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On regular and intra-regular Γ-semihypergroups

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ABSTRACT. In this paper we have investigated the intra-regular Γ-semihypergroup S and presented its characterizations while exploring properties of Γ-hyperideals of S. We have also studied the Γ-semihypergroup which is both regular and intra-regular and proved some results in this regard. We have investigated converse of some results and provided counterexamples as well.

1. INTRODUCTION

The notion of Γ-semigroup was first introduced by Sen and Saha [26] as a natural generalization of semigroup and ternary semigroup. This has inspired algebraist to extend and generalize many concepts, result and properties of classical theory of semigroups and rings to Γ-semigroups [1, 7, 8, 18]. N. Yaqoob and M. Aslam have studied the properties of prime (m,n)-bi-Γ-ideals in Γ-semigroups.

Davvaz et al. [6, 14, 15, 16] introduced the concept of Γ-semihypergroup as a generalization of three algebraic structures semigroup, semihypergroup and Γ-semigroup. They have given many examples and explored the properties of Γ-semihypergroups on a large scale. The notion of ordered Γ-semihypergroup is given by M. Kondo and N. Lekkoksung in [22] where they have introduced bi-hyperideals, quasi-hyperideals, left hyperideals and right hyperideals in ordered Γ-semihypergroups and characterized intra-regular ordered Γ-semihypergroups in terms of their bi-hyperideals and quasi-hyperideals. Recently S. Omidi et al. [24] defined and used the notion of regular ordered Γ- semihypergroups to examine some classical results and properties in ordered Γ-semihypergroups.

Theory of hyperstructure attracts the algebraists when it was first proposed by the French mathematician Marty [23] in the year 1934. He defined hypergroups based on the notion of hyperoperation at the 8^{th} Congress of Scandinavian Mathematicians. Since then a number of different algebraic hyperstructures like hypergroups, semihypergroups, Γ-semihypergroups, hyperrrings etc. are being studied. In a nonempty set equipped with binary operation, the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements yields a nonempty set, thus the hyperstructures represents the most natural extension of classical algebraic structures. There are many researchers across the world who study hyperstructures and extend their contributions through research articles and books. A useful review of various algebraic hyperstructures and their applications in different fields can be found in [9, 10, 11, 12, 13].

Some motivations for the study of hyperstructures comes mainly from inside mathematics, computer sciences, biological inheritance, cryptography, theoretical physics, physical phenomenon as the nuclear fission, chemical reactions and redox reactions and a lot of other fields. This wide range of applications in various fields has led to the expansion and generalization of hyperstructures in recent decades, such as H_v-structures

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and Γ-hyperstructures. A lot of work has been done in general on the theory of Γ-hyperstructures, in particular, Γ-semihypergroups by many algebraists, preparing the mathematical background for further applications. Hyperideal theory is important not only for the intrinsic interest and purity of its logical structure but because it is a necessary tool in many branches of mathematics and its applications. Several related results have been obtained in Γ-semihypergroup. However, the very fundamental results of Γ-hyperideals in different important classes of Γ-semihypergroups remained yet untouched. Therefore a need was felt to study further these classes in terms of hyperideals theory.

In this paper, we have generalized the results of Γ-semigroup and Γ-semiring to that of Γ-semihypergroups. We have studied the behaviour of Γ-hyperideals in intra-regular Γ-semihypergroup and in the Γ-semihypergroup which is both regular and intra-regular. We have investigated bi-Γ-hyperideals and quasi Γ-hyperideals in regular and intra-regular Γ-semihypergroup and used them to present the characterization of the regular and intra-regular Γ-semihypergroups. We have also explored the converse of few results and have given examples to support our claims.

2. Preliminaries

We start the section by recalling the notion of Γ-semihypergroup and some definitions and properties from [2, 6, 15] required to proceed with the article.

Definition 2.1. [6] Let H be a nonempty set and $\circ : H \times H \to \wp^*(H)$ be a hyperoperation, where $\wp^*(H)$ is the family of all nonempty subsets of H. The pair $(H, \circ)$ is called a $hypergroupoid$.

For any two nonempty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. [6] A hypergroupoid $(H, \circ)$ is called a $semihypergroup$ if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

In addition if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a $hypergroup$.

Let S be a nonempty set and Γ a nonempty set of binary operations on S. Then, S is called a Γ-semigroup if

(1) $s_1 \alpha s_2 \in S$,
(2) $(s_1 \alpha s_2) \beta s_3 = s_1 \alpha (s_2 \beta s_3)$

for all $s_1, s_2, s_3 \in S$ and all $\alpha, \beta \in \Gamma$. In [21], Kehayopulu added the following property in the definition:

(3) If $s_1, s_2, s_3, s_4 \in S$, $\gamma_1, \gamma_2 \in \Gamma$, are such that $s_1 = s_3, \gamma_1 = \gamma_2$ and $s_2 = s_4$, then $s_1 \gamma_1 s_2 = s_3 \gamma_2 s_4$.

Definition 2.3. [15, 16] Let S and Γ be two nonempty sets. Then S is called a Γ-$semihypergroup$ if every $\gamma \in \Gamma$ is a hyperoperation on S that is, $x\gamma y \subseteq S$ for every $x, y \in S$ such that, for $a, b, c, d \in S, \gamma, \gamma_1 \in \Gamma, a = c, b = d, \gamma = \gamma_1$ imply $a\gamma b = c\gamma_1 d$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property

$$x\alpha(y\beta z) = (x\alpha y)\beta z,$$

which means that, $\forall x, y, z \in S, \alpha, \beta \in \Gamma$, we have

$$\bigcup_{u\in x\alpha y} u\,\beta\,z = \bigcup_{v\in(y\beta z)} x\,\alpha\,v.$$

It is clear that, in the above definition, if every $\gamma \in \Gamma$ is an operation, then S is a Γ-semigroup.

Let A and B be two nonempty subsets of S and $\gamma \in \Gamma$, we denote the following:

$$A\gamma B = \bigcup_{a\in A, b\in B} a\gamma b$$

also,

$$A\Gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

A Γ-semihypergroup S is said to be commutative if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$. If (S, γ) is a hypergroup for every $\gamma \in \Gamma$, then S is called a Γ-hypergroup.

