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A modified Halpern extragradient method for equilibrium and fixed point problems in CAT(0) space

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ABSTRACT. In this paper, we introduce a modified Halpern extragradient-type algorithm for approximating an element in the intersection of the set of common solutions of equilibrium problems and common fixed points of family of nonexpansive mappings in a complete CAT(0) space. We establish strong convergence theorem that improve and generalize recently announced results in the literature.

1. INTRODUCTION

Let (X, d) be a metric space. For $x, y \in X$, let $d(x, y) = l$. A geodesic path from x to y is an isometry $\tilde{c} : [0, l] \to \tilde{c}([0, l]) \subset X$ satisfying $\tilde{c}(0) = x,\ \tilde{c}(l) = y$. The image of a geodesic path between two points is known as a geodesic segment. A Metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic segment.

A geodesic triangle, denoted by $\triangle(x_1, x_2, x_3)$, in a geodesic space consists of three points x_1, x_2, x_3 and three geodesic segments joining each pair of points. A comparison triangle of a geodesic triangle $\triangle(x_1, x_2, x_3)$, represented by $\overline{\triangle}(x_1, x_2, x_3)$ or $\triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$, is a triangle in the Euclidean plane $\mathbb{R}^2$ such that $d(\overline{x_i}, \overline{x_j}) = d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j \in \{1, 2, 3\}$. A geodesic segment joining two points x, y in a geodesic space (X, d) is denoted by $[x, y]$ and we represent every point say $z \in [x, y]$ by $\sigma x \oplus (1 - \sigma)y$ where $\sigma \in [0, 1]$, that is, $[x, y] := \{\sigma x \oplus (1 - \sigma)y : \sigma \in [0, 1]\}$ (see, [2]).

A geodesic space is a $CAT(0)$ space if for every geodesic triangle $\triangle$ and its comparison triangle $\overline{\triangle}$, the following inequality hold:

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})\ \forall x, y \in \triangle,\ \overline{x}, \overline{y} \in \overline{\triangle}.$$

A complete $CAT(0)$ space is known as Hadamard space. Examples of $CAT(0)$ spaces include Hilbert spaces, Euclidean spaces $\mathbb{R}^n$, $\mathbb{R}$-trees, the complex Hilbert ball equipped with hyperbolic metric among others. See for example [6, 22] for more details on these spaces .

Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for a bifunction f is find;

$$x^* \in C \text{ such that } f(x^*, z) \geq 0\ \forall z \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f, C)$. Problem (1.1) which was originally studied in [7] includes, as a special cases, many important Mathematical problems such as optimization problems, variational inequality problems, saddle point problems, Nash equilibrium problems and other problems of interest in many applications.

Definition 1.1. A mapping $T : X \to X$ is called nonexpansive if

$$d(Tx, Ty) \leq (x, y), \forall x, y \in X.$$

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A point $x \in X$ is called a fixed point of the map T if $Tx = x$. Denote by $F(T)$ the set of fixed point of the map T.

It is known, see for example [10], that for any $x \in X$ there exists a unique point $\tilde{x} \in C$ such that $d(x, \tilde{x}) = min_{y \in C} d(x, y)$. The mapping $P_C : X \to C$ defined by $P_C x = \tilde{x}$ is called the metric projection from X onto C. The basic properties of projection are summarized in the following results;

Theorem 1.1. ([4]) *Let C be a closed convex subset of a complete $CAT(0)$ space X. Then*

a) *for any $x \in X$, there exists a unique point $P_C x \in C$ such that*
$$d(x, P_C x) = d(x, C).$$

b) *for $x \in X$ and $y \in C$,*
$$d^2(x, P_C x) + d^2(P_C x, y) \leq d^2(x, y).$$

c) *the mapping P_C is a nonexpansive mapping from X onto C, that is, for any $x, y \in X$*
$$d(P_C x, P_C y) \leq d(x, y).$$

Definition 1.2. ([20]) A function $f : X \to (-\infty, +\infty]$ is called

i) convex if
$$f((1-\sigma)x \oplus \sigma y) \leq (1-\sigma)f(x) + \sigma f(y) \ \forall x, y \in X \text{ and } \sigma \in [0, 1].$$

ii) strictly convex if
$$f((1-\sigma)x \oplus \sigma y) < (1-\sigma)f(x) + \sigma f(y) \ \forall x, y \in X \ x \neq y \text{ and } \sigma \in [0, 1].$$

Remark 1.1. Observed that if f is strictly convex, then the minimizer of f is unique.

Definition 1.3. ([18]) Let X be a metric space and $\{x_n\}_{n=1}^{\infty}$ be any bounded sequence in X. For $x \in X$, set $r(x, \{x_n\}) := \limsup_{n \to \infty} d(x_n, x)$, then

- the asymptotic radius of the sequence $\{x_n\} \subseteq X$ denoted by $r(\{x_n\})$ is defined by
$$r(\{x_n\}) = \inf_{x \in X} r(\{x_n\}, x).$$
- the asymptotic center of $\{x_n\} \subseteq X$ is a set
$$A(\{x_n\}) = \{z \in X : r(z, \{x_n\}) = r(\{x_n\})\}.$$

A sequence $\{x_n\} \subseteq X$ is said to Δ-converge to x if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfies the condition that
$$A(\{x_{n_k}\}) = \{x\}.$$

That is to say a sequence $\{x_n\} \subseteq X$ Δ-converges to a point $x \in X$ if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and this is written as $\Delta - \lim_{n \to \infty} x_n = x$ (see [3]).

Concerning Δ-convergence and asymptotic center of a sequence $\{x_n\} \subseteq X$, we have the following result for nonexpansive mapping in $CAT(0)$ spaces.

Lemma 1.1. *([13]) Let C be a nonempty closed and convex subset of a complete $CAT(0)$ space X, $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n \to \infty} x_n = x^*$. Then $x^* = Tx^*$.*

In [5] Berg and Nikolaev introduced the notion of quasilinearizaton in $CAT(0)$ space.

Definition 1.4. ([5]) Let X be a $CAT(0)$ space and $(a,b) \in X \times X$. Then quasilinearization is a map $\langle , \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}d^2(a,d) + \frac{1}{2}d^2(b,c) - \frac{1}{2}d^2(a,c) - \frac{1}{2}d^2(b,d), \forall a,b,c,d \in X.$$

It can easily be checked that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ $\forall a,b,c,d,e \in X$. We say that the spacce X satisfies Chauchy-Schwartz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \ \forall a,b,c,d \in X.$$

Kakavandi and Amini [14], based on the work of Berg and Nikolaev [5], introduced the concept of duality in a complete $CAT(0)$ space X.

