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A new Krasnoselskii's type algorithm for zeros of strongly monotone and Lipschitz mappings

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ABSTRACT. For $q > 1$, let E be a q-uniformly smooth real Banach space with dual space E^*. Let $A : E \to E^*$ be a Lipschitz and strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For given $x_1 \in E$, let $\{x_n\}$ be generated iteratively by the algorithm :

$$x_{n+1} = x_n - \lambda J^{-1}(Ax_n),\ n \geq 1,$$

where J is the normalized duality mapping from E into E^* and λ is a positive real number choosen in a suitable interval. Then it is proved that the sequence $\{x_n\}$ converges strongly to x^*, the unique point of $A^{-1}(0)$. Our theorems are applied to the convex minimization problem. Futhermore, our technique of proof is of independent interest.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. An operator $A : H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle_H \geq 0 \; \forall \, x, y \in H, \tag{1.1}$$

and is called *strongly monotone* if there exists $k \in (0, 1)$ such that

$$\langle Ax - Ay, x - y \rangle_H \geq k \|x - y\|_H^2 \; \forall x, y \in H. \tag{1.2}$$

Interest in monotone operators stems mainly from their usefulness in numerous applications. Many problems in nonlinear analysis and optimization theory can be formulated as follows: *find u such that* $0 \in Au$. This problem has been investigated by many researchers (see for instance, Brézis and Lions [5], Martinet [26], Minty [29], Reich [41], Rockafellar [42], Takahashi and Ueda [45] and others). Such a problem is connected with the *convex minimization problem*. In fact, if $f : H \to (-\infty, +\infty]$ is a proper, lower-semicontinuous convex function, then, it is known that the multi-valued map $T := \partial f$, the subdifferential of f, is *maximal monotone* (see, e.g., [29], [42]), where for $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle \;\; \forall \, y \in H \\ &\Leftrightarrow x \in Argmin(f - \langle \cdot, w \rangle). \end{aligned}$$

In particular, the equation $0 \in \partial f(x)$ is equivalent to $f(x) = \min_{y \in H} f(y)$.
Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling [20], Pascali and Sburian [34]).

The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure

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of the monotonicity condition. The monotone mappings appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions. (Pascali and Sburian [34], p. 101).

The *first extension* involves mappings from E to E^*. Here and in the sequel, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between (a possible normed linear space) E and its dual E^*. A mapping $A : D(A) \subset E \to E^*$ is called *monotone* if for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay\rangle \quad \geq \quad 0, \tag{1.3}$$

where $\langle,\rangle$ denotes the duality pairing between elements of E and elements of E^*. It is said to be *strongly monotone* if there exists a positive constant k such that for all $x, y \in D(A)$,

$$\langle x - y, Ax - Ay\rangle \quad \geq \quad k\|x - y\|^2. \tag{1.4}$$

Note that if E is a real Hilbert space H, then $H = H^*$ and (1.3) coincides with (1.1).

The *second* extension of the notion of monotonicity to real normed spaces involves mappings E *into itself*. A mapping $A : D(A) \subset E \to E$ is called *accretive* if for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that the following inequality holds:

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \tag{1.5}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping of E defined by:

$$J(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\right\}.$$

Here, if E is a real Hilbert space, J becomes the identity map and condition (1.5) reduces to (1.1). Hence, in real Hilbert spaces, *accretive* operators become *monotone*. Consequently, accretive operators can be regarded as extension of Hilbert space monotonicity condition to real normed spaces.

A mapping $A : D(A) \subset E \to E$ is called *strongly accretive* if there exists a constant $k > 0$ such that for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq k\|x - y\|^2.$$

For approximating a solution of $Au = 0$, assuming existence, where $A : E \to E$ is of accretive-type, Browder [6] defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator *pseudo-contractive*. It is trivial to observe that zeros of A correspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume [12] proved the following theorem.

Theorem 1.1 (Theorem C1)**.** (Chidume, [12]) *Let $E = L_p,\ 2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n,\ n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique fixed point of T.*

By setting $T := I - A$ in Theorem C1, the following theorem for approximating a solution of $Au = 0$ where A is a strongly accretive and bounded operator can be proved.