Example 2.1. [6] Let $S = [0,1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $\gamma : S \times S \longrightarrow \wp^*(S)$ by $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$. Then γ is a hyperoperation on S and $x\alpha(y\beta z) = \left[0, \frac{xyz}{\alpha\beta}\right] = (x\alpha y)\beta z$. This means that S is a Γ-semihypergroup.

Definition 2.4. [15] A nonempty subset A of Γ-semihypergroup S is said to be a Γ-$subsemihyp-ergroup$ if $A\Gamma A \subseteq A$ i.e. $a\gamma b \subseteq A$ for every $a, b \in A$ and $\gamma \in \Gamma$.

Definition 2.5. [15] A nonempty subset A of a Γ-semihypergroup S is said to be a $left$ $(right)$ Γ-$hyperideal$ if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$).

A is said to be a $two\ sided$ Γ-$hyperideal$ or simply a Γ-$hyperideal$ if it is both left and right Γ-hyperideal.

S is called a $left$ $(right)$ $simple$ Γ-$semihypergroup$ if it has no proper left (right) Γ-hyperideal. S is said to be a $simple$ Γ-semihypergroup if it has no proper Γ-hyperideal.

Example 2.2. In Example 2.1, let $T = [0, t]$, where $t \in [0,1]$. Then T is left (right) Γ-hyperideal of S.

Definition 2.6. [15] A proper ideal P of a Γ-semihypergroup S is called a $prime\ ideal$, if for every ideal I, J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Theorem 2.1. [15] *Let S be a Γ-semihypergroup and P be a left ideal of S. Then P is prime if and only if for all $x, y \in S, x\Gamma S\Gamma y \subseteq P$ implies $x \in P$ or $y \in P$.*

Definition 2.7. [25] An element a of Γ-semihypergroup S is said to be an α-idempotent if $a \in a\alpha a$. An element a of Γ- semihypergroup S is said to be a Γ-idempotent or simply idempotent if $a \in a\alpha a$ for all $\alpha \in \Gamma$ i.e. $a \in a\Gamma a$.

Definition 2.8. [4] A Γ-semihypergroup S is said to be an idempotent Γ-semihypergroup if every element in S is a Γ-idempotent.

Definition 2.9. A nonempty subset A of a Γ-semihypergroup S is said to be a Γ-$idempotent$ subset of S if $A \subseteq A\Gamma A$. A is said to be globally idempotent if $A = A\Gamma A$.

Definition 2.10. [17] Let S be a Γ-semihypergroup and Q a nonempty subset of S. Then Q is called a $quasi$ Γ-$hyperideal$ of S if $Q\Gamma S \cap S\Gamma Q \subseteq Q$.

Theorem 2.2. [6] *Let S be a Γ-semihypergroup and R, L be right and left Γ-hyperideal of S, respectively. Then $R \cap L$ is a quasi Γ-hyperideal of S.*

Theorem 2.3. [6] *Every quasi Γ-hyperideal of S is the intersection of a right Γ-hyperideal and a left Γ-hyperideal of S.*

S. Abdullah et al. [2] defined bi-Γ-hyperideals of Γ-semihypergroup as follows.

Definition 2.11. [2] A nonempty subset B of a Γ-semihypergroup S is called *bi-Γ-hyperideal* of S if the following two conditions hold

(1) $B\Gamma B \subseteq B$
(2) $B\Gamma S\Gamma B \subseteq B$

A bi-Γ-hyperideal B of Γ-semihypergroups S is proper if $B \neq S$.

Example 2.3. [2] Let S=(0, 1), $\Gamma = \{\gamma_n | n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define the hyperoperation γ_n on S as follows

$$x\gamma_n y = \left\{\frac{xy}{2^k} | 0 \leq k \leq n\right\}, \text{for all } x, y \in S.$$

Then S is a Γ-semihypergroup. Let $S_i = \left(0, \frac{1}{2^i}\right)$ where $i \in \mathbb{N}$. Then S_i is a bi-Γ-hyperideal of S.

Example 2.4. [2] In example 2.2 T is a bi-Γ-hyperideal of S.

Proposition 2.1. [3] *In a* Γ*-semihypergroup* S *the following statements are true.*

(1) Every left (right, two sided) Γ*-hyperideal of* S *is a bi-*Γ*-hyperideal of* S*.*
(2) Intersection of a left Γ*-hyperideal of* S *and right* Γ*-hyperideal of* S *is a bi-*Γ*-hyperideal of* S*.*
(3) If B *is a bi-*Γ*-hyperideal of* S*, then* $S\Gamma B$ *and* $B\Gamma S$ *are also bi-*Γ*-hyperideals of* S*.*
(4) Every quasi Γ*-hyperideal of* S *is a bi-*Γ*-hyperideal of* S*.*

Readers are recommended to read [2, 3, 17] to know more about bi-Γ-hyperideals and quasi-Γ-hyperideals in Γ-semihypergroup.

3. Intra-regular Γ-semihypergroups

The concept of intra-regular Γ-semihypergroup generalizes the corresponding concept of intra-regular Γ-semigroup. In this section, we investigate intra-regular Γ- semihypergroup with examples and build its characterization in terms of left and right Γ-hyperideals and further in terms of bi-Γ-hyperideal and quasi Γ-hyperideal. Our work in this paper is inspired by the research of R.D. Jagtap and Y.S. Pawar [20].

Definition 3.12. [2] Let S be a Γ-semihypergroup. An element $x \in S$ is said to be an intra-regular element if there exits $y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x \in y\alpha x\beta x\gamma z$ or equivalently if $x \in S\Gamma x\Gamma x\Gamma S$.
A Γ-semihypergroup S is said to be an intra-regular Γ-semihypergroup if every element of S is intra-regular.

Example 3.5. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$ be the set of binary operations defined as follows:

α	e	a	b	c	d
e	e	e	e	e	e
a	e	$\{a,b\}$	b	b	b
b	e	b	b	b	b
c	e	c	c	c	c
d	e	d	d	d	d

β	e	a	b	c	d
e	e	e	e	e	e
a	e	a	a	a	a
b	e	a	$\{a,b\}$	a	a
c	e	c	c	c	c
d	e	d	d	d	d

then S is a Γ-semihypergroup [27]. It is easy to check that every element of S is intra-regular hence S is an intra-regular Γ-semihypergroup.