Definition 1.5. ([1]) Consider the map $G : \mathbb{R} \times X \times X \to C(X)$ defined by

$$G(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \ t \in \mathbb{R}, a, b, x \in X,$$

where $C(X,\mathbb{R})$ is a space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that the map $G(t,a,b)$ is a Lipschitz map with Lipschitz semi-norm $L(G(t,a,b)) = td(a,b), \forall\, t \in \mathbb{R}\; and\; a,b \in X$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,\ y \in X,$ $x \neq y\ \}$ is the semi-norm for any function $\varphi : X \to \mathbb{R}$.

Define a map $\hat{D}$ on $\mathbb{R} \times X \times X$ by

$$\hat{D}((t,a,b),(s,c,d)) = L(G(t,a,b) - G(s,c,d)), \forall\, t,s \in \mathbb{R}, a,b,c,d \in X.$$

Clearly $\hat{D}$ is a pseudometric.

A relation $\sim$ on $\mathbb{R} \times X \times X$ defined by $(t,a,b) \sim (s,c,d)$ if $\hat{D}((t,a,b),(s,c,d)) = 0$ is an equivalence relation, where the equivalence class of (t,a,b) is given as

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle, x, y \in X\}.$$

We denote by $X^* := \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ the set of all equivalence classes of (t,a,b). This together with the metric $\hat{D}$ on X^* is called the dual space of (X,d).

Techniques of solving Equilibrium problems and their generalizations have been very important tools for solving problems arising in the areas of linear or nonlinear programming, variational inequalities, optimization problems, fixed point problems and so on. It has been widely applied to physics, structural analysis, management sciences and economics, e.t.c., see for example [1, 9, 14, 23, 24]. Various methods have been used to study equilibrium problems, one of such is proximal point algorithm which was used in [17] to study the existence of solutions of equilibrium problems. Other methods includes extragradient method which was introduced in [25] by Quoc et al. in the setting of Hilbert spaces. They studied the following scheme;

$$\begin{cases} z_n \in \text{Argmin}_{z \in C}\{f(x_n,z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}, \\ x_{n+1} \in \text{Argmin}_{z \in C}\{f(z_n,z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}. \end{cases} \tag{1.2}$$

Under certain assumption, weak convergence of the sequence $\{x_n\}$ generated by (1.2) to a solution of some equilibrium problem has been established. In recent time, several authors have extended the notion of equilibrium to Hadamard spaces, see for example [8, 11, 12, 25, 26].

Khatibzadeh and Mohebbi [16] studied both $\Delta-$convergence and strong convergence of a sequence generated by the Extragradient Method for pseudo-monotone equilibrium problems in a complete $CAT(0)$ space.

In [17], the authors studied the existence of solutions of equilibrium problems associated with pseudo-monotone bifunctions with some conditions on the bifunctions in Hadamard spaces. They proved $\Delta-$convergence theorem of the sequence generated by the proximal point algorithm to an equilibrium point of the pseudo-monotone bifunction. They also, under some additional assumptions on the bifunction, proved strong convergence theorem.

Lusem and Mohebbi [15] proposed an Extragradient Method with line search for solving equilibrium problems of pseudo-monotone type in the setting of Hadamard space. They also proved $\Delta-$convergence and strong convergence theorems.

Very recently, Moharami and Eskandani [20] proposed the following extragradient type algorithm for finding a common element of the set of solutions of an equilibrium problem for a single bifunction f and a common zero of a finite family of monotone operators $A_1, A_2, \cdots, A_N$ in Hadamard spaces;

$$\begin{cases} w_n = J^{A_N}_{\beta_n^N} \circ J^{A_{N-1}}_{\beta_n^{N-1}} \circ \cdots \circ J^{A_1}_{\beta_n^1} x_n, \\ y_n = \underset{y \in K}{\operatorname{argmin}} \{f(w_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y)\}, \\ r_n = \underset{y \in K}{\operatorname{argmin}} \{f(y_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y)\}, \\ x_{n+1} = \alpha_n w \oplus (1-\alpha_n) r_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences satisfying some conditions. They proved strong convergence theorem of the sequence $\{x_n\}$ generated by the above scheme.

In this paper, motivated and inspired by the above works, we proposed and study an extragrdient type algorithm for approximating a common element of the set of solutions of equilibrium problems for finite family of bifunctions and the set of fixed points of family of nonexpansive mappings in a complete $CAT(0)$ space.

2. Preliminaries

The following notions and results are very vital in our subsquent discussion.

Definition 2.6. Let X be a complete $CAT(0)$ space and $f : D(f) \subseteq X \to \mathbb{R}$ be a function (D(f) denotes the domain of f). Then f is said to be $\Delta-$upper semicontinuous at some point $x_0 \in D(f)$ if

$$f(x_0) \geq \limsup f(x_n)$$

for every sequence $\{x_n\} \subseteq D(f)$ satisfying the condition that $\Delta - \lim_{n \to \infty} x_n = x_0$. We say that f is $\Delta-$upper semicontinuous on $D(f)$ if it is $\Delta-$upper semicontinuous at every point in $D(f)$.

Definition 2.7. ([20]) Let X be a complete $CAT(0)$ space. A bifunction $f : X \times X \to \mathbb{R}$ is said to be

1. monotone if
$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in X.$$
2. pseudo-monotone if for every $x, y \in X$
$$f(x, y) \geq 0 \text{ implies } f(y, x) \leq 0.$$

In [20], f is assumed to satisfy the following conditions;

$A_1:$ $f(x,.) : X \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in X$.

$A_2:$ $f(.,y) : X \to \mathbb{R}$ is $\Delta-$upper semicontinuous for all $y \in X$.

$A_3:$ f is Lipschitz-type continuous, that is there exist two positive constant d_1 and d_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - d_1 d^2(x,y) - d_2 d^2(y,z), \; \forall x,y,z \in X.$$

$A_4:$ f is pseudo-monotone.

In this paper, we assume f satisfies $A_1 - A_4$.

