprime ring if for $x \in R$, $xRx = 0$ then $x = 0$. Here, we notice that every prime ring is a semiprime ring but converse need not be true in general. One of the other characterizations of semiprime is that the center of semiprime ring does not contain nonzero nilpotent elements. Let $n \geq 2$ be an integer. A ring R is said to be n-torsion free if for all $x \in R$, $nx = 0$ then $x = 0$. The commutator of x and y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. Let S be a nonempty subset of R. An additive mapping $f : R \to R$ is said to be centralizing (resp. commuting) on S if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$. In [6], Deng and Bell extended this notion to n-centralizing (resp. n-commuting) for positive integer n, i. e. a mapping f is said to be n-centralizing (resp. n-commuting) on S if $[f(x), x^n] \in Z(R)$ (resp. $[f(x), x^n] = 0$) for all $x \in S$. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Posner [16], has initiated the study of centralizing and commuting mappings and prove that the existence of a nonzero centralizing derivation on a prime ring forces the ring is to be a commutative ring. Further, several authors have extended 'derivation' in various directions such as generalized derivation, Jordan derivation, (α, β)-derivation, multiplicative derivation, multiplicative generalized derivation, . . . ,etc. and have studied the structure of rings as well as structure of additive mappings.

One more generalization of derivation is a skew derivation. By skew derivation, we mean an additive mapping $d : R \to R$ associated with an automorphism α on R such that

$$d(xy) = d(x)y + \alpha(x)d(y),$$

for all $x, y \in R$. A family of additive mappings $(d_i)_{i\in\mathbb{N}}$ of a ring R is said to be a higher derivation if for every $m \in \mathbb{N}$,

$$d_m(xy) = d_m(x)y + \sum_{\substack{i+j=m\\ i,j\geq 1}} d_i(x)d_j(y) + xd_m(y),$$

for all $x, y \in R$ (for more detail see [9] exercise 4, page 540). A family of additive mappings $(d_i)_{i\in\mathbb{N}}$ of a ring R is said to be a skew higher derivation associated with an automorphism

Received: 11.05.2018. In revised form: 27.07.2019. Accepted: 01.09.2019
2010 *Mathematics Subject Classification.* 16W25, 16N60, 16R50.
Key words and phrases. *semiprime ring, extended centroid, skew higher derivation.*

α if for every $m \in \mathbb{N}$, $d_m(xy) = d_m(x)y + \sum_{\substack{i+j=m \\ i,j\geq 1}} d_i(x)d_j(y) + \alpha(x)d_m(y)$ for all $x, y \in R$.

Let $(d_i)_{i\in\mathbb{N}}$ be a higher derivation of a ring T and D be a ring such that either $D \subseteq T$ or $T \subseteq D$. We say that $(d_i)_{i\in\mathbb{N}}$ satisfy a D-linear relation on T if there exist $a_0, a_1, \ldots, a_n \neq 0$ in D such that $\sum_{i=0}^{n} a_i d_i(x) = 0$ for every $x \in T$. (for more detail see [8], Definition 1.2).

In [2], Bell and Martindale proved that if a semiprime ring R possesses derivation d centralizing on a nonzero left ideal I of R then R contains a nonzero central ideal provided $d(I) \neq 0$. In this direction several other results have been done by number of authors see [3, 12, 17], where further references can be found. Recently, in [7], Dhara and Ali proved several results on an n-commuting generalized derivation of a semiprime ring. We extend some of these results for skew higher derivation on a semiprime ring.

2. Preliminaries

In this section, we invoke some known results those are useful in the proof of our results. The Martindale's quotient ring and the extended centroid of a semiprime ring R will be denoted by Q and C respectively. We have the following facts:

Fact 1. $Z(Q) \cap R = Z(R)$ ([10, Proposition 14.17]). A ring R is said to be von Neumann regular ring if for every $a \in R$ there exists $x \in R$ such that $axa = a$.

Fact 2. The Martindale's quotient ring Q of a semiprime ring R is a semiprime ring ([1, page 65]).

Fact 3. I, Q and R satisfy same generalized polynomial identity (GPI) ([4]).

Fact 4. The extended centroid C of a semiprime ring R is a commutative von Neumann regular self injective ring ([1, Theorem 2, page 66]).

An element $e \in R$ is said to be a nontrivial idempotent if $e \neq 0, 1$ and $e^2 = e$.