Proposition 3.2. *Every* Γ*-hyperideal of an intra-regular* Γ*-semihypergroup* S *is globally idempotent.*

Proof. Proof is straightforward. □

Definition 3.13. Let S be a Γ-semihypergroup and $x \in S$. Then the intersection of all left Γ-hypeideals (respectively right Γ-hyperideals) of S containing x is a left Γ-hyperideal (respectively right Γ-hyperideal) generated by x and is denoted by $< x >_l$ (respectively $< x >_r$).

By definition it is clear that $< x >_l$ (respectively $< x >_r$) is the smallest left (respectively right) Γ-hyperideal of S containing x.

Proposition 3.3. *Let* S *be a* Γ*-semihypergroup. Then for* $x \in S$*, the left and the right* Γ*-hyperideals generated by* x *are given as* $< x >_l = x \cup S\Gamma x$ *and* $< x >_r = x \cup x\Gamma S$ *respectively.*

Proof. Proof is elementary. □

Definition 3.14. [15] Let A be a nonempty subset of a Γ-semihypergroup S. Then intersection of all ideals of S containing A is an ideal of S generated by S, and denoted by $< A >$.

Lemma 3.1. [15] *Let* S *be a* Γ*-semihypergroup. If* A *is a nonempty subset of* S*, then* $< A >= A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S$.

Proposition 3.4. *Let* I *be a* Γ*-hyperideal of* S *and* A *be a* Γ*-hyperideal of* I*. Then* $< A > \Gamma < A > \Gamma < A >\subseteq A$.

Proof. Let S be a Γ-semihypergroup and A be a Γ-hyperideal of a Γ-hyperideal I of S. Consider $< A > \Gamma < A > \Gamma < A >\subseteq I\Gamma < A > \Gamma I = I\Gamma(A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S)\Gamma I = (I\Gamma A \cup I\Gamma A\Gamma S \cup I\Gamma S\Gamma A \cup I\Gamma S\Gamma A\Gamma S)\Gamma I \subseteq (A \cup A\Gamma S \cup I\Gamma A \cup I\Gamma A\Gamma S)\Gamma I \subseteq (A \cup A\Gamma S \cup A \cup A\Gamma S)\Gamma I = A\Gamma I \cup A\Gamma S\Gamma I \cup A\Gamma I \cup A\Gamma S\Gamma I \subseteq A \cup A\Gamma I \cup A \cup A\Gamma I \subseteq A \cup A \cup A \cup A = A$ because A is Γ-hyperideal of I and I is Γ-hyperideal of S. Thus, $< A > \Gamma < A > \Gamma < A >\subseteq A$. □

Remark 3.1. A Γ-hyperideal of a Γ-hyperideal of a Γ-semihypergroup S need not be a Γ-hyperideal of S.

To see this consider the following example.

Example 3.6. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$ be the set of binary hyperoperations defined as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	$\{b, c\}$

A mapping $S \times \Gamma \times S \to P^*(S)$ defined by $x\gamma y = x \cdot y$ for all $x, y \in S$ and for all $\gamma \in \Gamma$. Then S is a Γ-semihypergroup. Here $I = \{a, b, c\}$ is a Γ-hyperideal of S because $S\Gamma I = I\Gamma S = \{a, b\} \subseteq I$. For $A = \{a, c\}$, A is a Γ-hyperideal of I because $I\Gamma A = A\Gamma I = \{a\} \subseteq A$ but A is not a Γ-hyperideal of S, for $S\Gamma A = A\Gamma S = \{a, b\} \nsubseteq A$.

In an intra-regular Γ-semihypergroup S we have the following observation.

Theorem 3.4. *Let* I *be a* Γ*-hyperideal of an intra-regular* Γ*-semihypergroup* S*. If* A *is a* Γ*-hyperideal of* I*, then* A *is* Γ*-hyperideal of* S *also.*

Proof. Proof is elementary. □

Converse of the Theorem 3.4 need not be true. To see this consider the following example.

Example 3.7. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$ be the set of binary hyperoperations defined as follows:

α	a	b	c	d	e
a	$\{a, b\}$	$\{b, c\}$	c	$\{d, e\}$	e
b	$\{b, c\}$	c	c	$\{d, e\}$	e
c	c	c	c	$\{d, e\}$	e
d	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	d	e
e	e	e	e	e	e

α	a	b	c	d	e
a	$\{b, c\}$	c	c	$\{d, e\}$	e
b	c	c	c	$\{d, e\}$	e
c	c	c	c	$\{d, e\}$	e
d	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	d	e
e	e	e	e	e	e

Then S is a Γ-semihypergroup [27]. All Γ-hyperideals of S are given by $I_1 = \{e\}$, $I_2 = \{d, e\}$, $I_3 = \{c, d, e\}$ and $I_4 = \{b, c, d, e\}$. It is easy to check that every Γ-hyperideal of Γ-hyperideal I_1, I_2, I_3 and I_4 is a Γ-hyperideal of S but S is not an intra-regular Γ-semihypergroup because $b \notin S\Gamma b\Gamma b\Gamma S$.

Theorem 3.5. *Let I, J be two Γ-hyperideals of a Γ-semihypergroup S with $I\Gamma I \subseteq J$. If S is an intra-regular Γ-semihypergroup, then $I \subseteq J$.*

Proof. Straightforward. □

Converse of the above result need not be true. That is, for any two Γ-hyperideals I, J of a Γ-semihypergroup S, if $I\Gamma I \subseteq J$ implies that $I \subseteq J$, then S need not be an intra-regular Γ-semihypergroup.

Example 3.8. Consider the example of Γ-semihypergroup given in example 3.7, we see that $I_1\Gamma I_1 \subseteq I_1, I_2, I_3, I_4$, $I_2\Gamma I_2 \subseteq I_2, I_3, I_4$, $I_3\Gamma I_3 \subseteq I_3, I_4$ and $I_4\Gamma I_4 \subseteq I_4$ implies that $I_1 \subseteq I_1, I_2, I_3, I_4$, $I_2 \subseteq I_2, I_3, I_4$, $I_3 \subseteq I_3, I_4$, $I_4 \subseteq I_4$. But S is not an intra-regular Γ-semihypergroup.

