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General iterative algorithm for demicontractive-type mapping in real Hilbert spaces

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ABSTRACT. In this paper, we investigate the problem of finding a solution to fixed point problem involving demicontractive mappings in the framework of Hilbert spaces. Inspired by general iterative algorithm, a new iterative method for solving the problem is introduced. Strong convergence theorem of the proposed method is established without any compactness assumption. Our theorems are significant improvements on several important recent results.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle .,.\rangle$ and induced norm $\|\cdot\|$. An operator $A : H \to H$ is said to be *strongly positive bounded linear* if there exists a constant $k > 0$ such that

$$\langle Ax, x\rangle \geq k\|x\|^2, \ \forall\, x \in H.$$

An operator $A : H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall\, x, y \in H,$$

and it is called *k-strongly monotone* if there exists $k \in (0, 1)$ such that for each $x, y \in H$

$$\langle Ax - Ay, x - y\rangle \geq k\|x - y\|^2.$$

Remark 1.1. From the defintion of A, we note that strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k- strongly monotone operator.

Let K be a nonempty subset of H. A map $T : K \to K$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \ \forall x, y \in K, \tag{1.1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $Fix(T)$ the set of fixed points of the mapping T, that is $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. If T is nonexpansive mapping, it is well known $Fix(T)$ is closed and convex. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in Fix(T)$. The mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

A mapping $T : K \to K$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2, \forall x, y \in K.$$

A map T is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

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$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \forall x \in K, \;\; p \in Fix(T).$$

We note that the following inclusions hold for the classes of the mappings:
firmly nonexpansive$\subset$ nonexpansive$\subset$ quasi-nonexpansive$\subset$ k-strictly pseudo-contractive $\subset$ k-demi-contractive.

The function T in the following example is k-demi-contractive mapping but is not a k-strictly pseudo-contractive mapping.

Example 1.1. [5] Let $H = \mathbb{R}$ and $K = [-1, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{2}{3} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ \\ 0, & x = 0. \end{cases} \tag{1.2}$$

Clearly $Fix(T) = \{0\}$. For $x \in K$, we have

$$|Tx - 0|^2 = \left|\frac{2}{3} x \sin(\frac{1}{x})\right|^2 \leq \left|\frac{2}{3} x\right|^2 \leq |x|^2 \leq |x - 0|^2 + k|x - Tx|^2 \;\; \forall k \in [0, 1).$$

Thus T is k demi-contratcive for $k \in [0, 1)$. To see that T is not k strictly pseudo-contractive, choose $x = \dfrac{2}{\pi}$ and $y = \dfrac{2}{3\pi}$, then

$$|Tx - Ty|^2 > |x - y|^2 + k|x - y - (Tx - Ty)|^2.$$

Hence, T is not k strictly pseudo-contractive mapping for $k \in [0, 1)$.

For several years, fixed point problem involving demicontractive mappings has attracted, and continues to attract, the interest of several well known mathematicians due to the fact that many nonlinear problems can be reformulated as fixed point equations of demicontractive mappings (see, for example, Hicks and Kubicek [5], Wang et al. [14], Chidume and Maruster [3], Maruster [6], Boonchari and Saejung [1], Osilike [10] and the references therein).
On other hand, iterative methods for nonexpansive mappings have been applied to solve convex minimization problems; see, e.g., [11, 7] and the references therein. A typical problem is to minimize a quadratic function over the set of fixed point of nonexpansive mapping in a real Hilbert space:

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle. \tag{1.3}$$

In [11], Xu proved that the sequence $\{x_n\}$ defined iteratively from arbitrary initial guess $x_0 \in H$ by:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.4}$$

converges strongly to the unique solution of the minimization problem (1.3), where T is a nonexpansive mapping in H and A is a strongly positive bounded linear operator. In 2006, Marino and Xu [7] improved the result of Moudafi [8] by considering a general iterative method for nonexpansive mappings : let f be a contraction map on H and A : $H \to H$ be a strongly positive bounded linear operator. Let $\{x_n\}$ be the sequence defined iteratively from arbitrary initial guess $x_0 \in H$ by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \; n \geq 0. \tag{1.5}$$

They proved that the sequence $\{x_n\}$ converges strongly to the fixed point of T, which is the unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T),$$

under some appropriate conditions on γ and $\{\alpha_n\}$.