Theorem 1.2 (Theorem C2)**.** *Let $E = L_p,\ 2 \leq p < \infty$. Let $A : E \to E$ be a strongly accretive and bounded map. Assume $A^{-1}(0) \neq \emptyset$. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = x_n - \lambda_n Ax_n,\ n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: (i) $\sum_{n=1}^{\infty} \lambda_n = \infty$, (ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then, $\{x_n\}$ converges strongly to the unique solution of $Au = 0$.*

The main tool used in the proof of Theorem C1 is an inequality of Bynum [7]. This theorem signalled the return to extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations. Consequently, Theorem C1 has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g.,Censor and Reich [8], Chidume [12], Chidume [10, 11], Chidume and Ali [13], Chidume and Chidume [15, 16], Chidume and Osilike [17], Deng [19], Zhou and Jia [57], Liu [24], Qihou [35], Reich [36, 37, 38], Reich and Sabach [39, 40], Weng [46], Xiao [48], Xu [50, 55, 54], Berinde *et al.* [4], Moudafi [30, 31, 32], Moudafi and Thera [33], Xu and Roach [51], Xu *et al.* [52], Zhu [58] and a host of other authors). Recent monographs emanating from these researches include those by Berinde [3], Chidume [9], Goebel and Reich [22], and William and Shahzad [47].

Unfortunately, the success achieved in using geometric properties developed from the mid 1980s to early 1990s in approximating zeros of *accretive-type mappings* has not carried over to approximating zeros of *monotone-type operators* in general Banach spaces. Part of the problem is that since A maps E to E^*, for $x_n \in E$, Ax_n is in E^*. Consequently, a recursion formula containing x_n and Ax_n may not be well defined.

Attemps have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.

In the case of Banach spaces, for finding zeros point of a maximal monotone mappings by using the proximal point algorithm, Kohshada and Takahashi [44] introduced the following iterative sequence for a monotone mapping $A : E \to 2^{E^*}$:

$$x_1 = u \in E,\ x_{n+1} = J^{-1}\Big(\alpha_n Ju + (1-\alpha)JJ_{r_n}x_n\Big),\ n \geq 1, \tag{1.6}$$

where $J_{r_n} := (J + r_n A)^{-1}$, and J the duality mapping from E into E^*, $\{\alpha_n\} \in (0, 1)$ and $\{r_n\} \in (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$ and $\lim_{n\to\infty} r_n = \infty$. They proved that if E is smooth and uniformly convex and A maximal monotone with $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}(0)$. However, the algorithm requires the computation of $(J+r_nA)^{-1}x_n$ at each step of the process, which make difficult its implementation for applications.

Following the work of Kohshada and Takahashi [44], in [56], Zegeye introduced an iterative scheme for approximating zeros of maximal monotone mappings defined in uniformly smooth and 2- uniformly convex real Banach spaces. In fact, he proved the following theorem.

Theorem 1.3 (Theorem Z)**.** (Zegeye [56]) *Let E be a uniformly smooth and 2- uniformly convex real Banach space with dual E^*. Let $A : E \to E^*$ be a Lipschitz continuous monotone mappings with constant $L > 0$ and $A^{-1}(0) \neq \emptyset$. For given $u, x_1 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = J^{-1}\Big(\beta_n Ju + (1-\beta_n)(Jx_n - \alpha_n Ax_n)\Big),\ n \geq 1$$

where J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying (i) $\lim_{n\to\infty} \beta_n = 0$, (ii) $\sum \beta_n = \infty$ and (iii) $\alpha_n = o(\beta_n)$. Suppose that $B_{min} \cap \Big(AJ^{-1}\Big)^{-1}(0) \neq \emptyset$. Then $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$ and that $R(Ju) = Jx^* \in \Big(AJ^{-1}\Big)^{-1}(0)$, where R is a sunny generalized nonexpansive retraction of E^* onto $\Big(AJ^{-1}\Big)^{-1}(0)$.

Motivated by approximating zeros of monotone mappings, Chidume et. al. [14] proposed a krasnoselskii type scheme and proved a strong convergence theorem in $L_p,\ 2 \le p < \infty$. In fact, they obtained the following result.

Theorem 1.4 (Theorem CA)**.** (Chidume et. al. [14]). *Let $X= L_p,\ 2 \le p < \infty$ and $A : X \to X^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0, 1)$ such that A satisfies the following condition:*

$$\langle Ax - Ay, x - y\rangle \ge k\|x - y\|^{\frac{p}{p-1}}, \tag{1.7}$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ iteratively by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \ge 1, \tag{1.8}$$

where $\lambda \in (0, \delta_p)$ and δ_p is some positive constant. Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

In [14], they also proved a similar result for the class of Lipschitzian strongly monotone mappings in L_p-spaces for $1 < p \le 2$.