Lemma 2.2. (see [13]) *Let $(X,\ d)$ be a complete $CAT(0)$ space, $r,x,y,z \in X$ and $t \in [0,1]$. Then,*

1. $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$,
2. $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z)$,
3. $d(tx \oplus (1-t)y, tr \oplus (1-t)z) \leq td(x,r) + (1-t)d(y,z)$,
4. $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$.

Lemma 2.3. (see [28]) *Let $(X,\ d)$ be a $CAT(0)$ space, $r,z \in X$ and $\lambda_i \in [0,1]$, $i = 1,2,3,...,n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then,*

1. $d(\bigoplus_{i=1}^{n} \lambda_i r_i, z) \leq \sum_{i=1}^{n} \lambda_i d(r_i, z)$,
2. $d^2(\bigoplus_{i=1}^{n} \lambda_i r_i, z) \leq \sum_{i=1}^{n} \lambda_i d^2(r_i, z) - \lambda_i \lambda_j d^2(r_i, r_j)$, *for* $i,j \in \{1,2,\cdots,n\}$.

Lemma 2.4. ([18]) *Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence .*

Lemma 2.5. ([29]) *Let $\{b_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\delta_n\}$ be a sequence of real numbers. Suppose*

$$b_{n+1} \leq (1-\alpha_n)b_n + \alpha_n \delta_n, n \geq 0 \text{ and } \limsup_{n \to \infty} \delta_n \leq 0.$$

Then $\lim_{n \to \infty} b_n = 0.$

Lemma 2.6. ([21]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ satisfying $a_{n_j} < a_{n_j+1}$ $\forall j \geq 0$. Let $\{m_k\} \subset \mathbb{N}$ be defined by*

$$m_k = max\{i \leq k : a_i < a_{i+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence satisfying $\lim_{k \to \infty} m_k = \infty$ and for all $k \geq n_0$, the following two estimates hold;

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

3. Main results

In this section, we propose the following Halpern extragradient algorithm for family of nonexpansive mappings and equilibrium problems.

$$\begin{cases} u, x_1 \in X \text{chosen arbitrarily}, \\ z_n^i = \underset{y \in C}{\text{argmin}}\{f_i(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1,2,3,\cdots,N\}, \\ y_n^i = \underset{y \in C}{\text{argmin}}\{f_i(z_n^i, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1,2,3,\cdots,N\}, \\ i_n = \text{argmax}\{d^2(y_n^i, x_n),\ i = 1,2,3,\cdots,N\}, \overline{y}_n = y_n^{i_n}, \\ w_n = \gamma_{n,0}\overline{y}_n \bigoplus_{j=1}^{m} \gamma_{n,j} T_j x_n, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) w_n, n \geq 1, \end{cases} \tag{3.3}$$

where $\{\beta_n\}$, $\{\gamma_{n,j}\}$ and $\{\lambda_n\}$ are sequences satisfying the following conditions;

$C1 : \{\beta_n\} \subset (0,1)$ such that $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,
$C2 : \{\gamma_{n,j}\} \subset [\sigma, 1-\sigma]$ for some $\sigma \in (0, \frac{1}{2})$ with $\gamma_{n,0} + \sum_{j=1}^{m} \gamma_{n,j} = 1$,
$C3 : 0 < \gamma \leq \lambda_n \leq \beta < min\{\frac{1}{2d_{1,i}}, \frac{1}{2d_{2,i}}\}$ and $n = 1, 2, 3, \cdots$.

Lemma 3.7. *Let C be a nonempty closed convex subset of a Complete $CAT(0)$ space X. For each $i = 1, 2, \cdots, N$, let $f_i : C \times C \to \mathbb{R}$ be bifunctions satisfying the conditions $A_1 - A_4$ and $T_j : C \to C$, $j \in \{1, 2, \cdots, m\}$ be a family of nonexpansive mappings with $T_0 = I$ (the identity mapping) such that $F = (\bigcap_{j=1}^{m} F(T_j)) \bigcap (\bigcap_{i=1}^{N} EP(f_i, C)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by algorithm* (3.3). *Then*

$$d^2(\overline{y}_n, \hat{q}) \leq d^2(x_n, \hat{q}) - (1-2d_{1,i}\lambda_n) d^2(x_n, z_n^i) - (1-2d_{2,i}\lambda_n) d^2(z_n^i, \overline{y}_n),\ \forall\, i \in \{1, 2, \cdots, N\}.$$

Proof. Let $\hat{q} \in F$. Since for each $i = 1, 2, 3, \cdots, N$ y_n^i solves the minimization problem in algorithm (3.3). Then, setting $y = ty_n^i \oplus (1-t)\hat{q}$ for $i \in \{1, 2, 3, \cdots, N\}$, $t \in [0, 1)$ and using Lemma 2.2(2), we have

$$\begin{aligned} f_i(z_n^i, y_n^i) + \frac{1}{2\lambda_n} d^2(x_n, y_n^i) &\leq f_i(z_n^i, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \\ &\leq t f_i(z_n^i, y_n^i) + (1-t) f_i(z_n^i, \hat{q}) + \frac{1}{2\lambda_n}\Big(t d^2(x_n, y_n^i) \\ &\quad + (1-t) d^2(x_n, \hat{q}) - t(1-t) d^2(y_n^i, \hat{q})\Big), \end{aligned} \tag{3.4}$$

for each $i \in \{1, 2, \cdots, N\}$.

Observe here that for each $i \in \{1, 2, \cdots, N\}$ $f_i(\hat{q}, z_n^i) \geq 0$, so it follows from pseudomonotonicity property of $f_i^{\prime s}$ that

$$(1-t) f_i(z_n^i, y_n^i) \leq \frac{1}{2\lambda_n}\Big(- (1-t) d^2(x_n, y_n^i) + (1-t) d^2(x_n, \hat{q}) - t(1-t) d^2(y_n^i, \hat{q})\Big),$$

for each $i \in \{1, 2, \cdots, N\}$, from which we obtain

$$f_i(z_n^i, y_n^i) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, \hat{q}) - d^2(x_n, y_n^i) - t d^2(y_n^i, \hat{q})\Big), \forall i \in \{1, 2, \cdots, N\}. \tag{3.5}$$

Allowing $t \to 1^-$ in (3.5), we get

$$f_i(z_n^i, y_n^i) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, \hat{q}) - d^2(x_n, y_n^i) - d^2(y_n^i, \hat{q})\Big), \forall i \in \{1, 2, \cdots, N\}. \tag{3.6}$$