In the following result a very important characterization of an intra-regular Γ- semihypergroup is presented which will be of great help to prove that a Γ-semihyergroup is intra-regular in forthcoming part of the paper.

Theorem 3.6. *A Γ-semihypergroup S is intra-regular if and only if for any left Γ-hyperideal I and for any right Γ-hyperideal J of S, $I \cap J \subseteq I\Gamma J$.*

Proof. Let S be an intra-regular Γ-semihypergroup and I, J be the left and right Γ- hyperideals of S respectively. For $x \in I \cap J$ we have $x \in S\Gamma x\Gamma x\Gamma S$ because S is an intra-regular Γ-semihypergroup. Therefore, $x \in S\Gamma x\Gamma x\Gamma S \subseteq S\Gamma I\Gamma J\Gamma S = (S\Gamma I)\Gamma(J\Gamma S) \subseteq I\Gamma J$ for all $x \in I \cap J$. Thus we obtain $I \cap J \subseteq I\Gamma J$. Conversely, assume that for any left Γ-hyperideal I and for any right Γ-hyperideal J of S we have $I \cap J \subseteq I\Gamma J$. We know that for any $x \in S, < x >_l$ and $< x >_r$ are the left and right Γ-hyperideals of S respectively, and both contain x hence by assumption we have $< x >_l \cap < x >_r \subseteq < x >_l \Gamma < x >_r$. By Proposition 3.3 we have

$$\begin{aligned} < x >_l \cap < x >_r &\subseteq < x >_l \Gamma < x >_r \\ &= (x \cup S\Gamma x)\Gamma(x \cup x\Gamma S) \\ &= x\Gamma(x \cup x\Gamma S) \cup S\Gamma x\Gamma(x \cup x\Gamma S) \\ &= x\Gamma x \cup x\Gamma x\Gamma S \cup S\Gamma x\Gamma x \cup S\Gamma x\Gamma x\Gamma S \qquad \text{.............(1)} \end{aligned}$$

Consider S as a right Γ-hyperideal of S, by hypothesis we have

$$x \in < x >_l = < x >_l \cap S \subseteq < x >_l \Gamma S = (x \cup S\Gamma x)\Gamma S = x\Gamma S \cup S\Gamma x\Gamma S$$

this implies that $x \in x\Gamma S \cup S\Gamma x\Gamma S$. Similarly treating S as left Γ-hyperideal of S we can write

$$x \in <x>_r = S\cap <x>_r \subseteq S\Gamma <x>_r = S\Gamma(x \cup x\Gamma S) = S\Gamma x \cup S\Gamma x\Gamma S$$

this implies that $x \in S\Gamma x \cup S\Gamma x\Gamma S$. Consider

$$\begin{aligned} S\Gamma x \cup S\Gamma x\Gamma S &\subseteq S\Gamma(x\Gamma S \cup S\Gamma x\Gamma S) \cup S\Gamma x\Gamma S \\ &= S\Gamma x\Gamma S \cup S\Gamma S\Gamma x\Gamma S \cup S\Gamma x\Gamma S \\ &\subseteq S\Gamma x\Gamma S \cup S\Gamma x\Gamma S \cup S\Gamma x\Gamma S \\ &= S\Gamma x\Gamma S \qquad \text{.............(2)} \end{aligned}$$

from (1) we have

$$\begin{aligned} S\Gamma x\Gamma S &\subseteq S\Gamma(x\Gamma x \cup x\Gamma x\Gamma S \cup S\Gamma x\Gamma x \cup S\Gamma x\Gamma x\Gamma S)\Gamma S \\ &= (S\Gamma x\Gamma x \cup S\Gamma x\Gamma x\Gamma S \cup S\Gamma S\Gamma x\Gamma x \cup S\Gamma S\Gamma x\Gamma x\Gamma S)\Gamma S \\ &\subseteq (S\Gamma x\Gamma x \cup S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x \cup S\Gamma x\Gamma x\Gamma S)\Gamma S \\ &= S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x\Gamma S\Gamma S \cup S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x\Gamma S\Gamma S \\ &\subseteq S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x\Gamma S \cup S\Gamma x\Gamma x\Gamma S \\ &= S\Gamma x\Gamma x\Gamma S \qquad \text{.............(3)} \end{aligned}$$

Thus from (2) and (3) we have $x \in S\Gamma x \cup S\Gamma x\Gamma S \subseteq S\Gamma x\Gamma S \subseteq S\Gamma x\Gamma x\Gamma S$, that is for every $x \in S, x \in S\Gamma x\Gamma x\Gamma S$, hence x is an intra-regular element. Therefore S is an intra-regular Γ-semihypergroup. □

Now in the next two results characterization of intra-regular Γ-semihypergroup is presented in terms of quasi Γ-hyperideals and bi-Γ-hyperideals.

Theorem 3.7. *Let S be a Γ-semihypergroup. The following statements are equivalent in S.*

(1) S is an intra-regular Γ-semihypergroup.

(2) For any two bi-Γ-hyperideals B_1 and B_2 of S,
$B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.

(3) For a bi-Γ-hyperideal B and a quasi Γ-hyperideal Q of S,
$B \cap Q \subseteq (S\Gamma B\Gamma Q\Gamma S) \cap (S\Gamma Q\Gamma B\Gamma S)$.

(4) For any two quasi Γ-hyperideals Q_1 and Q_2 of S,
$Q_1 \cap Q_2 \subseteq S\Gamma Q_1\Gamma Q_2\Gamma S$.