Fact 5. The extended centroid C of a semiprime ring R has many idempotents ([1, page 66]).

Fact 6. For any ring R, R/P is a prime ring if and only if P is a prime ideal of R. Moreover, if R is a semiprime ring then $\bigcap \{P \mid P \text{ is a prime ideal of R}\} = \{0\}$ ([14, Corollary 4.16]).

Fact 7. Let n be a fixed positive integer and R is an $n!$-torsion free ring. Suppose $x_1, x_2, \ldots, x_n \in R$ satisfy $kx_1 + k^2x_2 + \ldots + k^n x_n = 0$ $(resp.\ kx_1 + k^2x_2 + \ldots + k^n x_n \in Z(R))$ for $k = 1, 2, \ldots, n$. Then $x_i = 0$ $(resp.\ x_i \in Z(R))$ for all $i = 1, 2, \ldots, n$. ([5, Lemma 1], [15, Lemma 2.4])

3. Main results

Theorem 3.1. *Let R be an $(n+1)!$-torsion free semiprime ring and $(d_i)_{i\in\mathbb{N}}$ be a skew higher derivation such that $[d_m(x), x^n] \in Z(R)$ for all $x \in I$, where I be an ideal of R, then $[d_m(x), x^n] = 0$ for all $x \in I$.*

Proof. Replacing x by $x + ky$ in $[d_m(x), x^n] \in Z(R)$ we get

$$[d_m(x) + kd_m(y), x^n + k(x^{n-1}y + x^{n-2}yx + \ldots + xyx^{n-2} + yx^{n-1}) + \ldots$$
$$+k^{n-1}(xy^{n-1} + yxy^{n-2} + \ldots + y^{n-2}xy + y^{n-1}x) + k^n y^n] \in Z(R)$$

which can be written as

$$[d_m(x), x^n] + k\{[d_m(y), x^n] + [d_m(x), x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1})]\} + \ldots$$
$$+k^n\{[d_m(x), y^n] + [d_m(y), xy^{n-1} + \ldots + x^{n-1}y] + k^{n+1}[d_m(y), y^n]\} \in Z(R).$$

Since $[d_m(x), x^n] \in Z(R)$ for all $x \in R$ above expression becomes

$$\begin{aligned} &k\{[d_m(y), x^n] + [d_m(x), x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1})]\} + \ldots \\ &+k^n\{[d_m(x), y^n] + [d_m(y), xy^{n-1} + \ldots + x^{n-1}y]\} \in Z(R). \end{aligned} \tag{3.1}$$

Using fact 2 in (3.1), we get

$$[d_m(y), x^n] + [d_m(x), x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1}] \in Z(R) \tag{3.2}$$

for all $x, y \in I$. Replacing y by x^{n+1} in (3.2), we obtain

$$[d_m(x^{n+1}), x^n] + [d_m(x), x^{2n} + x^{2n} + \ldots + x^{2n}] \in Z(R) \tag{3.3}$$

for all $x \in I$. (3.3) can be rewritten as

$$[d_m(x)x^n + A_m(x, x^n) + \alpha(x)d_m(x^n), x^n] + n[d_m(x), x^{2n}] \in Z(R) \tag{3.4}$$

for all $x \in I$, where $A_m(x, x^n) = d_{m-1}(x)d_1(x^n) + d_{m-2}(x)d_2(x^n) + \ldots + d_1(x)d_{m-1}(x^n)$. That is

$$[d_m(x), x^n]x^n + [A_m(x, x^n) + \alpha(x)d_m(x^n), x^n] + nx^n[d_m(x), x^n] + n[d_m(x), x^n]x^n \in Z(R) \tag{3.5}$$

for all $x \in I$. Since $[d_m(x), x^n] \in Z(R)$, from (3.5), we have

$$(2n+1)x^n[d_m(x), x^n] + [A_m(x, x^n) + \alpha(x)d_m(x^n), x^n] \in Z(R) \tag{3.6}$$

for all $x \in I$. Suppose $s \in Z(R)$ such that

$$s = (2n+1)x^n[d_m(x), x^n] + [A_m(x, x^n) + \alpha(x)d_m(x^n), x^n] \in Z(R) \tag{3.7}$$

for all $x \in I$. Put x^{n+1} in place of x in our hypothesis, we have $[d_m(x^{n+1}), x^{n(n+1)}] \in Z(R)$ for all $x \in I$. This gives