Similarly, since z_n^i solves the minimization problem in algorithm (3.3) for each $i = 1, 2, 3, \cdots, N$, then setting $y = tz_n^i \oplus (1-t) y_n^i$ for $i \in \{1, 2, 3, \cdots, N\}$, $t \in [0, 1)$ and using Lemma 2.2(2) we obtain

$$\begin{aligned} f_i(x_n, z_n^i) + \frac{1}{2\lambda_n} d^2(x_n, z_n^i) \leq t f_i(x_n, z_n^i) + (1-t) f_i(x_n, y_n^i) + \frac{1}{2\lambda_n}\Big(t d^2(x_n, z_n^i) \\ + (1-t) d^2(x_n, y_n^i) - t(1-t) d^2(z_n^i, y_n^i)\Big), \end{aligned}$$

for each $i \in \{1, 2, \cdots, N\}$, so that

$$f_i(x_n, z_n^i) - f_i(x_n, y_n^i) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, y_n^i) - d^2(x_n, z_n^i) - t d^2(z_n^i, y_n^i)\Big), \tag{3.7}$$

for each $i \in \{1, 2, \cdots, N\}$.

Setting $t \to 1^-$ in (3.7), we obtain

$$f_i(x_n, z_n^i) - f_i(x_n, y_n^i) \le \frac{1}{2\lambda_n}\Big(d^2(x_n, y_n^i) - d^2(x_n, z_n^i) - d^2(z_n^i, y_n^i)\Big), \tag{3.8}$$

for each $i \in \{1, 2, \cdots, N\}$.

By condition A_3, each $f_i^{'s}$ is Lipschitz-type continuous and so there exists two positive constants $d_{1,i}$, $d_{2,i}$ such that

$$f_i(x_n, z_n^i) - f_i(x_n, y_n^i) \ge -f_i(z_n^i, y_n^i) - d_{1,i}d^2(x_n, z_n^i) - d_{2,i}d^2(z_n^i, y_n^i), \forall i \in \{1, 2, \cdots, N\}.$$

Hence,

$$-d_{1,i}d^2(x_n, z_n^i) - d_{2,i}d^2(z_n^i, y_n^i) - f_i(z_n^i, y_n^i) \le f_i(x_n, z_n^i) - f_i(x_n, y_n^i), \forall i \in \{1, 2, \cdots, N\}. \tag{3.9}$$

Combining (3.8) and (3.9) we have

$$-d_{1,i}d^2(x_n, z_n^i) - d_{2,i}d^2(z_n^i, y_n^i) + \frac{1}{2\lambda_n}\Big(d^2(x_n, z_n^i) + d^2(z_n^i, y_n^i) - d^2(x_n, y_n^i)\Big) \le f_i(z_n^i, y_n^i),$$

for each $i \in \{1, 2, \cdots, N\}$.

This implies

$$(\frac{1}{2\lambda_n} - d_{1,i})d^2(x_n, z_n^i) + (\frac{1}{2\lambda_n} - d_{2,i})d^2(z_n^i, y_n^i) - \frac{1}{2\lambda_n}d^2(x_n, y_n^i) \le f_i(z_n^i, y_n^i), \tag{3.10}$$

for each $i \in \{1, 2, \cdots, N\}$.

(3.6) and (3.10) give

$$(\frac{1 - 2d_{1,i}\lambda_n}{2\lambda_n})d^2(x_n, z_n^i) + (\frac{1 - 2d_{2,i}\lambda_n}{2\lambda_n})d^2(z_n^i, y_n^i) \le \frac{1}{2\lambda_n}\Big(d^2(x_n, \hat{q}) - d^2(y_n^i, \hat{q})\Big)$$

for each $i \in \{1, 2, \cdots, N\}$, from which we get

$$d^2(y_n^i, \hat{q}) \le d^2(x_n, \hat{q}) - (1 - 2d_{1,i}\lambda_n)d^2(x_n, z_n^i) - (1 - 2d_{2,i}\lambda_n)d^2(z_n^i, y_n^i),$$

for each $i \in \{1, 2, \cdots, N\}$.

Thus,

$$d^2(\overline{y}_n, \hat{q}) \le d^2(x_n, \hat{q}) - (1 - 2d_{1,i}\lambda_n)d^2(x_n, z_n^i) - (1 - 2d_{2,i}\lambda_n)d^2(z_n^i, \overline{y}_n), \tag{3.11}$$

for each $i \in \{1, 2, \cdots, N\}$.

□

Lemma 3.8. *Let C be a nonempty closed convex subset of a Complete $CAT(0)$ space X. For each $i = 1, 2, \cdots, N$, let $f_i : C \times C \to \mathbb{R}$ be bifunctions satisfying the conditions $A_1 - A_4$ and $T_j : C \to C$, $j \in \{1, 2, \cdots, m\}$ be a family of nonexpansive mappings with $T_0 = I$ (the identity mapping) such that $\digamma = (\bigcap_{j=1}^m F(T_j)) \bigcap (\bigcap_{i=1}^N EP(f_i, C)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by algorithm* (3.3). *Then $\{x_n\}$, $\{w_n\}$, $\{\overline{y}_n\}$, $\{T_j x_n\}$ are all bounded.*

Proof. Let $\hat{q} \in \digamma$. Using (3.11), Lemma 2.3(1), condition $C2$ and the fact that T_j, $j = 1, 2, 3, \cdots, m$, are nonexpansive mappings, we have

$$\begin{aligned} d(w_n,\ \hat{q}) &= (\gamma_{n,0}\overline{y}_n \textstyle\bigoplus_{j=1}^m \gamma_{n,j}T_j x_n,\ \hat{q}) \\ &\le \gamma_{n,0}d(\overline{y}_n,\ \hat{q}) + \textstyle\sum_{j=1}^m \gamma_{n,j}d(T_j x_n, \hat{q}) \\ &\le \gamma_{n,0}d(x_n,\ \hat{q}) + \textstyle\sum_{j=1}^m \gamma_{n,j}d(x_n, \hat{q}) \\ &\le d(x_n,\ \hat{q}). \end{aligned} \tag{3.12}$$