In this paper, motivated by above results, the fact that the class of demi-contractive mappings contains those of quasi-nonexpansive and strictly pseudo-contractive mappings as subclasses and general iterative algorithm is remarkably useful for solving most important problems with nonlinear operators, we construct and study an explicit iterative method and prove strong convergence theorems for approximating fixed points of demi-contractive mappings in the setting of a real Hilbert space which is a solution of some varitional inequality problems. Our result extends and improves the results of Marino and Xu [7], Xu [11] and many other authors.

2. Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Let H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H, and let $x \in H$. Weak convergence of x_n to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K , assigns to each $x \in H$ the unique $P_K x$ with the property

$$\|x - P_K x\| \leq \|y - x\|$$

for all $y \in K$. It is well know that P_K satisfies

$$\langle x - P_K x, y - P_K x\rangle \leq 0 \tag{2.6}$$

for all $y \in K$.

Definition 2.1. Let H be a real Hilbert space and $T : D(T) \subset H \to H$ be a mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.

Lemma 2.1 ([2]). *Let H be a real Hilbert space. Then, for any $x, y \in H$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$
$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2, \ \ \lambda \in (0, 1).$$

Lemma 2.2 (Xu, [12]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\displaystyle\limsup_{n\to\infty} \sigma_n \leq 0$ or $\displaystyle\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. Then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [13] *Let K be a nonempty, closed convex subset of be a real Hilbert space H. Let $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\Big\{1, \dfrac{1}{\tau}\Big\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|, \ \ x, y \in K.$$

Lemma 2.4 ([9], Proposition 2.1). *Assume K is a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a self-mapping of C. If T is a k-demicontractive mapping, then the fixed point set $Fix(T)$ is closed and convex.*

Lemma 2.5. [9] *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|.$$

(ii) If T is a k-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at 0.

3. Main results

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty, closed convex subset of a real Hilbert space H and $A : K \to H$ be an k-strongly monotone and L-Lipschitzian operator. Let $f : K \to H$ be an b-Lipschitzian mapping with a constant $b > 0$. Let $T : K \to K$ be a k- demi-contractive mapping such that $Fix(T) \neq \emptyset$. Assume that $0 < \eta < \dfrac{2k}{L^2}$, $0 < \gamma b < \tau$, where $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$ and $I - T$ is demiclosed at the origin. Let $\{x_n\}$ and $\{y_n\}$ be sequences defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = P_K\Big(\alpha_n \gamma f(x_n) + (I - \eta\alpha_n A)y_n\Big), \end{cases} \tag{3.7}$$

with $\beta_n \in]k, 1[$ such that
$\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lim_{n\to\infty} \inf(\beta_n - k)(1-\beta_n) > 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.7) converge strongly to $x^* \in Fix(T)$, which is a unique solution of the following variational inequality:

$$\langle \eta Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T). \tag{3.8}$$

Proof. We first show that the uniqueness of a solution of variational inequality (3.8). Suppose both $x^* \in Fix(T)$ and $x^{**} \in Fix(T)$ are solutions to (3.8), then

$$\langle \eta Ax^* - \gamma f(x^*), x^* - x^{**}\rangle \leq 0 \tag{3.9}$$

and

$$\langle \eta Ax^{**} - \gamma f(x^{**}), x^{**} - x^*\rangle \leq 0. \tag{3.10}$$

Adding up (3.9) and (3.10) yields

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), x^{**} - x^*\rangle \leq 0. \tag{3.11}$$

$$\frac{L^2\eta}{2} > 0 \Longleftrightarrow k - \frac{L^2\eta}{2} < k \Longleftrightarrow \eta\Big(k - \frac{L^2\eta}{2}\Big) < k\eta \Longleftrightarrow \tau < k\eta.$$