$$\begin{aligned} [d_m(x^{n+1}), x^{n(n+1)}] &= [d_m(x)x^n, x^{n^2+n}] + [A_m(x, x^n) + \alpha(x)d_m(x^n), x^{n^2+n}] \\ &= [d_m(x), x^{n^2+n}]x^n + [A_m(x, x^n) + \alpha(x)d_m(x^n), x^{n^2+n}] \end{aligned} \tag{3.8}$$

for all $x \in I$. From (3.7) we notice that $[A_m(x, x^n) + \alpha(x)d_m(x^n), x^n]$ commutes with x and using the hypothesis $[d_m(x), x^n] \in Z(R)$ for all $x \in I$, we get

$$[d_m(x), x^{n^2+n}]x^n = (n+1)x^{n^2+n}[d_m(x), x^n]$$

and

$$[A_m(x, x^n) + \alpha(x)d_m(x^n), x^{n^2+n}] = (n+1)x^{n^2}[A_m(x, x^n) + \alpha(x)d_m(x^n), x^n].$$

Hence, (3.8) implies that

$$\begin{aligned} [d_m(x^{n+1}), x^{n(n+1)}] &= (n+1)x^{n^2+n}[d_m(x), x^n] \\ &+ (n+1)x^{n^2}[A_m(x, x^n) + \alpha(x)d_m(x^n), x^n] \in Z(R) \end{aligned} \tag{3.9}$$

for all $x \in I$. Application of (3.7), (3.9) reduces to

$$\begin{aligned} [d_m(x^{n+1}), x^{n(n+1)}] &= (n+1)x^{n^2+n}[d_m(x), x^n] \\ &+ (n+1)x^{n^2}\Big\{s - (2n+1)x^n[d_m(x), x^n]\Big\} \in Z(R) \end{aligned} \tag{3.10}$$

for all $x \in I$. That is

$$\begin{aligned} [d_m(x^{n+1}), x^{n(n+1)}] &= -2n(n+1)x^{n^2+n}[d_m(x), x^n] \\ &+ (n+1)x^{n^2}s \in Z(R) \end{aligned} \tag{3.11}$$

for all $x \in I$.

Now commuting x^{kn} with $d_m(x)$, we get

$$[d_m(x), x^{kn}] = [d_m(x), x^n.x^n.\ldots.x^n] = k[d_m(x), x^n]x^{(k-1)n}.$$

Again commuting the last expression with $d_m(x)$ we get

$$[d_m(x), [d_m(x), x^{kn}]] = \frac{k!}{(k-2)!}[d_m(x), x^n]^2 x^{(k-2)n}.$$

Thus commuting x^{kn} with $d_m(x)$ successively m times, we get

$$[d_m(x), \ldots, [d_m(x), x^{kn}] \ldots] = \frac{k!}{(k-m)!}[d_m(x), x^n]^m x^{(k-m)n} \tag{3.12}$$

for all $x \in I$.

Now commuting both sides of (3.11) with $d_m(x)$ and using (3.12), we get

$$-2n(n+1)\frac{(n+1)!}{1!}[d_m(x), x^n]^{n+1}x^n + (n+1)n!s[d_m(x), x^n]^n = 0. \tag{3.13}$$

Again commuting (3.13) with $d_m(x)$, we get

$$-2n(n+1)(n+1)![d_m(x), x^n]^{n+2} = 0 \tag{3.14}$$

for all $x \in I$. By using $(n+1)!$-torsion freeness and semiprimness of R, we obtained $[d_m(x), x^n] = 0$. □

Corollary 3.1. *Let R be an $(n+1)!$-torsion free semiprime ring and d be a skew derivation such that $[d(x), x^n] \in Z(R)$ for all $x \in I$, where I be an ideal of R, then $[d(x), x^n] = 0$ for all $x \in I$.*

Theorem 3.2. *Let R be a 2 and 3-torsion free semiprime ring and $(d_i)_{i\in\mathbb{N}}$ be a skew higher derivation such that $[[d_m(x), x], x] \in Z(R)$ for all $x \in I$, where I be an ideal of R, then $[[d_m(x), x], x] = 0$ for all $x \in I$.*