Now, from scheme (3.3), Lemma 2.2(1) and (3.12), we have

$$\begin{aligned} d(x_{n+1},\ \hat{q}) &= d(\beta_n u \oplus (1-\beta_n)w_n,\ \hat{q}) \\ &\leq \beta_n d(u,\ \hat{q}) + (1-\beta_n)d(w_n,\ \hat{q}) \\ &\leq \beta_n d(u,\ \hat{q}) + (1-\beta_n)d(x_n,\ \hat{q}) \\ &\leq Max\{d(u,\ \hat{q}), d(x_n,\ \hat{q})\}. \end{aligned}$$

By induction, we get

$$d(x_{n+1},\ \hat{q}) \leq Max\{d(u,\ \hat{q}), d(x_1,\ \hat{q})\}.$$

This implies that $\{x_n\}$ is bounded and consequently $\{w_n\}$, $\{\overline{y}_n\}$ and $\{T_j x_n\}$ are also bounded. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Complete $CAT(0)$ space X. For each $i = 1, 2, \cdots, N$, let $f_i : C \times C \to \mathbb{R}$ be bifunctions satisfying the conditions $A_1 - A_4$ and $T_j : C \to C$, $j \in \{1, 2, \cdots, m\}$ be a family of nonexpansive mappings with $T_0 = I$ (the identity mapping) such that $\digamma = (\bigcap_{j=1}^{m} F(T_i)) \bigcap (\bigcap_{i=1}^{N} EP(f_i, C)) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by* (3.3) *converges strongly to $\hat{q} = P_\digamma u$.*

Proof. Let $\hat{q} = P_\digamma u$. We then divide the proof here into two cases.

Case 1: Suppose that $\{d(x_n,\ \hat{q})\}$ is a monotone decreasing sequence, then $\lim_{n\to\infty} d(x_n,\ \hat{q})$ exists and consequently

$$\lim_{n\to\infty} \big(d(x_{n+1},\ \hat{q}) - d(x_n,\ \hat{q})\big) = 0.$$

Thus, from algorithm (3.3), Lemma 2.2(4) and (3.12), we have

$$\begin{aligned} 0 &= \liminf_{n\to\infty} (d^2(x_{n+1},\ \hat{q}) - d^2(x_n, \hat{q})) \\ &\leq \liminf_{n\to\infty} (\beta_n d^2(u,\ \hat{q}) + (1-\beta_n)d^2(w_n,\ \hat{q}) - \beta_n(1-\beta_n)d^2(u, w_n) - d^2(x_n, \hat{q})) \\ &\leq \liminf_{n\to\infty} (\beta_n(d^2(u,\ \hat{q}) - d^2(w_n,\ \hat{q})) + d^2(w_n,\ \hat{q}) - d^2(x_n, \hat{q})) \\ &\leq \limsup_{n\to\infty} \beta_n(d^2(u,\ \hat{q}) - d^2(w_n,\ \hat{q})) + \liminf_{n\to\infty} (d^2(w_n,\ \hat{q}) - d^2(x_n, \hat{q})) \\ &= \liminf_{n\to\infty} (d^2(w_n,\ \hat{q}) - d^2(x_n, \hat{q})) \\ &\leq \limsup_{n\to\infty} (d^2(w_n,\ \hat{q}) - d^2(x_n, \hat{q})) \\ &\leq 0. \end{aligned}$$

This implies

$$\lim_{n\to\infty} (d^2(w_n,\ \hat{q}) - d^2(x_n, \hat{q})) = 0. \tag{3.13}$$

On the other hand, from (3.11) and condition $C3$, we have

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}(d^2(w_n,\ \hat{q}) - d^2(x_n,\hat{q})) \\
&\leq \limsup_{n\to\infty}(d^2(w_n,\ \hat{q}) - d^2(x_n,\hat{q})) \\
&\leq \limsup_{n\to\infty}(\gamma_{n,0}d^2(\overline{y}_n,\ \hat{q}) + (1-\gamma_{n,0})d^2(x_n,\hat{q}) - d^2(x_n,\hat{q})) \\
&\leq \limsup_{n\to\infty}(\gamma_{n,0}(d^2(\overline{y}_n,\ \hat{q}) - d^2(x_n,\hat{q}))) \\
&\leq 0,
\end{aligned}$$

from which we get

$$\lim_{n\to\infty}(d^2(\overline{y}_n,\ \hat{q}) - d^2(x_n,\hat{q})) = 0. \tag{3.14}$$

Thus, for any $j \in \{1, 2, \cdots, m\}$ we have by using Lemma 2.2(2), Lemma 2.3(2) and algorithm (3.3) that

$$\begin{aligned}
d^2(w_n,\ \hat{q}) &= d^2(\gamma_{n,0}\overline{y}_n \bigoplus_{j=1}^{m} \gamma_{n,j}T_jx_n, \hat{q}) \\
&\leq \gamma_{n,0}d^2(\overline{y}_n,\ \hat{q}) + \sum_{j=1}^{m}\gamma_{n,j}d^2(T_jx_n,\ \hat{q}) - \gamma_{n,0}\gamma_{n,j}d^2(\overline{y}_n, T_jx_n) \\
&\leq \gamma_{n,0}d^2(\overline{y}_n,\ \hat{q}) + \sum_{j=1}^{m}\gamma_{n,j}d^2(x_n,\ \hat{q}) - \gamma_{n,0}\gamma_{n,j}d^2(\overline{y}_n, T_jx_n) \\
&\leq \gamma_{n,0}d^2(x_n,\ \hat{q}) + (1-\gamma_{n,0})d^2(x_n,\ \hat{q}) - \gamma_{n,0}\gamma_{n,j}d^2(\overline{y}_n, T_jx_n),
\end{aligned}$$

from which we obtain

$$\sigma^2 d^2(\overline{y}_n, T_jx_n) \leq d^2(x_n,\ \hat{q}) - d^2(w_n,\ \hat{q}). \tag{3.15}$$

Using (3.13), we get from (3.15) that

$$\lim_{n\to\infty} d(\overline{y}_n, T_jx_n) = 0,\ j \in \{1, 2, \cdots, m\}. \tag{3.16}$$

Also from (3.11), (3.14) and $\liminf_{n\to\infty}(1 - 2d_{k,i}\lambda_n) > 0$, for $k = 1, 2$, we obtain

$$\lim_{n\to\infty} d^2(x_n,\ z_n^i) = \lim_{n\to\infty} d^2(z_n^i,\ \overline{y}_n) = \lim_{n\to\infty} d^2(x_n, \overline{y}_n) = 0. \tag{3.17}$$

Now,

$$d(x_n,\ T_jx_n) \leq d(x_n, \overline{y}_n) + d(\overline{y}_n, T_jx_n),\ \text{for each } j \in \{1, 2, \cdots, m\}. \tag{3.18}$$

Hence, (3.16), (3.17) and (3.18) give

$$\lim_{n\to\infty} d(x_n,\ T_jx_n) = 0,\ \text{for each } j \in \{1, 2, \cdots, m\}. \tag{3.19}$$

By Lemma 3.8 $\{x_n\} \subseteq X$ is bounded. Since X is a complete $CAT(0)$ space, then it follows from Lemma 2.4 that $\{x_n\}$ has Δ-convergence subsequence $\{x_{n_k}\}$ such that

$$\Delta - \lim_{k\to\infty} x_{n_k} = w^*.$$

This together with (3.19) gives $w^* \in \bigcap_{j=1}^m F(T_j)$.