Remark 1.1. Theorem CA is proved in L_p-spaces, $2 \le p < \infty$ with Lipschitz mapping satisfying condition (1.7). The method of proof used in (1.7) is not extendable to the class of strongly monotone mappings.

Remark 1.2. In [14], the authors posed the following open problem: If $E = L_p,\ 2 \le p < \infty$, attempts to obtain strong convergence of the *Krasnoselskii-type sequence* defined for $x_0 \in E$, by:

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \ge 0, \lambda \in (0, 1) \tag{1.9}$$

to a solution of the equation $Au = 0$, where A is strongly monotone and Lipschitz, have not yielded any positive result. It is, therefore, of interest to find out if a *Krasnoselskii-type sequence* will converge strongly to a solution of $Au = 0$ *in this space*.

Following the works of Chidume et. al., [14] and motivating by approximating zeros of monotone type mappings, several strong convergence results have been established by various authors using the algorithm (1.8) proposed by Chidume et. al in [14] (see, e.g., Diop et. al [21], Mendy et. al [27], Mendy et. al [28], Sow et.al [43]).
Recently Mendy et. al [27] study the *Krasnoselskii-type* algorithm introduced by Chidume et.al [14] and they prove strong convergence theorems to approximate the unique zero of *a Lipschitz strongly monotone mapping* 2-uniformly smooth and p-uniformly convex real Banach space for $p \ge 2$. In fact, they prove the following theorem.

Theorem 1.5 (Theorem MA)**.** (Mendy et. al. [27]). *For $p \ge 2$, let E be a 2-uniformly smooth and p-uniformly convex real Banach space and let $A : X \to X^*$ be a Lipschitz strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}(Jx_n - \lambda Ax_n), n \ge 1, \tag{1.10}$$

where λ is a postive real number and J is the duality mapping of E. Then there exists a positive real number δ_p such that if $\lambda \in (0, \delta_p)$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

Remark 1.3. The results obtained by Mendy et. al [27] extend and generalize recent works by various authors. In particular, Mendy et. al provide an affirmative answer to the Chidume et al. open problem in [14].

In this paper, we introduce a new type Krasnoselskii algorithm to approximate the unique zero of *a Lipschitz strongly monotone mapping* defined in some class of Banach spaces including the L_p and Sobolev spaces. The algorithm proposed in this work is simpler than the one used by Chidume et. al in [14]. Applications are also given for convex minimisation problem.

2. Preliminaries

Let E be a normed linear space. E is said to be smooth if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.11}$$

exist for each $x, y \in S_E$ (here $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1 : \|x\|=1, \|y\|=\tau\right\}; \quad \tau>0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

It is well known (see, e.g., [2] p.95) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*.
A normed linear space E is said to be strictly convex if:

$$\|x\|=\|y\|=1,\ x\neq y\ \Rightarrow\ \left\|\frac{x+y}{2}\right\|<1.$$

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf\left\{1-\frac{1}{2}\|x+y\| \ : \|x\|=\|y\|=1,\ \|x-y\|\geq\epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $p > 1$. Then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex.

Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } 2\leq p<\infty; \\ p-\text{uniformly smooth and } 2-\text{uniformly convex} & \text{if } 1<p<2. \end{cases}$$

It is well known that E is smooth if and only if J is single valued. Moreover, if E is a reflexive smooth and strictly convex Banach space, then J^{-1} is single valued, one-to-one, surjective and it is the duality mapping from E^* into E. Finally, if E has uniform Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on bounded sets.

Remark 2.4. Note also that a duality mapping exists in each Banach space. We recall from [18] some of the examples of this mapping in $\ell_p, L_P, W^{m,p}$-spaces, $1 < p < \infty$.

(i) ℓ_p : $Jx = \|x\|_{l_p}^{2-p} y \in \ell_q$, $x = (x_1, x_2, \cdots, x_n, \cdots)$, $y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots)$,

(ii) $L_p :\ Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q$,

(iii) $W^{m,p} :\ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|}(-1)^{|\alpha|}D^{\alpha}\Big(|D^{\alpha}u|^{p-2}D^{\alpha}u\Big) \in W^{-m,q}$,

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Lemma 2.1 (Xu, [53]). *Let $q > 1$ be a real number and E be a Banach space. Then the following assertions are equivalent.*

(i) E is q-uniformly smooth.
(ii) There exists a constant $d_q > 0$ such that for all $x, y \in E$, one has,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + d_q\|y\|^q.$$

As examples,

i) the L_p, ℓ_p and $W^{m,p}$ spaces for $1 < p < \infty$ are q-uniformly smooth real Banach spaces with q given by $q = \min\{2, p\}$ and $d_q \geq 1$ is given by:

$$d_q = \begin{cases} \frac{1+\tau^{q-1}}{(1+\tau)^{q-1}}, & \text{if } 1 < p < 2, \\ p-1, & \text{if } 2 \leq p < \infty, \end{cases}$$

where $\tau \in (0,1)$ is the unique solution of the equation $(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0$.

ii) Hilbert spaces are q-uniformly smooth Banach spaces with $q = 2$ and $d_q = 1$.