It follows that

$$0 < b\gamma < \tau < k\eta.$$

Noticing that

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), x^{**} - x^*\rangle \geq (k\eta - b\gamma)\|x^* - x^{**}\|^2,$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. Let t_0 be a fixed real number such that $t_0 \in \left(0, min\left\{1\ , \frac{1}{\tau}\right\}\right)$. We observe that $P_{Fix(T)}(I + (t_0\gamma f - t_0\eta A))$ is a contraction. Indeed, for all $x, y \in K$, by Lemma 2.3, we have

$$\|P_{Fix(T)}(I + (t_0\gamma f - t_0\eta A))x - P_{Fix(T)}(I + (t_0\gamma f - t_0\eta A))y\|$$

$$\leq \|(I + (t_0\gamma f - t_0\eta A))x \quad - (I + (t_0\gamma f - t_0\eta A))y\|$$

$$\leq t_0\gamma\|f(x) - f(y)\| \quad + \|(I - t_0\eta A)x \quad - (I - t_0\eta A)y\| \leq (1 - t_0(\tau - b\gamma))\|x - y\|.$$

Banach's Contraction Mapping Principle guarantees that $P_{Fix(T)}(I + (t_0\gamma f - t_0\eta A))$ has a unique fixed point, say $x_1 \in K$. That is, $x_1 = P_{Fix(T)}(I + (t_0\gamma f - t_0\eta A))x_1$. Thus, by inequality (2.6), it is equivalent to the following variational inequality problem

$$\langle \eta A x_1 - \gamma f(x_1), x_1 - p\rangle \leq 0, \quad \forall \ \ p \in Fix(T).$$

By the uniqueness of the solution of (3.8), we have $x_1 = x^*$.
In what follows, we denote x^* to be the unique solution of (3.8). Without loss of generality, we can assume $\alpha_n \in \left(0, min\{1\ , \frac{1}{\tau}\}\right)$. We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. By using (3.7) and Lemma 2.1, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \left\|\beta_n(x_n - x^*) + (1 - \beta_n)(Tx_n - x^*)\right\|^2 \\ &= \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|Tx_n - x^*\|^2 - \beta_n(1 - \beta_n)\|Tx_n - x_n\|^2.\end{aligned}$$

Using the fact that, T is k-demi-contractive, we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\Big(\|x_n - x^*\|^2 + k\|Tx_n - x_n\|^2\Big) - \beta_n(1 - \beta_n)\|Tx_n - x_n\|^2 \\ &\leq \|x_n - x^*\|^2 - (1 - \beta_n)(\beta_n - k)\|Tx_n - x_n\|^2. \qquad \text{(A)}\end{aligned}$$

Since $\beta_n \in]k, 1[$, we have,

$$\|y_n - x^*\Big\| \leq \|x_n - x^*\Big\|. \tag{3.12}$$

By Lemma 2.3 and (3.12), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|P_K\big(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n\big) - x^*\| \\ &\leq \|\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n - x^*\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(x^*)\| + (1 - \tau\alpha_n)\|y_n - x^*\| + \alpha_n\|\gamma f(x^*) - \eta A x^*\| \\ &\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - \eta A x^*\| \\ &\leq \max\{\|x_n - x^*\|, \frac{\|\gamma f(x^*) - \eta A x^*\|}{\tau - b\gamma}\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \frac{\|\gamma f(x^*) - \eta A x^*\|}{\tau - b\gamma}\}, \quad n \geq 1.$$

Hence, $\{x_n\}$ is bounded also are $\{f(x_n)\}$, and $\{Ax_n\}$.
Consequently, by Lemma 2.3, inequality (A) and property of β_n, we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \|\alpha_n(\gamma f(x_n)-\eta Ax^*)+(I-\eta\alpha_n A)(y_n-x^*)\|^2\\ &\leq \alpha_n^2\|\gamma f(x_n)-\eta Ax^*\|^2+(1-\tau\alpha_n)^2\|y_n-x^*\|^2\\ &\quad +2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta Ax^*\|\|y_n-x^*\|\\ &\leq \alpha_n^2\|\gamma f(x_n)-\eta Ax^*\|^2+(1-\tau\alpha_n)^2\|x_n-x^*\|^2\\ &\quad -(1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-k)\|Tx_n-x_n\|^2\\ &\quad +2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta Ax^*\|\|x_n-x^*\|.\end{aligned}$$