Proof. (1) $\Longrightarrow$ (2)
Assume that S is an intra-regular Γ-semihypergroup and B_1 , B_2 be the bi-Γ-hyperideals of S. Let $x \in B_1 \cap B_2$ implies that $x \in S$ and S is an intra-regular Γ-semihypergroup hence $x \in S\Gamma x\Gamma x\Gamma S \subseteq S\Gamma B_1\Gamma B_2\Gamma S$ for every $x \in B_1 \cap B_2$. Therefore $B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.
(2) $\Longrightarrow$ (3)
Assume that (2) holds in a Γ-semihypergroup S and let Q be a quasi Γ-hyperideal of S by (4) of Proposition 2.1 we see that Q is a bi-Γ-hyperideal of S, therefore by hypothesis, $B \cap Q \subseteq S\Gamma B\Gamma Q\Gamma S$ and $Q \cap B \subseteq S\Gamma Q\Gamma B\Gamma S$. But $B \cap Q = Q \cap B$ therefore $B \cap Q \subseteq S\Gamma B\Gamma Q\Gamma S \cap S\Gamma Q\Gamma B\Gamma S$.
(3) $\Longrightarrow$ (4)
Assume that (3) holds in a Γ-semihypergroup S and let Q_1, Q_2 be the two quasi Γ-hyperideals of S. By (4) of Proposition 2.1 we see that Q_1 and Q_2 are bi-Γ-hyperideals of S. Therefore from assumption we have $Q_1 \cap Q_2 \subseteq S\Gamma Q_1\Gamma Q_2\Gamma S$.
(4) $\Longrightarrow$ (1)
Assume that the statement (4) holds in a Γ-semihyperoup S and let I, J be the left and

right Γ-hyperideals of S respectively. By Theorem 2.2 we see that $I \cap J$ is a quasi Γ-hyperideal of S. Therefore by assumption we have $I \cap J = (I \cap J) \cap (I \cap J) \subseteq S\Gamma(I \cap J)\Gamma(I \cap J)\Gamma S \subseteq S\Gamma I\Gamma J\Gamma S \subseteq I\Gamma J$, that is $I \cap J \subseteq I\Gamma J$. By Theorem 3.6 we conclude that S is an intra-regular Γ-semihypergroup. □

Proposition 3.5. *Every one sided Γ-hyperideal of a Γ-semihypergroup S is a quasi Γ-hyperideal.*

Proof. Straightforward. □

Theorem 3.8. *In a Γ-semihypergroup S following statements are equivalent.*

(1) S is an intra-regular Γ-semihypergroup.

(2) For a left Γ-hyperideal I and bi-Γ-hyperideal B of S, $I \cap B \subseteq I\Gamma B\Gamma S$.

(3) For a left Γ-hyperideal I and a quasi Γ-hyperideal Q of S, $I \cap Q \subseteq I\Gamma Q\Gamma S$.

(4) For a right Γ-hyperideal J and a bi-Γ-hyperideal B of S, $J \cap B \subseteq S\Gamma B\Gamma J$.

(5) For a right Γ-hyperideal J and a quasi Γ-hyperideal Q of S, $J \cap Q \subseteq S\Gamma Q\Gamma J$.

Proof. $(1) \Longrightarrow (2)$
Straightforward.
$(2) \Longrightarrow (3)$
Assume that (2) holds in a Γ-semihypergroup S and let I, Q be the left Γ-hyperideal and quasi Γ-hyperideal of S respectively. By (4) of Proposition 2.1 we know that every quasi Γ-hyperideal is a bi-Γ-hyperideal hence Q is a bi-Γ-hyperideal of S and by (2) we have $I \cap Q \subseteq I\Gamma Q\Gamma S$.
$(3) \Longrightarrow (1)$
Assume that (3) holds in a Γ-semihypergroup S and let I, J be the left Γ-hyperideal and right Γ-hyperideal of S respectively. By Proposition 3.5 J is a quasi Γ-hyperideal of S therefore by (3) we have $I \cap J \subseteq I\Gamma J\Gamma S \subseteq I\Gamma J$ because J is a right Γ-hyperideal of S. Thus we obtained $I \cap J \subseteq I\Gamma J$ for every left Γ-hyperideal I and every right Γ-hyperideal J of S hence by Theorem 3.6 we conclude that S is an intra-regular Γ-semihypergroup.
$(1) \Longrightarrow (4)$
Let S be an intra-regular Γ-semihypergroup and J, B be the right Γ-hyperideal and bi-Γ-hyperideal of S respectively. Take $x \in J \cap B$ implies that $x \in S$ and S is an intra-regular Γ-semihypergroup therefore $x \in S\Gamma x\Gamma x\Gamma S \subseteq S\Gamma B\Gamma J\Gamma S \subseteq S\Gamma B\Gamma J$ for all $x \in J \cap B$. Hence $J \cap B \subseteq S\Gamma B\Gamma J$.
$(4) \Longrightarrow (5)$
Assume that (4) holds in a Γ-semihypergroup S and let J, Q be the right Γ-hyperideal and quasi Γ-hyperideal of S respectively. Because every quasi Γ-hyperideal is a bi-Γ-hyperideal of S by (4) of Proposition 2.1, we see that Q is a bi-Γ-hyperideal of S hence by (4) we obtain $J \cap Q \subseteq S\Gamma Q\Gamma J$.
$(5) \Longrightarrow (1)$
Assume that (5) holds in a Γ-semihypergroup S and let I, J be the left Γ-hyperideal and right Γ-hyperideal of S respectively. By Proposition 3.5 it is known that every left Γ-hyperideal is a quasi Γ-hyperideal hence I is a quasi Γ-hyperideal of S and by (4) we have $I \cap J = J \cap I \subseteq S\Gamma I\Gamma J \subseteq I\Gamma J$ because I is a left Γ-hyperideal of S. Thus $I \cap J \subseteq I\Gamma J$, therefore by Theorem 3.6 S is an intra-regular Γ-semihypergroup.
Thus it is established that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and $(1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ and this proves that all the statements are equivalent. □

4. Regular and Intra-Regular Γ-Semihypergroups

Regular and Intra-regular Γ-semihypergroups have been introduced by S. Abdullah et al. in [2]. We begin this section by presenting the definition of regular Γ-semihypergroup and it characterizations which will be needed as the section unfolds. We have discussed properties of Γ-semihypergroup which is both regular and intra-regular and later presented few characterizations as well.

Definition 4.15. [2] An element a of the Γ-semihypergroup S is called $regular$ if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. If every element of Γ-semihypergroup S is regular, then S is called a $regular$ Γ-$semihypergroup$.

Theorem 4.9. [5] *S is a regular* Γ*-semihypergroup if and only if for any left* Γ*-hyperideal I and for any right* Γ*-hyperideal J of S,* $I \cap J = J\Gamma I$.