Proof. Replacing x by $x+y$ and using torsion free restriction on R and fact 2, we get

$$[[d_m(y), x], x] + [[d_m(x), y], x] + [[d_m(x), x], y] \in Z(R) \tag{3.15}$$

for all $x, y \in I$. Substituting x^2 for y in (3.15), we get

$$\begin{aligned} &[[d_m(x^2), x], x] + [[d_m(x), x^2], x] + [[d_m(x), x], x^2] \\ &= [[d_m(x)x + A_m(x,x) + \alpha(x)d_m(x), x], x] + x[[d_m(x), x], x] \\ &\quad + [[d_m(x), x], x]x + x[[d_m(x), x], x] \\ &\quad + [[d_m(x), x], x]x \in Z(R) \end{aligned} \tag{3.16}$$

for all $x \in I$. Since $[[d_m(x), x], x] \in Z(R)$, (3.16) reduces to

$$[[A_m(x,x) + \alpha(x)d_m(x), x], x] + 5x[[d_m(x), x], x] \in Z(R) \tag{3.17}$$

for all $x \in I$. Substituting x^2 for x in our hypothesis, we get

$$\begin{aligned} [[d_m(x^2), x^2], x^2] &= [[d_m(x)x + A_m(x,x) + \alpha(x)d_m(x), x^2], x^2] \\ &= [[d_m(x)x, x^2], x^2] + [[A_m(x,x) + \alpha(x)d_m(x), x^2], x^2] \\ &= [[d_m(x), x], x]x^3 + 2x[[d_m(x), x], x]x^2 \\ &+ x^2[[d_m(x), x], x]x + [[A_m(x,x) + \alpha(x)d_m(x), x], x]x^2 \\ &+ 2x[[A_m(x,x) + \alpha(x)d_m(x), x], x]x \\ &+ x^2[[A_m(x,x) + \alpha(x)d_m(x), x], x] \in Z(R) \end{aligned} \tag{3.18}$$

for all $x \in I$. Using the fact that $[[d_m(x), x], x] \in Z(R)$, we get

$$\begin{aligned}[][[d_m(x^2), x^2], x^2] &= 4x^3[[d_m(x), x], x] + [[A_m(x, x) + \alpha(x)d_m(x), x], x]x^2 \\ &+ 2x[[A_m(x, x) + \alpha(x)d_m(x), x], x]x \\ &+ x^2[[A_m(x, x) + \alpha(x)d_m(x), x], x] \in Z(R) \end{aligned} \tag{3.19}$$

for all $x \in I$. From (3.17), we notice that $[[A_m(x, x)+\alpha(x)d_m(x), x], x] = s-5x[[d_m(x), x], x]$, where $s \in Z(R)$. It implies that $[[A_m(x, x) + \alpha(x)d_m(x), x], x]$ commutes with x. Hence, (3.19) reduces to

$$\begin{aligned}[][[d_m(x^2), x^2], x^2] &= 4x^3[[d_m(x), x], x] \\ &+ 4x^2[[A_m(x, x) + \alpha(x)d_m(x), x], x] \in Z(R). \end{aligned} \tag{3.20}$$

Substituting the value of $[[A_m(x, x) + \alpha(x)d_m(x), x], x]$ in (3.20), we get

$$\begin{aligned}[][[d_m(x^2), x^2], x^2] &= 4x^3[[d_m(x), x], x] + 4x^2\{s - 5x[[d_m(x), x], x]\} \\ &= -16x^3[[d_m(x), x], x] + 4x^2 s \in Z(R) \end{aligned} \tag{3.21}$$

for all $x \in I$. Now commuting both sides of (3.21) with $[d_m(x), x]$ and we notice that $[[d_m(x), x], x^k] = k[[d_m(x), x], x]x^{k-1}$, we get $-48[[d_m(x), x], x]^2x^2 + 8[[d_m(x), x], x]xs \in Z(R)$. Again commuting last expression consecutive two times with $[d_m(x), x]$ we get, $-96[[d_m(x), x], x]^4 = 0$. Now using torsion restriction and semiprimness of R, we get $[[d_m(x), x], x] = 0$ for all $x \in I$. □

In particular, for $m = 1$ and $d_1 = d$, we have following corollary:

Corollary 3.2. *Let R be a 2 and 3-torsion free semiprime ring and d be a skew derivation on R such that $[[d(x), x], x] \in Z(R)$ for all $x \in I$, where I be an ideal of R, then $[[d(x), x], x] = 0$ for all $x \in I$.*