We now show that $w^* \in EP(f_i, C)$. It follows from (3.10), (3.6) and (3.17) that

$$\lim_{n\to\infty} f_i(z_n^i,\ y_n^i) = 0,\ \text{for each } i \in \{1, 2, \cdots, N\}. \tag{3.20}$$

Since for each $i = 1, 2, 3, \cdots, N$ y_n^i solves the minimization problem in algorithm (3.3), therefore using similar computation as in (3.4) with $y = ty_n^i \oplus (1-t)q$ for $i \in \{1, 2, 3, \cdots, N\}$, $t \in [0, 1)$ and $q \in C$, we have

$$f_i(z_n^i, y_n^i) - f_i(z_n^i, q) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, q) - d^2(x_n, y_n^i) - td^2(y_n^i, q)\Big),$$

for each $i \in \{1, 2, \cdots, N\}$ and by setting $t \to 1^-$ here, we obtain

$$f_i(z_n^i, y_n^i) - f_i(z_n^i, q) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, q) - d^2(x_n, y_n^i) - d^2(y_n^i, q)\Big),$$

for each $i \in \{1, 2, \cdots, N\}$.

This implies

$$\frac{1}{2\lambda_n}\Big(d^2(x_n, y_n^i) + d^2(y_n^i, q) - d^2(x_n, q)\Big) \leq f_i(z_n^i, q) - f_i(z_n^i, y_n^i),$$

for each $i \in \{1, 2, \cdots, N\}$.

Thus,

$$-\frac{1}{2\lambda_n}d^2(x_n, y_n^i)\Big(d^2(y_n^i, q) + d^2(x_n, q)\Big) \leq f_i(z_n^i, q) - f_i(z_n^i, y_n^i), \tag{3.21}$$

for each $i \in \{1, 2, \cdots, N\}$.

Using (3.17), (3.20) and the fact that $\Delta - \lim_{n\to\infty} z_n^i = w^*$ we get from (3.21) that

$$0 \leq \limsup_{n\to\infty}(f_i(z_n^i, q) - f_i(z_n^i, y_n^i)) \leq \limsup_{n\to\infty} f_i(z_n^i, q) \leq f_i(w^*, q), \forall\, q \in C,$$

$i \in \{1, 2, \cdots, N\}$.

Therefore $w^* \in \bigcap_{i=1}^N EP(f_i, C)$ and so $w^* \in \digamma$.

Next, using algorithm (3.3) and Lemma 2.2(4) we have

$$\begin{aligned} d^2(x_{n+1},\ \hat{q}) &= d^2(\beta_n u \oplus (1-\beta_n)w_n,\ \hat{q}) \\ &\leq \beta_n d^2(u,\ \hat{q}) + (1-\beta_n)d^2(w_n,\ \hat{q}) - \beta_n(1-\beta_n)d^2(u, w_n) \\ &\leq (1-\beta_n)d^2(x_n,\ \hat{q}) + \beta_n[d^2(u,\ \hat{q}) - (1-\beta_n)d^2(u, w_n)]. \end{aligned} \tag{3.22}$$

Claim: $d^2(x_n,\ \hat{q}) \to 0$ as $n \to \infty$ such that $\hat{q} = P_\digamma u$.

It suffices, by Lemma 2.5, to show that

$$\limsup_{n\to\infty}(d^2(u,\ \hat{q}) - (1-\beta_n)d^2(u, w_n)) \leq 0.$$

Now, from algorithm (3.3) we have

$$\begin{aligned} d(w_n,\ \overline{y}_n) &= d(\gamma_{n,0}\overline{y}_n \bigoplus_{j=1}^{m} \gamma_{n,j}T_j x_n,\ \overline{y}_n) \\ &\leq \sum_{j=1}^{m} \gamma_{n,j} d(\overline{y}_n, T_j x_n). \end{aligned} \tag{3.23}$$

Therefore, (3.23) and (3.16) give

$$\lim_{n\to\infty} d(w_n, \overline{y}_n) = 0 \tag{3.24}$$

Also,

$$d(x_n,\ w_n) \leq d(x_n,\ \overline{y}_n) + d(\overline{y}_n,\ w_n) \tag{3.25}$$

Thus, from (3.17), (3.24) and (3.25) we obtain

$$\lim_{n\to\infty} d(x_n, w_n) = 0. \tag{3.26}$$

Since $\{w_n\}$ is bounded, $\Delta - \lim_{k\to\infty} x_{n_k} = w^*$, then we get by using (3.26) that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\Delta - \lim_{k\to\infty} w_{n_k} = w^* \in C$.

Therefore, by $\Delta-$lower semicontinuity of $d^2(u,\ .)$, we have

$$\begin{aligned} \limsup_{n\to\infty}(d^2(u,\ \hat{q}) - (1-\beta_n)d^2(u, w_n)) &= \lim_{k\to\infty}(d^2(u,\ \hat{q}) - (1-\beta_{n_k})d^2(u, w_{n_k})) \\ &\leq d^2(u,\ \hat{q}) - d^2(u, w^*). \end{aligned} \tag{3.27}$$

It remains now to show that $d^2(u,\ \hat{q}) \leq d^2(u, w^*)$.