Theorem 4.10. [5] *In a* Γ*-semihypergroup* S*, if one of the following statements is true, then so are the others.*

(1) S is a regular Γ*-semihypergroup.*

*(2) For any bi-*Γ*-hyperideal B of S,* $B\Gamma S\Gamma B = B$.

(3) For any quasi Γ*-hyperideal Q of S,* $Q\Gamma S\Gamma Q = Q$.

Remark 4.2. A regular Γ-semihypergroup need not be an intra-regular. To see this, consider the following example.

Example 4.9. *Let* $S = \{a, b, c, d, e, f\}$ *and* $\Gamma = \{\alpha, \beta\}$ *be the set of binary operations defined as follows.*

α	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

Then S is a Γ*-semihypergroup* [27]. *Moreover as* $a \in a\alpha c\beta a$, $b \in b\beta e\alpha b$, $c \in c\alpha f\alpha c$, $d \in d\alpha f\beta d$, $e \in e\alpha f\beta e$ *and* $f \in f\beta d\alpha f$*, that is every element of S is a regular element, we see that S is a regular* Γ*-semihypergroup. But S is not an intra-regular* Γ*-semihypergroup because* $e \notin S\Gamma e\Gamma e\Gamma S$.

Remark 4.3. There exists a Γ-semihypergroup which is neither regular nor intra-regular. To see this consider the following example.

Example 4.10. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$ be the set of binary hyperoperations defined as follows.

α	a	b	c	d	e
a	$\{a,b\}$	$\{b,e\}$	c	$\{c,d\}$	e
b	$\{b,e\}$	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

β	a	b	c	d	e
a	$\{b,e\}$	e	c	$\{c,d\}$	e
b	e	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

Then S is a Γ-semihypergroup [28]. S is not regular because $b \notin b\Gamma S\Gamma b$. Similarly, $b \notin S\Gamma b\Gamma b\Gamma S$ hence S is not intra-regular.

Example 4.11. The Γ-semihyeprgroup given in example 3.7 is neither regular nor intra-regular because $b \notin b\Gamma S\Gamma b$ and $b \notin S\Gamma b\Gamma b\Gamma S$.

Remark 4.4. There do exist a Γ-semihypergroup which is both regular and intra-regular. For this consider the following examples.

Example 4.12. Let $S = \{e, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$ be the set of binary operations defined as follows.

α	e	a	b	c
e	e	$\{a,b\}$	$\{a,b\}$	c
a	$\{a,b\}$	c	c	e
b	$\{a,b\}$	c	c	e
c	c	e	e	$\{a,b\}$

β	e	a	b	c
e	c	e	e	$\{a,b\}$
a	e	$\{a,b\}$	$\{a,b\}$	c
b	e	$\{a,b\}$	$\{a,b\}$	c
c	$\{a,b\}$	c	c	e

Then S is a Γ-semihypergroup [19], where $e \in e\alpha a\beta e$, $a \in a\alpha e\beta a$, $b \in b\alpha e\beta b$ and $c \in c\alpha c\beta c$, that is every element of S is reular hence S is a regular Γ-semihypergroup. Similarly, $e \in a\alpha e\beta e\beta a$, $a \in e\beta a\alpha a\beta b$, $b \in e\alpha b\beta b\beta a$ and $c \in a\alpha c\alpha c\beta b$, that is every element of S is intra-regular therefore S is an intra-regular Γ-semihypergroup.

Example 4.13. The Γ-semihypergroup discussed in example 3.5 is both regular and intra-regular.

Remark 4.5. At this point, based upon intuition we state that an intra-regular Γ-semihypergroup need not be regular. To find example for this is left upon the reader as an open exercise.

Now in the next few results we will see the properties of Γ-semihypergroup when it is both regular and intra-regular.

Proposition 4.6. *Let I, J and Q be the left Γ-hyperideal, right Γ-hyperideal and quasi Γ-hyperideal of a Γ-semihypergroup S. If S is both regular and intra-regular Γ-semihypergroup, then $I \cap Q \cap J \subseteq I\Gamma Q\Gamma J$.*

Proof. Let S be regular as well as an intra-regular Γ-semihypergroup and I, J and Q be the left Γ-hyperideal, right Γ-hyperideal and quasi Γ-hyperideal of S respectively. Take $x \in I \cap Q \cap J$ implies that $x \in S$ and S is both regular and intra-regular hence $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$. Consider $x\Gamma S\Gamma x \subseteq (S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S) = (S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma I)\Gamma(Q\Gamma S\Gamma Q)\Gamma(J\Gamma S)$. Now using the fact that I is left Γ-hyperideal and J is right Γ-hyperideal and by Theorem 4.10 we have $x\Gamma S\Gamma x \subseteq I\Gamma Q\Gamma J$. Thus for any $x \in I \cap Q \cap J$, $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq I\Gamma Q\Gamma J$ implies that $I \cap Q \cap J \subseteq I\Gamma Q\Gamma J$. □

Proposition 4.7. *Let S be a regular as well as an intra-regular Γ-semihypergroup. Then for a left Γ-hyperideal I, a right Γ-hyperideal J and a bi-Γ-hyperideal B of S, $I \cap B \cap J \subseteq I\Gamma B\Gamma J$.*

Proof. Assume that S is both regular and an intra-regular Γ- semihypergroup. Let I, J and B be the left Γ-hyperideal, right Γ-hyperideal and bi-Γ-hyperideal of S respectively. For $x \in I \cap B \cap J$, $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ because S is regular as well as an intra-regular Γ-semihypergroup. Therefore $x\Gamma S\Gamma x \subseteq (S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S) = (S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma x)\Gamma(x\Gamma S\Gamma x)\Gamma(x\Gamma S) \subseteq (S\Gamma I)\Gamma(B\Gamma S\Gamma B)\Gamma(J\Gamma S) \subseteq I\Gamma B\Gamma J$ because I and J are left and right Γ-hyperideals of S respectively and $B\Gamma S\Gamma B = B$ in a regular Γ-semihypergroup by Theorem 4.10. □

Following are the characterizations of Γ-semihypergroup which is both regular and intra-regular.