Theorem 3.3. *Let R be an $(n + 1)!$-torsion free semiprime ring, I an ideal of R and Q be its Martindale's ring of quotient with extended centroid C. Let $(d_i)_{i\in\mathbb{N}}$ be a skew higher derivation of R such that $[d_m(x), x^n] \in Z(R)$ for all $x \in I$, then either R is commutative ring or some linear combination of $(d_i)_{i\in\mathbb{N}}$ maps center of R to zero.*

Proof. Linearizing $[d_m(x), x^n] \in Z(R)$ and using torsion free restriction and fact 2, we get

$$[d_m(y), x^n] + [d_m(x), x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1}] \in Z(R) \tag{3.22}$$

for all $x, y \in I$. Replace y by yz in (3.22), where $z \in Z(R)$, we get

$$\begin{aligned} &[d_m(y)z + A_m(y, z) + \alpha(y)d_m(z), x^n] \\ + \quad &[d_m(x), \{x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1}\}z] \in Z(R) \end{aligned} \tag{3.23}$$

for all $x, y \in I$. From (3.22) and (3.23), we obtain

$$[A_m(y, z) + \alpha(y)d_m(z), x^n] \in Z(R) \tag{3.24}$$

for all $x, y \in I$. That is $[[A_m(y, z) + \alpha(y)d_m(z), x^n], x] = [\delta(x^n), x] = 0$, where $\delta(x) = [A_m(y, z) + \alpha(y)d_m(z), x]$, then by [11] (main theorem), we have either $\delta(I) = 0$ or $\delta(I)$ and $\delta(R)I$ are contained in a nonzero central ideal of R. If $\delta(I)$ and $\delta(R)I$ are contained in a nonzero central ideal of R then R is commutative. If $\delta(I) = 0$, then $[A_m(y, z) + \alpha(y)d_m(z), x] = 0$ for all $x, y \in I$. Since I and Q satisfy same generalized polynomial identity, we have

$$[A_m(y, z) + \alpha(y)d_m(z), x] = 0 \tag{3.25}$$

for all $x, y \in Q$.

If P be a prime ideal of Q then $\overline{Q} = Q/P$ is a prime ring. Define additive mappings $(\bar{d}_i)_{i\in\mathbb{N}}$ and $\bar{\alpha}$ from $\overline{Q} \to \overline{Q}$ such that $\bar{d}_{i|Q} = d_i$ and $\bar{\alpha} = \alpha$ and for each $m \in \mathbb{N}$,

$\bar{d}_m(\bar{x}\bar{y}) = \bar{d}_m(\bar{x})\bar{y} + \sum_{i+j=m, i\geq 1} \bar{d}_i(\bar{x})\bar{d}_j(\bar{y}) + \bar{\alpha}(\bar{x})\bar{d}_m(\bar{y})$, then $(\bar{d}_i)_{i\in\mathbb{N}}$ is a skew higher derivation associated with automorphism $\bar{\alpha}$. Now (3.25) can be rewritten as

$$[A_m(\bar{y},\bar{z}) + \bar{\alpha}(\bar{y})\bar{d}_m(\bar{z}), \bar{x}] = 0 \tag{3.26}$$

for all $\bar{x},\bar{y} \in \overline{Q}$ and $\bar{z} \in Z(\overline{Q})$. Now replacing the value of $A_m(\bar{y},\bar{z})$ in (3.26) we get,

$$\begin{gathered}[\bar{d}_{m-1}(\bar{y})\bar{d}_1(\bar{z}) + \bar{d}_{m-2}(\bar{y})\bar{d}_2(\bar{z}) \\ + \ldots + \bar{d}_1(\bar{y})\bar{d}_{m-1}(\bar{z}) + \bar{\alpha}(\bar{y})\bar{d}_m(\bar{z}), \bar{x}] = 0\end{gathered} \tag{3.27}$$

for all $\bar{x},\bar{y} \in \overline{Q}$. Since $\bar{d}_i(\bar{z}) \in Z(\overline{Q})$ for all $i \geq 1$, (3.27) can be rewritten as

$$\begin{gathered}[\bar{d}_{m-1}(\bar{y}),\bar{x}]\bar{d}_1(\bar{z}) + [\bar{d}_{m-2}(\bar{y}),\bar{x}]\bar{d}_2(\bar{z}) \\ + \ldots + [\bar{d}_1(\bar{y}),\bar{x}]\bar{d}_{m-1}(\bar{z}) + [\bar{\alpha}(\bar{y}),\bar{x}]\bar{d}_m(\bar{z}) = 0.\end{gathered} \tag{3.28}$$