3. Main results

Let E be a 2-uniformly smooth Banach space. For the remainder of this paper, d_2 denotes the constant appearing in Lemma 2.1.

Let $A : E \to E^*$ be a map. We assume that :

(i) A is Liptchitzian, that is, there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$, $\forall\, x, y \in E$..

(ii) A is strongly monotone, that is, there exists $k > 0$ such that is $\langle x - y, Ax - Ay\rangle \geq k\|x - y\|^2$, $\forall\, x, y \in E$.

(iii) $L^2(d_2 - 1) < k^2$.

With these assumptions, we prove the following theorem..

Theorem 3.6. *Let E be a 2-uniformly smooth Banach space. Let $A : E \to E^*$ be a mapping with $A^{-1}(0) \neq \emptyset$ and such that the assumptions (i), (ii) and (iii) are satisfied. For given $x_1 \in E$, let the sequence $\{x_n\}$ be defined as follows:*

$$x_{n+1} = x_n - \lambda J^{-1}(Ax_n),\ n \geq 1, \tag{3.12}$$

where $\lambda \in (\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{k}{L^2}$ and $\alpha_2 = \frac{k+\sqrt{k^2-L^2(d_2-1)}}{L^2}$. Then, the sequence $\{x_n\}$ converges strongly to x^*, the unique solution of $Au = 0$.

Proof. Let $x^* \in E$ be the unique solution of $Ax = 0$. Using (3.12), Lemma 2.1 and the fact that $\|J^{-1}v\| = \|v\|$ for all $v \in E^*$, we have the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|x_n - x^* - \lambda J^{-1}(Ax_n)\|^2 \\ &\leq \|\lambda J^{-1}(Ax_n)\|^2 - 2\langle x_n - x^*, J(\lambda J^{-1}(Ax_n))\rangle + d_2\|x_n - x^*\|^2 \\ &= \lambda^2\|Ax_n\|^2 - 2\lambda\langle x_n - x^*, Ax_n\rangle + d_2\|x_n - x^*\|^2. \end{aligned}$$

Using the assumptions (i) and (ii) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \lambda^2 L^2\|x_n - x^*\|^2 - 2k\lambda\|x_n - x^*\|^2 + d_2\|x_n - x^*\|^2. \\ &= \left(\lambda^2 L^2 - 2k\lambda + d_2\right)\|x_n - x^*\|^2. \end{aligned}$$

Using the fact that $\lambda \in (\alpha_1, \alpha_2)$, it follows that $0 < \lambda^2 L^2 - 2k\lambda + d_2 < 1$ and we have,

$$\|x_{n+1} - x^*\| \leq \delta(\lambda)\|x_n - x^*\|, \tag{3.13}$$

where $\delta(\lambda) := \left(\lambda^2 L^2 - 2k\lambda + d_2\right)^{\frac{1}{2}}$. Therefore the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Convergence in L_p, $2 \leq p < \infty$.

Observing that L_p spaces, for $2 \leq p < \infty$ are 2-uniformly smooth real banach spaces, using the fact that $d_2 = p - 1$, then the following corollary is immediate.

Corollary 3.1. *Let $E = L_p$, $p \geq 2$ and let $A : E \to E^*$ be a mapping, with $A^{-1}(0) \neq \emptyset$ and such that (i) and (ii) are satisfied. Assume that $2 \leq p < 2 + \frac{k^2}{L^2}$. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be defined as follows:*

$$x_{n+1} = x_n - \lambda J^{-1}(Ax_n),\ n \geq 1, \tag{3.14}$$

where $\lambda \in (\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{k}{L^2}$ and $\alpha_2 = \frac{k + \sqrt{k^2 - L^2(p-2)}}{L^2}$.
Then, the sequence $\{x_n\}$ converges strongly to x^*, the unique solution of $Au = 0$.