Theorem 4.11. *In a* Γ*-semihypergroup S the following statements are equivalent.*

(1) S is regular and intra-regular Γ*-semihypergroup.*

(2) For any left Γ*-hyperideal I and any right* Γ*-hyperideal J of S, $I \cap J = J\Gamma I \subseteq I\Gamma J$.*

*(3) For any bi-*Γ*-hyperideal B of S, $B\Gamma B = B$.*

(4) For any quasi Γ*-hyperideal Q of S, $Q\Gamma Q = Q$.*

Proof. $(1) \Longrightarrow (2)$

Let S be a regular as well as an intra-regular Γ-semihypergroup and I, J be the left and right Γ-hyperideals of S respectively. By Theorem 4.9 we have $I \cap J = J\Gamma I$ and by the characterization Theorem 3.6 of intra-regular Γ-semihypergroup we have $I \cap J \subseteq I\Gamma J$. Combining these two results we can write $I \cap J = J\Gamma I \subseteq I\Gamma J$.

$(2) \Longrightarrow (1)$

Assume that in a Γ-semihypergroup S we always have $I \cap J = J\Gamma I \subseteq I\Gamma J$ for any left Γ-hyperideal I and right Γ-hyperideal J of S. That is $I \cap J = J\Gamma I$ which implies that S is a regular Γ-semihypergroup by Theorem 4.9 and $I \cap J \subseteq I\Gamma J$ implies that S is an intra-regular Γ-semihypergroup by Theorem 3.6.

$(1) \Longrightarrow (3)$

Suppose that S is both regular and intra-regular Γ-semihypergroup and B be the bi-Γ-hyperideal of S. For any $x \in B$, $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ because S is both regular and intra-regular. Therefore $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq (x\Gamma S)\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma(S\Gamma x) = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (B\Gamma S\Gamma B)\Gamma(B\Gamma S\Gamma B) = B\Gamma B$ since in a regular Γ-semihypergroup, we have $B\Gamma S\Gamma B = B$ by Theorem 4.10. Thus we obtain whenever $x \in B$, $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq B\Gamma B$, therefore $B \subseteq B\Gamma B$. By definition of bi-Γ-hyperideal we have $B\Gamma B \subseteq B$, hence $B\Gamma B = B$.

$(3) \Longrightarrow (4)$

Assume that (3) holds in a Γ-semihypergroup S and let Q be the quasi Γ-hyperideal of S. We know that every quasi Γ-hyperideal is a bi-Γ-hyperideal by Proposition 2.1, therefore by (3) we have $Q\Gamma Q = Q$. $(4) \Longrightarrow (1)$

Assume that (4) holds in a Γ-semihypergroup S and let I, J be the left Γ-hyperideal and right Γ-hyperideal of S respectively. By Theorem 2.2 we see that $I \cap J$ is also a quasi Γ-hyperideal of S. Therefore by (4) we can write $I \cap J = (I \cap J)\Gamma(I \cap J) \subseteq I\Gamma J$ that is $I \cap J \subseteq I\Gamma J$ for any left Γ-hyperideal I and any right Γ-hyperideal J of S, hence by Theorem 3.6 we conclude that S is an intra-regular Γ-semihypergroup. Similarly $I \cap J = (I \cap J)\Gamma(I \cap J) \subseteq J\Gamma I$ and $J\Gamma I \subseteq J\Gamma S \subseteq J$, $J\Gamma I \subseteq S\Gamma I \subseteq I$ implies that $J\Gamma I \subseteq I \cap J$. Thus we obtain $I \cap J = J\Gamma I$ therefore by Theorem 4.9 we conclude that S is a regular Γ-semihypergroup. It is proved that if (4) holds in a Γ-semihypergroup S, then S is both regular and intra-regular Γ-semihypergroup. Thus we have proved $(1) \Leftrightarrow (2), (1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ hence all the four statements are equivalent. □

Theorem 4.12. *In a* Γ*-semihypergroup S the following statements are equivalent.*

(1) S is both regular and intra-regular Γ*-semihypergroup.*

*(2) For any two bi-*Γ*-hyperideals B_1 and B_2 of S,*
$B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$

*(3) For any bi-*Γ*-hyperideal B and any quasi* Γ*-hyperideal Q of S,*
$B \cap Q \subseteq (B\Gamma Q) \cap (Q\Gamma B)$

(4) *For any two quasi* Γ*-hyperideals* Q_1 *and* Q_2 *of* S,
$Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1)$

(5) *For any left* Γ*-hyprideal* I *and a bi-*Γ*-hyperideal* B *of* S,
$I \cap B \subseteq (I\Gamma B) \cap (B\Gamma I)$

(6) *For any left* Γ*-hyprideal* I *and a quasi* Γ*-hyperideal* Q *of* S,
$I \cap Q \subseteq (I\Gamma Q) \cap (Q\Gamma I)$

(7) *For any right* Γ*-hyprideal* J *and a bi-*Γ*-hyperideal* B *of* S,
$J \cap B \subseteq (J\Gamma B) \cap (B\Gamma J)$

(8) *For any right* Γ*-hyprideal* J *and a quasi* Γ*-hyperideal* Q *of* S,
$J \cap Q \subseteq (J\Gamma Q) \cap (Q\Gamma J)$

(9) *For any left* Γ*-hyperideal* I *and any right* Γ*-hyperideal* J *of* S,
$I \cap J \subseteq (I\Gamma J) \cap (J\Gamma I)$