Replace $\bar{x}$ by $\bar{x}\bar{t}$ in (3.28) and using (3.28), we get

$$\begin{gathered}[\bar{d}_{m-1}(\bar{y}),\bar{x}]\bar{t}\bar{d}_1(\bar{z}) + [\bar{d}_{m-2}(\bar{y}),\bar{x}]\bar{t}\bar{d}_2(\bar{z}) \\ + \ldots + [\bar{d}_1(\bar{y}),\bar{x}]\bar{t}\bar{d}_{m-1}(\bar{z}) + [\bar{\alpha}(\bar{y}),\bar{x}]\bar{t}\bar{d}_m(\bar{z}) = 0\end{gathered} \tag{3.29}$$

for all $\bar{x},\bar{y},\bar{t} \in \overline{Q}$.
By [13] (see corollary, page 444), either $\{[\bar{\alpha}(\bar{y}),\bar{x}], [\bar{d}_1(\bar{y}),\bar{x}], \ldots, [\bar{d}_{m-1}(\bar{y}),\bar{x}]\}$ is linearly dependent over extended centroid $\overline{C}$ of $\overline{Q}$ or $\{\bar{d}_1(\bar{z}), \bar{d}_2(\bar{z}), \ldots, \bar{d}_m(\bar{z})\}$ is linearly dependent over $\overline{C}$.

Since $\overline{C}$ is a field of dimension 1, the set $\{\bar{d}_1(\bar{z}), \bar{d}_2(\bar{z}), \ldots, \bar{d}_m(\bar{z})\}$ is always linearly dependent. So, there are scalers, not all zero, $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m \in \overline{C}$ such that

$\sum_{i=1}^{m} \bar{\lambda}_i\bar{d}_i(\bar{z}) = 0$ or $\sum_{i=1}^{m} \lambda_i d_i(z) = 0 \bmod P$.

Since P is an arbitrary prime ideal and $\cap\{P \mid P$ is a prime ideal of Q$\} = \{0\}$, we have $\sum_{i=1}^{m} \lambda_i d_i(z) = 0$ for all $z \in Z(R)$. Thus $\sum_{i=1}^{m} \lambda_i d_i(Z(R)) = 0$.

If $\{[\bar{\alpha}(\bar{y}),\bar{x}], [\bar{d}_1(\bar{y}),\bar{x}], \ldots, [\bar{d}_{m-1}(\bar{y}),\bar{x}]\}$ is linearly dependent over $\overline{C}$, then there are scalars, not all zero, $\bar{\lambda}_0, \bar{\lambda}_1, \ldots, \bar{\lambda}_{m-1} \in \overline{C}$ such that

$$\bar{\lambda}_0[\bar{\alpha}(\bar{y}),\bar{x}] + \bar{\lambda}_1[\bar{d}_1(\bar{y}),\bar{x}] + \ldots + \bar{\lambda}_{m-1}[\bar{d}_{m-1}(\bar{y}),\bar{x}] = 0.$$

Suppose j is the highest index such that $\bar{\lambda}_j \neq 0$. Then the last expression can be rewritten as

$$\bar{\lambda}_0[\bar{\alpha}(\bar{y}),x] + \sum_{i=1}^{j} \bar{\lambda}_i[\bar{d}_i(\bar{y}),\bar{x}] = \bar{\lambda}_0\delta^0_{\bar{\alpha}(\bar{y})}(\bar{x}) + \sum_{i=1}^{j} \bar{\lambda}_i\delta^i_{\bar{d}_i(\bar{y})}(\bar{x}) = 0 \tag{3.30}$$

for all $\bar{x},\bar{y} \in \overline{Q}$, where $\delta^0_{\bar{\alpha}(\bar{y})}(\bar{x}) = [\bar{\alpha}(\bar{y}),(\bar{x})]$ and $\delta^i_{\bar{d}_i(\bar{y})}(\bar{x}) = [\bar{d}_i(\bar{y}),\bar{x}]$. From (3.30), we notice that $\{\delta^0_{\bar{\alpha}(\bar{y})}, \delta^1_{\bar{d}_1(\bar{y})}, \ldots, \delta^j_{\bar{d}_j(\bar{y})}\}$ satisfies $\overline{C}$ linear relation over $\overline{Q}$ of length $j+1$, then by [8] (corollary 1.4), there are $\bar{q}_0 = \bar{1}, \bar{q}_1, \ldots, \bar{q}_j \in \overline{Q}$ such that

$$\bar{q}_j\delta^0_{\bar{\alpha}(\bar{y})} + \sum_{i=1}^{j} \bar{q}_{j-i}\delta^i_{\bar{d}_i(\bar{y})} = 0. \tag{3.31}$$