Proof. Since L_p-spaces, $2 \leq p < \infty$ are 2-uniformly Banach spaces and observing that from $2 \leq p < 2 + \frac{k^2}{L^2}$, the condition (iii) is satisfied, then the proof follows from Theorem 3.6. □

Hilbert spaces are 2-uniformly smooth real Banach spaces. For a real Hilbert space, the duality mapping is the identity map and in addition, $d_2 = 1$, so the condition (iii), $L^2(d_2 - 1) < k^2$ is satisfied.
With these observations, the following corollary is immediate.

Corollary 3.2. *Let H be a real Hilbert space and let $A : H \to H$ be a mapping with $A^{-1}(0) \neq \emptyset$ and such that (i) and (ii) are satisfied. For given $x_1 \in E$, let the sequence $\{x_n\}$ be defined as follows:*

$$x_{n+1} = x_n - \lambda Ax_n,\ n \geq 1, \tag{3.15}$$

where $\lambda \in (\frac{k}{L^2}, 2\frac{k}{L^2})$, Then, the sequence $\{x_n\}$ converges strongly to x^*, the unique solution of $Au = 0$.

Convergence in L_p, $1 < p \leq 2$.

In the sequel, we need the following result.

Lemma 3.2. [7] *Let $E = L_p, 1 < p \leq 2$, one has the following inequality:*

$$(p-1)\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \|y\|^2, \ \forall\, x, y \in E. \tag{3.16}$$

We now prove the following result.

Theorem 3.7. *Let $E = L_p$, $1 < p \leq 2$ and let $A : E \to E^*$ be a mapping, with $A^{-1}(0) \neq \emptyset$ and such that (i) and (ii) are satisfied. Assume that $2 - \frac{k^2}{L^2} < p \leq 2$. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be defined as follows:*

$$x_{n+1} = x_n - \lambda J^{-1}(Ax_n), \ n \geq 1, \tag{3.17}$$

where $\lambda \in (\beta_1, \beta_2)$ with $\beta_1 = \frac{k}{L^2}$ and $\beta_2 = \frac{k+\sqrt{k^2-L^2(2-p)}}{L^2}$.
Then, the sequence $\{x_n\}$ converges strongly to x^*, the unique solution of $Au = 0$.

Proof. Let $x^* \in E$ such that $Ax^* = 0$. Using (3.17), Lemma 3.2 and the fact that $\|J^{-1}v\| = \|v\|$ for all $v \in E^*$, we have the following estimates:

$$\begin{aligned}(p-1)\|x_{n+1} - x^*\|^2 &= (p-1) \quad \|x_n - x^* - \lambda J^{-1}(Ax_n)\|^2 \\ &\leq \quad \|\lambda J^{-1}(Ax_n)\|^2 - 2\langle x_n - x^*, J(\lambda J^{-1}(Ax_n))\rangle + 1\|x_n - x^*\|^2 \\ &= \quad \lambda^2\|Ax_n\|^2 - 2\lambda\langle x_n - x^*, Ax_n\rangle + 1\|x_n - x^*\|^2.\end{aligned}$$

Using (i) and (ii), we obtain

$$(p-1)\|x_{n+1} - x^*\|^2 \quad \leq \quad \lambda^2 L^2\|x_n - x^*\|^2 - 2k\lambda\|x_n - x^*\|^2 + \|x_n - x^*\|^2,$$

which implies that

$$\|x_{n+1} - x^*\|^2 \leq \left(\frac{\lambda^2L^2 - 2k\lambda + 1}{p-1}\right)\|x_n - x^*\|^2.$$

Using the fact that $\lambda \in (\beta_1, \beta_2)$, it follows that $0 < \frac{\lambda^2L^2-2k\lambda+1}{p-1} < 1$ and we have,

$$\|x_{n+1} - x^*\| \leq \delta(\lambda)\|x_n - x^*\|, \tag{3.18}$$

where $\delta(\lambda) := \left(\frac{\lambda^2L^2-2k\lambda+1}{p-1}\right)^{\frac{1}{2}}$ Therefore the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof.

□

4. Application to convex minimization problems

In this section, we study the problem of finding a minimizer of a convex function f defined from a real Banach space E to $\mathbb{R}$.

The following basic results are well known.

Remark 4.5. It is well known that if $f : E \to \mathbb{R}$ is a real-valued differentiable convex function and $a \in E$, then the point a is a minimizer of f on E if and only if $df(a) = 0$.