Since $\hat{q} = P_F u$, we have $d(u, \hat{q}) \leq d(u, y),\ \forall\ y \in F$. As $w^* \in F$, we obtain

$$d(u, \hat{q}) \leq d(u, w^*). \tag{3.28}$$

Thus, (3.27) together with (3.28) gives

$$\limsup_{n\to\infty}(d^2(u,\ \hat{q}) - (1-\beta_n)d^2(u, w_n)) \leq 0.$$

Hence,

$$x_n \to \hat{q} \text{ as } n \to \infty.$$

Case 2: Assume $\{d(x_n,\ \hat{q})\}$ is not monotonically decreasing sequence and let $\Phi_n = d(x_n,\ \hat{q})$, that is, there exists a subsequence $\{\Phi_{n_r}\}$ of $\{\Phi_n\}$ such that $\Phi_{n_r} \leq \Phi_{n_r+1}\ \forall r \in \mathbb{N}$. Then by Lemma 2.6 and for large $\overline{N}$ such that $n \geq \overline{N}$, let $\varphi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\varphi(n) = max\{k \leq n : \Phi_k \leq \Phi_{k+1}\}.$$

Then $\{\varphi(n)\}$ is nondecreasing sequence satisfying $\varphi(n) \to \infty$ as $n \to \infty$ and

$$\Phi_{\varphi(n)} \leq \Phi_{\varphi(n)+1} \text{ and } \Phi_n \leq \Phi_{\varphi(n)+1}\ \forall\ n \in \overline{N}.$$

Therefore, for $n \geq \overline{N}$

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}(\Phi^2_{\varphi(n)+1} - \Phi^2_{\varphi(n)})\\
&\leq \limsup_{n\to\infty}(\beta_{\varphi(n)}d^2(u,\ \hat{q}) + (1-\beta_{\varphi(n)})d^2(w_{\varphi(n)},\ \hat{q}) - \beta_{\varphi(n)}(1-\beta_{\varphi(n)})d^2(u, w_{\varphi(n)}) - d^2(x_{\varphi(n)}, \hat{q}))\\
&\leq \limsup_{n\to\infty}(\beta_{\varphi(n)}(d^2(u,\ \hat{q}) - d^2(w_{\varphi(n)},\ \hat{q})) + d^2(w_{\varphi(n)},\ \hat{q}) - d^2(x_{\varphi(n)}, \hat{q}))\\
&\leq \limsup_{n\to\infty}\beta_{\varphi(n)}(d^2(u,\ \hat{q}) - d^2(w_{\varphi(n)},\ \hat{q})) + \liminf_{n\to\infty}(d^2(w_{\varphi(n)},\ \hat{q}) - d^2(x_{\varphi(n)}, \hat{q}))\\
&\leq \limsup_{n\to\infty}(d^2(w_{\varphi(n)},\ \hat{q}) - d^2(x_{\varphi(n)}, \hat{q}))\\
&\leq 0,
\end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}(d^2(w_{\varphi(n)},\ \hat{q}) - d^2(x_{\varphi(n)}, \hat{q})) = 0. \tag{3.29}$$

Also from (3.14), (3.17), (3.19), (3.20) and (3.26) we can show that

$$\lim_{n\to\infty}(d^2(\overline{y}_{\varphi(n)},\ \hat{q}) - d^2(x_{\varphi(n)}, \hat{q})) = 0,$$

$$\lim_{n\to\infty} d^2(x_{\varphi(n)},\ z^i_{\varphi(n)}) = \lim_{n\to\infty} d^2(z^i_{\varphi(n)},\ \overline{y}_{\varphi(n)}) = \lim_{n\to\infty} d^2(x_{\varphi(n)}, \overline{y}_{\varphi(n)}) = 0,$$

$$\lim_{n\to\infty} d(x_{\varphi(n)},\ T_j x_{\varphi(n)}) = 0,\ \text{for each } j \in \{1, 2, \cdots, m\},$$

$$\lim_{n\to\infty} f_i(z^i_{\varphi(n)},\ y^i_{\varphi(n)}) = 0,\ \text{for each } i \in \{1, 2, \cdots, N\} \text{ and}$$

$$\lim_{n\to\infty} d(x_{\varphi(n)}, w_{\varphi(n)}) = 0.$$

In similar fashion as in case 1 we also get

$$\limsup_{n\to\infty}(d^2(u,\ \hat{q}) - (1-\beta_{\varphi(n)})d^2(u, w_{\varphi(n)})) \leq 0. \tag{3.30}$$

It also follows from (3.22) that

$$\Phi^2_{\varphi(n)+1} \leq (1-\beta_{\varphi(n)})\Phi^2_{\varphi(n)} + \beta_{\varphi(n)}[d^2(u,\ \hat{q}) - (1-\beta_{\varphi(n)})d^2(u, w_{\varphi(n)})]. \tag{3.31}$$

Since $\Phi_{\varphi(n)} \leq \Phi_{\varphi(n)+1}$, we get from (3.31) that

$$\Phi^2_{\varphi(n)} \leq \big(d^2(u,\ \hat{q}) - (1-\beta_{\varphi(n)})d^2(u, w_{\varphi(n)})\big).$$

Hence, this together with (3.30) give

$$\lim_{n\to\infty} \Phi_{\varphi(n)} = 0. \tag{3.32}$$

Furthermore, since $\Phi_n \leq \Phi_{\varphi(n)+1}$, it follows from (3.32) that

$$\lim_{n\to\infty} \Phi_n = 0,$$

$$\text{i.e} \lim_{n\to\infty} d(x_n, \hat{q}) = 0.$$

Thus from the two cases, Case 1 and Case 2, we conclude that $x_n \to \hat{q} = P_F u$.

This completes the proof.

□

If we set $N = 1$ and $w_n = y_n$ in the theorem 3.2 then we get the following result which is the result of Khatibzadeh and Mohebbi [16].

Corollary 3.1. *Let C be a nonempty closed convex subset of a Complete $CAT(0)$ space X and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $A_1 - A_4$. If $\digamma = EP(f, C)) \neq \emptyset$, then the sequence $\{x_n\}$ generated by $u,\ x_1 \in C$,*

$$\begin{cases} z_n = \underset{y\in C}{argmin}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n)\}, \\ y_n = \underset{y\in C}{argmin}\{f(z_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n)\}, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) y_n, n \geq 1, \end{cases} \tag{3.33}$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences satisfying the following conditions;
$C1 : \{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,
$C2 : 0 < \gamma \leq \lambda_n \leq \beta < min\{\frac{1}{2d_1}, \frac{1}{2d_2}\}$ and $n = 0, 1, 2, \cdots$,
converges strongly to $\hat{q} = P_\digamma u$.