Moreover,

$$\delta^0_{\bar{\alpha}(\bar{y})} = \delta_{\bar{q}_1} \text{ and } \delta^k_{\bar{d}_k(\bar{y})}(\bar{t}) = \delta_{\bar{q}_k}(\bar{t}) - \sum_{i=1}^{k-1} \bar{q}_i\delta^{k-i}_{\bar{d}_{k-i}(\bar{y})}(\bar{t}) \tag{3.32}$$

for all $\bar{t} \in \overline{Q}$, $2 \leq k \leq j$. If $\delta^0_{\bar{\alpha}(\bar{y})} = \delta_{\bar{q}_1}$, then $[\bar{\alpha}(\bar{y}), \bar{t}] = [\bar{q}_1, \bar{t}]$ for all $\bar{y}, \bar{t} \in \overline{Q}$. Replace $\bar{y}$ by $\bar{\alpha}^{-1}(\bar{t})$, we get $[\bar{q}_1, \bar{t}] = 0$ for all $\bar{t} \in \overline{Q}$ and thus $\bar{q}_1 \in \overline{C}$. Second expression of (3.32) can be rewritten as

$$[\bar{d}_k(\bar{y}), \bar{t}] = [\bar{q}_k, \bar{t}] - \sum_{i=1}^{k-1} \bar{q}_i [\bar{d}_{k-i}(\bar{y}), \bar{t}],$$

for all $\bar{y}, \bar{t} \in \overline{Q}$.

Replace $\bar{y}$ by $\bar{e}$ and using $\bar{d}_j(\bar{e}) = 0$ for all $j \geq 1$, we get $[\bar{q}_k, \bar{t}] = 0$ for all $\bar{t} \in \overline{Q}$.

Thus $\bar{q}_k \in \overline{C}$ for all $2 \leq k \leq j$.

So, for $k = j$ second expression of (3.32) reduces to

$$\sum_{i=0}^{j-1} \bar{q}_i \delta^{j-i}_{\bar{d}_{j-i}(\bar{y})}(\bar{t}) = 0. \tag{3.33}$$

Subtracting (3.33) from (3.31), we get

$$\bar{q}_j [\bar{\alpha}(\bar{y}), \bar{t}] = 0,$$

for all $\bar{y}, \bar{t} \in \overline{Q}$.

Since $\bar{q}_j \in \overline{C}$, we get $[\bar{\alpha}(\bar{y}), \bar{t}] = 0$ or $[\bar{y}, \bar{t}] = 0$ or $\bar{y}\bar{t} = \bar{t}\bar{y}$ in $\overline{Q}$. That is $yt - ty \in P$ for all $y, t \in Q$. Since P was an arbitrary prime ideal of Q and $\cap\{P \mid P$ is a prime ideal of Q$\} = \{0\}$ we get $yt - ty = 0$ for all $y, t \in Q$, thus Q is commutative. Since $R \subseteq Q$, we get R is commutative. □

Corollary 3.3. *Let n be a fixed positive integer and R be an $n!$-torsion free semiprime ring. Let d be a skew derivation of R such that $[d(x), x^n] \in Z(R)$ for all $x \in I$, where I be an ideal of R, then either $d(Z(R)) = 0$ or R is commutative.*

Proof. By putting $m = 1$ and $d_1 = d$ in Theorem (3.3), we get either R is commutative or $\{d(z)\}$, $z \in Z(R) \subseteq Z(Q) = C$ is linearly dependent over C. Since C is von Neumaan regular free module over C with dimension 1, we conclude that $d(z) = 0$. Since $z \in Z(R)$ was arbitrary we get $d(Z(R)) = 0$. □

Acknowledgments. The author is highly thankful to the referee for his/her several useful suggestions. This work is supported by the research grant DST-SERB EMR/2016/001550.

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