Definition 4.1. A function $f : E \to \mathbb{R}$ is said to be strongly convex if there exists $\alpha > 0$ such that for every $x, y \in E$ with $x \neq y$ and $\lambda \in (0,1)$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{\lambda(1-\lambda)\alpha}{2}\|x-y\|^2. \tag{4.19}$$

Lemma 4.3. *Let E be normed linear space and $f : E \to \mathbb{R}$ a real-valued differentiable convex function. Assume that f is strongly convex. Then the differential map $df : E \to E^*$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle df(x) - df(y), x - y\rangle \geq k\|x-y\|^2 \ \forall\, x, y \in E. \tag{4.20}$$

We now prove the following theorem.

Theorem 4.8. *Let $E = L_p$,$p \geq 2$ and $f : E \to \mathbb{R}$ be a differentiable, strongly convex real-valued function which satisfies the growth condition: $f(x) \to +\infty$ as $\|x\| \to +\infty$. Assume that the differential map $df : E \to E^*$ is Lipschitz. Let k and L denote the constants from the strong convexity of f and the Lipschtz property of the differential map df of f. Assume that $2 \leq p < 2 + \frac{k^2}{L^2}$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be a sequence defined iteratively by:*

$$x_{n+1} = x_n - \lambda J^{-1}(df(x_n)),\ n \geq 1, \tag{4.21}$$

where $\lambda \in (\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{k}{L^2}$ and $\alpha_2 = \frac{k+\sqrt{k^2-L^2(p-2)}}{L^2}$.

Then, f has a unique minimizer $a^* \in E$ and the sequence $\{x_n\}$ converges strongly to a^*.

Proof. Since E is reflexive, then from the growth condition, the continuity and the strict convexity of f, it follws that f has a unique minimizer a^* characterized by $df(a^*) = 0$ (Remark 4.5). Finally, from Lemma 4.3 and the fact that the differential map $df : E \to E^*$ is Lipschitz, the proof follows from Corollary 3.1. □

Theorem 4.9. *Let $E = L_p$,$1 < p \leq 2$ and $f : E \to \mathbb{R}$ be a differentiable, strongly convex real-valued function which satisfies the growth condition: $f(x) \to +\infty$ as $\|x\| \to +\infty$. Assume that the differential map $df : E \to E^*$ is Lipschitz. Let k and L denote the constants from the strong convexity of f and the Lipschtz property of the differential map df of f. Assume that $2 - \frac{k^2}{L^2} < p \leq 2$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be a sequence defined iteratively by:*

$$x_{n+1} = x_n - \lambda J^{-1}(df(x_n)),\ n \geq 1, \tag{4.22}$$

where $\lambda \in (\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{k}{L^2}$ and $\alpha_2 = \frac{k+\sqrt{k^2-L^2(2-p)}}{L^2}$.

Then, f has a unique minimizer $a^* \in E$ and the sequence $\{x_n\}$ converges strongly to a^*.

Proof. Since E is reflexive, then from the growth condition, the continuity and the strict convexity of f, it follws that f has a unique minimizer a^* characterized by $df(a^*) = 0$ (Remark 4.5). Finally, from Lemma 4.3 and the fact that the differential map $df : E \to E^*$ is Lipschitz, the proof follows from Theorem 3.7. □

5. Illustration of the proposed algorithm in L^p spaces

From [1], the duality mapping J is known precisely in $L^p(\Omega)$ for $1 < p < \infty$ and is given by :

$$Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u,\ \ \forall\, u \in L^p(\Omega).$$

For $1 < p < \infty$, $L^p(\Omega)$ is a smooth, reflexive and strictly convex real Banach space, then its duality mapping J is one-to-one, surjective and its inverse J^{-1} is the duality mapping of $L^q(\Omega)$ with$1/p + 1/q = 1$. Therefore,

$$J^{-1}v = \|v\|_{L_q}^{2-q}|v|^{q-2}v, \ \ \forall\, v \in L^q(\Omega),$$

In this settings, the sequences $\{x_n\}$ defined in (3.12) is given iteratively from $x_1 \in L^p(\Omega)$ by:

$$x_{n+1} = x_n - \lambda\|Ax_n\|_{L_q}^{2-q}|Ax_n|^{q-2}Ax_n,\ n \geq 1. \tag{5.23}$$

From Corollary 3.1 and Theorem 3.7, it follows that the sequence $\{x_n\}$ given by (5.23) converges strongly to some $x^* \in L^p(\Omega)$, where x^* is the unique zero of the mapping A.

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