Thus,

$$\begin{aligned}(1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-k)\|Tx_n-x_n\|^2 \leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n^2\|\gamma f(x_n)-\eta Ax^*\|^2\\ +2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta Ax^*\|\|x_n-x^*\|.\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, then there exists a constant $C>0$ such that

$$(1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-k)\|Tx_n-x_n\|^2 \leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n C. \quad (3.13)$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n-x^*\|\}$ is monotonically decreasing sequence. Then $\{\|x_n-x^*\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\Big[\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\Big]=0.$$

It then implies from (3.13) that

$$\lim_{n\to\infty}(1-\beta_n)(\beta_n-k)\|Tx_n-x_n\|^2=0. \quad (3.14)$$

Since $\beta_n\in]k,1[$ and $\lim_{n\to\infty}\inf(\beta_n-k)(1-\beta_n)>0$, we have

$$\lim_{n\to\infty}\Big\|x_n-Tx_n\Big\|=0. \quad (3.15)$$

Next, we prove that $\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),x^*-x_n\rangle\leq 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),x^*-x_n\rangle=\lim_{j\to+\infty}\langle\eta Ax^*-\gamma f(x^*),x^*-x_{n_j}\rangle.$$

From (3.15) and the fact that $I-T$ is demiclosed, we obtain $a\in Fix(T)$. On other hand, the fact that x^* solves (3.8), we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),x^*-x_n\rangle &= \lim_{j\to+\infty}\langle\eta Ax^*-\gamma f(x^*),x^*-x_{n_j}\rangle\\ &= \langle\eta Ax^*-\gamma f(x^*),x^*-a\rangle\leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$.

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|P_K(\alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n)-x^*\|^2\\
&\le \langle \alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n-x^*, x_{n+1}-x^*\rangle\\
&= \langle \alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)y_n-x^*-\alpha_n\gamma f(x^*)+\alpha_n\gamma f(x^*)-\alpha_n\eta Ax^*\\
&\quad +\alpha_n\eta Ax^*, x_{n+1}-x^*\rangle\\
&\le \Big(\alpha_n\gamma\|f(x_n)-f(x^*)\|+\|(I-\alpha_n\eta A)(y_n-x^*)\|\Big)\|x_{n+1}-x^*\|\\
&\quad +\alpha_n\langle \eta Ax^*-\gamma f(x^*), x^*-x_{n+1}\rangle\\
&\le (1-\alpha_n(\tau-b\gamma))\|x_n-x^*\|\|x_{n+1}-x^*\|+\alpha_n\langle \eta Ax^*-\gamma f(x^*), x^*-x_{n+1}\rangle\\
&\le (1-\alpha_n(\tau-b\gamma))\|x_n-x^*\|^2+2\alpha_n\langle \eta Ax^*-\gamma f(x^*), x^*-x_{n+1}\rangle.
\end{aligned}$$

From Lemma 2.2, its follows that $x_n \to x^*$.
Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing sequence. Set $B_n=\|x_n-x^*\|^2$ and $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) by $\tau(n)=\max\{k\in\mathbb{N}: k\le n,\ \ B_k\le B_{k+1}\}$.
We have τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $B_{\tau(n)}\le B_{\tau(n)+1}$ for $n\ge n_0$. Let $i\in\mathbb{N}^*$, from (3.13), we have

$$(1-\tau\alpha_{\tau(n)})^2(1-\beta_{\tau(n)})(\beta_{\tau(n)}-k)\|x_{\tau(n)}-Tx_{\tau(n)}\|^2\le \alpha_{\tau(n)}C.$$