Also setting $m = 1$ in theorem 3.2 we obtain the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Complete $CAT(0)$ space X. For each $i = 1, 2, \cdots, N$, let $f_i : C \times C \to \mathbb{R}$ be bifunctions satisfying the conditions $A_1 - A_4$ and $T : C \to C$ be a nonexpansive mapping with $T_0 = I$ (the identity mapping) such that $\digamma = F(T) \bigcap (\bigcap_{i=1}^{N} EP(f_i, C)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u,\ x_1 \in C$ chosen arbitrarily,*

$$\begin{cases} z_n^i = \underset{y\in C}{argmin}\{f_i(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ y_n^i = \underset{y\in C}{argmin}\{f_i(z_n^i, y) + \frac{1}{2\lambda_n} d^2(y, x_n),\ i = 1, 2, 3, \cdots, N\}, \\ i_n = argmax\{d^2(y_n^i, x_n),\ i = 1, 2, 3, \cdots, N\}, \overline{y}_n = y_n^{i_n}, \\ w_n = \gamma_n \overline{y}_n \oplus \gamma_n T x_n, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) w_n, n \geq 1, \end{cases} \tag{3.34}$$

where $\{\beta_n\}$, $\{\gamma_{n,i}\}$ and $\{\lambda_n\}$ are sequences satisfying the following conditions;
$C1 : \{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,
$C2 : \{\gamma_n\} \subset [\sigma, 1-\sigma]$ for some $\sigma \in (0, \frac{1}{2})$,
$C3 : 0 < \gamma \leq \lambda_n \leq \beta < min\{\frac{1}{2d_{1,i}}, \frac{1}{2d_{2,i}}\}$ and $n = 0, 1, 2, \cdots$.

Then the sequence $\{x_n\}$ converges strongly to an element $\hat{q} = P_\digamma u$.

4. NUMERICAL EXAMPLE

In this section, we give a numerical example to illustrate the convergence nature of our algorithm to a common solution.

Example 4.1. Let $X = \mathbb{R}$ with the usual metric and $C = [-5, 5]$. Then $\mathbb{R}$ is a complete $CAT(0)$ space and C is a nonempty closed convex subset of $\mathbb{R}$. For $i = 1, 2$, we define $f_i : C \times C \to \mathbb{R}$ by $f_i(x, y) = y^2 + 6xy - 7x^2$. It is easy to see that $0 \in \cap_{i=1}^{2} EP(f_i, C)$

and that f_{i^s} satisfy conditions A_1, A_2 and A_4. Moreover, f_{i^s} satisfy condition A_3 with $d_1 = d_2 = 3$. Indeed for any $x, y, z \in X$

$$\begin{aligned} f_i(x,y) + f_i(y,z) =& y^2 + 6xy - 7x^2 + z^2 + 6yz - 7y^2 \\ =& z^2 + 6xy - 7x^2 + 6yz - 6y^2 \\ =& f_i(x,z) - 6xz + 6xy + 6yz - 6y^2 \\ =& f_i(x,z) - 3(y-x)^2 - 3(z-y)^2 + 3(z-x)^2 \\ \geq& f_i(x,z) - 3d^2(y-x) - 3d^2(z-y). \end{aligned}$$

Let $j = 1,2$, define $T_j : C \to C$ by $T_j(x) = \frac{x}{j+1}$ $\forall x \in C$. Clearly T_j is nonexpansive mapping for each $j \in \{1,2\}$ and $0 \in \cap_{j=1}^{2} F(T_j)$. Thus, the solution set $F = (\bigcap_{j=1}^{2} F(T_j)) \bigcap (\bigcap_{i=1}^{2} EP(f_i, C)) = \{0\} \neq \emptyset$. Therefore our proposed algorithm (3.3) takes the following form;

$$\begin{cases} z_n^i = \frac{1-6\lambda_n}{1+2\lambda_n} x_n, \ i = 1,2, \\ y_n^i = \frac{x_n - 6\lambda_n z_n^i}{1+2\lambda_n}, \ i = 1,2, \\ \overline{y}_n = y_n^i, \ i = 1,2, \\ w_n = \gamma_{n,0}\overline{y}_n + \frac{3\gamma_{n,1}+2\gamma_{n,2}}{6} x_n, \\ x_{n+1} = \beta_n u + (1-\beta_n) w_n, n \geq 1. \end{cases} \tag{4.35}$$

Set $\beta_n = \frac{1}{5n+2}$, $\lambda_n = \frac{1}{n+9}$ and $\gamma_{n,0} = \gamma_{n,1} = \gamma_{n,2} = \frac{1}{3}$. It can be observed that all assumptions of Theorem 3.2 are clearly satisfied. Let $\{x_n\}$ be a sequence generated by algorithm (4.35).

Case I. Take $x_1 = 1$, $u = 0.25$.

Case II. Take $x_1 = -1$, $u = -2$.

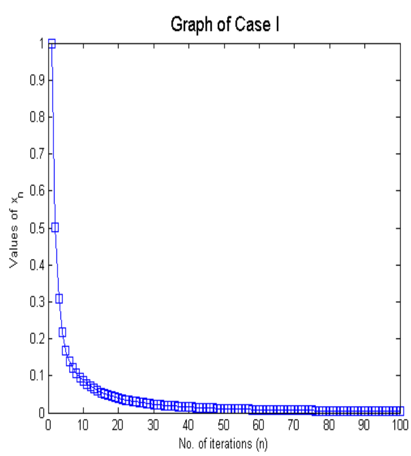

FIGURE 1. The graph of sequence $\{x_n\}$ generated by algorithm (4.35) versus number of iterations (Case I).

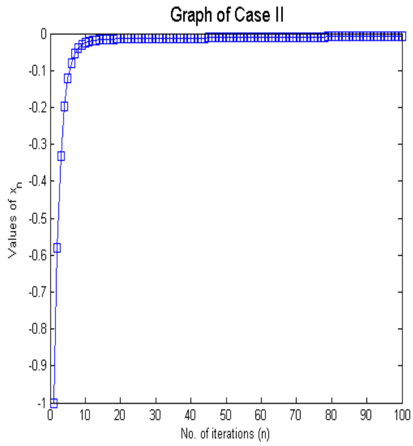

FIGURE 2. The graph of sequence $\{x_n\}$ generated by algorithm (4.35) versus number of iterations (Case II).

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