Since $\beta_n\in]k,1[$, it follows that

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0. \tag{3.16}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in H and $\limsup_{\tau(n)\to+\infty}\langle \eta Ax^*-\gamma f(x^*), x^*-x_{\tau(n)})\rangle\le 0$. We have for all $n\ge n_0$,

$$0\le\|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2\le\alpha_{\tau(n)}[-(\tau-b\gamma)\|x_{\tau(n)}-x^*\|^2+2\langle \eta Ax^*-\gamma f(x^*), x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2\le\frac{2}{\tau-b\gamma}\langle \eta Ax^*-\gamma f(x^*), x^*-x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2=0.$$

Therefore,

$$\lim_{n\to\infty}B_{\tau(n)}=\lim_{n\to\infty}B_{\tau(n)+1}=0.$$

Furthermore, for all $n\ge n_0$, we have $B_{\tau(n)}\le B_{\tau(n)+1}$ if $n\ne\tau(n)$ (that is, $n>\tau(n)$); because $B_j>B_{j+1}$ for $\tau(n)+1\le j\le n$. As consequence, we have for all $n\ge n_0$,

$$0\le B_n\le\max\{B_{\tau(n)},\ B_{\tau(n)+1}\}=B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty}B_n=0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

By using Theorem 3.1, we have the following strong convergence results for computing fixed point of strictly pseudo-contractive mappings without demiclosedness assumption.

Theorem 3.2. *Let K be a nonempty, closed convex subset of a real Hilbert space H and $A:K\to H$ be an k-strongly monotone and L-Lipschitzian operator. Let $f:K\to H$ be an b-Lipschitzian mapping with a constant $b\ge 0$. Let $T:K\to K$ be a k- strictly pseudo-contractive mapping such* *that* $Fix(T) \neq \emptyset$. *Assume that* $0 < \eta < \dfrac{2k}{L^2}$, $0 < \gamma b < \tau$, *where* $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = P_K\Big(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n\Big), \end{cases} \tag{3.17}$$

with $\beta_n \in]k,1[$ *such that*
$\{\alpha_n\}$ *be a real sequence in* $(0,1)$ *satisfying:*
$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty,$
$(iii)\ \lim_{n\to\infty} \inf(\beta_n - k)(1-\beta_n) > 0$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by* (3.17) *converge strongly to* $x^* \in Fix(T)$, *which is a unique solution of the following variational inequality:*

$$\langle \eta Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T). \tag{3.18}$$

Proof. Since every strictly pseudo-contractive mapping is demi-contractive, then the proof follows Lemma 2.5 and Theorem 3.1. □

We now apply Theorem 3.1 for solving constrained minimization problem over the set of fixed points of demi-contractive mappings.

Theorem 3.3. *Let* K *be a nonempty, closed convex subset of a real Hilbert space* H *and, let* $A : K \to H$ *be strongly bounded linear operator with coefficient* $k > 0$. *Let* $f : K \to H$ *be an b-Lipschitzian mapping with a constant* $b > 0$. *Let* $T : K \to K$ *be a k- strictly pseudo-contractive mapping such that* $Fix(T) \neq \emptyset$. *Assume that* $0 < \eta < \dfrac{2k}{\|A\|^2}$, $0 < \gamma b < \tau$, *where* $\tau = \eta\Big(k - \dfrac{\|A\|^2\eta}{2}\Big)$ *and* $I - T$ *is demiclosed at the origin. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences defined iteratively from arbitrary* $x_0 \in K$ *by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = P_K\Big(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)y_n\Big), \end{cases} \tag{3.19}$$

with $\beta_n \in]k,1[$ *such that*
$\{\alpha_n\}$ *be a real sequence in* $(0,1)$ *satisfying:*
$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty,$
$(iii)\ \lim_{n\to\infty} \inf(\beta_n - k)(1-\beta_n) > 0$. *Then, the sequence* $\{x_n\}$ *generated by* (3.19) *converges strongly to* $x^* \in Fix(T)$, *which satisfies the optimality condition of the minimization problem:*

$$\min_{x\in Fix(T)} \frac{\eta}{2}\langle Ax, x\rangle - h(x), \tag{3.20}$$

where h *is a potential function for* γf *(i.e.* $h'(x) = \gamma f(x)$ *on* K *).*

Proof. We note that strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k- strongly monotone operator. The proof follows Theorem 3.1. □

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