Proof. $(1) \Longrightarrow (2)$
Assume that S is both regular and intra-regular Γ-semihypergroup and B_1, B_2 be any two bi-Γ-hyperideals of S. Let $x \in B_1 \cap B_2$ implies that $x \in S$ and S is both regular and intra-regular, hence $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$. Consider $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq (x\Gamma S)\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma(S\Gamma x) = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (B_1\Gamma S\Gamma B_1)$ $\Gamma(B_2\Gamma S\Gamma B_2) = B_1\Gamma B_2$ since in a regular Γ-semihypergroup, for a bi-Γ-hyperideal B, we have $B\Gamma S\Gamma B = B$ by Theorem 4.10. Thus we obtain whenever $x \in B_1 \cap B_2$, $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq B_1\Gamma B_2$, therefore $B_1 \cap B_2 \subseteq B_1\Gamma B_2$. Similarly consider $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq (x\Gamma S)\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma(S\Gamma x) = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq (x\Gamma S\Gamma x)\Gamma$ $(x\Gamma S\Gamma x) \subseteq B_2\Gamma S\Gamma B_2)\Gamma(B_1\Gamma S\Gamma B_1) = B_2\Gamma B_1$ implies that $B_1 \cap B_2 \subseteq B_2\Gamma B_1$. Hence $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.
$(2) \Longrightarrow (3)$
Assume that (2) holds in a Γ-semihypergroup S and let B, Q be the bi-Γ-hyperideal and quasi Γ-hyperideal of S respectively. As every quasi Γ-hyperideal is bi-Γ-hyperideal by Proposition 2.1, considering Q as bi-Γ-hyperideal we have $B \cap Q \subseteq (B\Gamma Q) \cap (Q\Gamma B)$ by (2).
$(3) \Longrightarrow (4)$
Assume that (3) holds in a Γ-semihypergroup S and let Q_1, Q_2 be any two quasi Γ-hyperideals of S. Because every quasi Γ-hyperideal is bi-Γ-hyperideal by Proposition 2.1, considering Q_1 as bi-Γ-hyperideal (3) implies that $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1)$.
$(4) \Longrightarrow (1)$
Assume that (4) holds in a Γ-semihypergroup S and let I, J be the left Γ-hyperideal and right Γ-hyperideal of S respectively. By Proposition 3.5 it is known that every one sided Γ-hyperideal is a quasi Γ-hyperideal, therefore by (4) we can write $I \cap J \subseteq (I\Gamma J) \cap (J\Gamma I)$. But $(I\Gamma J) \cap (J\Gamma I) \subseteq I\Gamma J$ and we get $I \cap J \subseteq I\Gamma J$, therefore by Theorem 3.6, S is intra-regular Γ-semihypergroup. Similarly, $(I\Gamma J) \cap (J\Gamma I) \subseteq J\Gamma I$ implies that $I \cap J \subseteq J\Gamma I$. Also $J\Gamma I \subseteq I \cap J$ always holds and this implies that $I \cap J = J\Gamma I$ hence S is regular Γ-semihypergroup by Theorem 4.9.
$(1) \Longrightarrow (5)$
Let S be a regular as well as intra-regular Γ-semihypergroup and I, B be the left Γ-hyperideal and bi-Γ-hyperideal of S respectively. Let $x \in I \cap B$ implies that $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ because$S$ is both regular and intra-regular Γ-semihypergroup. Consider $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq (x\Gamma S)\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma(S\Gamma x) = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (S\Gamma S\Gamma I)\Gamma(B\Gamma S\Gamma B) \subseteq (S\Gamma I)\Gamma B \subseteq I\Gamma B$ since I is a left Γ-hyperideal of S and by Theorem 4.10, $B\Gamma S\Gamma B = B$. Thus for any $x \in I\cap B$, $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq$

$I\Gamma B$ therefore $I\cap B \subseteq I\Gamma B$. Similarly, $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq (x\Gamma S)\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma(S\Gamma x)$ $= (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq (x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (B\Gamma S\Gamma B)\Gamma(S\Gamma S\Gamma I) \subseteq B\Gamma(S\Gamma I) \subseteq$ $B\Gamma I$ since I is a left Γ-hyperideal of S and by Theorem 4.10, $B\Gamma S\Gamma B = B$. Thus for any $x \in I \cap B$, $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq B\Gamma I$ therefore $I \cap B \subseteq B\Gamma I$. Thus we get $I \cap B \subseteq (I\Gamma B) \cap (B\Gamma I)$.

$(5) \Longrightarrow (6)$

Assume that (5) holds in a Γ-semihypergroup S and let I, Q be the left Γ-hyperideal and quasi Γ-hyperideal of S respectively. By Proposition 2.1 we know that every quasi Γ-hyperideal is bi-Γ-hyperideal so Q is also a bi-Γ-hyperideal of S and (5) implies that $I \cap Q \subseteq (I\Gamma Q) \cap (Q\Gamma I)$.

$(6) \Longrightarrow (1)$

Assume that (6) holds in a Γ-semihypergroup S and let I, J be the left Γ-hyperideal and right Γ-hyperideal of S. By Proposition 3.5 every right Γ-hyperideal is a quasi Γ-hyperideal, by (6) we have $I \cap J \subseteq (I\Gamma J) \cap (J\Gamma I)$ and $(I\Gamma J) \cap (J\Gamma I) \subseteq I\Gamma J$ implies that $I \cap J \subseteq I\Gamma J$ hence by Theorem 3.6 S is an intra-regular Γ-semihypergroup. Similarly $(I\Gamma J) \cap (J\Gamma I) \subseteq J\Gamma I$ and we always have $J\Gamma I \subseteq I \cap J$ yields $I \cap J = J\Gamma I$ which implies that S is a regular Γ-semihypergroup by Theorem 4.9.

$(1) \Longrightarrow (7)$

Suppose that S is both regular and intra-regular Γ-semihypergroup and J, B be the right Γ-hyperideeal and bi-Γ-hyperideal of S respectively. Let $x \in J\cap B$ implies that $x \in x\Gamma S\Gamma x$ and $x \in S\Gamma x\Gamma x\Gamma S$ because S is both regular and intra-regular Γ-semihypergroup. We have $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq x\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma x = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq$ $(x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (J\Gamma S\Gamma S)\Gamma(B\Gamma S\Gamma B) \subseteq (J\Gamma S)\Gamma B \subseteq J\Gamma B$ because J is right Γ-hyperideal of S and $B\Gamma S\Gamma B = B$ in a regular Γ- semihypergroup by Theorem 4.10. Similarly, $x\Gamma S\Gamma x \subseteq (x\Gamma S\Gamma x)\Gamma S\Gamma x \subseteq x\Gamma S\Gamma(S\Gamma x\Gamma x\Gamma S)\Gamma S\Gamma x = (x\Gamma S\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma S\Gamma x) \subseteq$ $(x\Gamma S\Gamma x)\Gamma(x\Gamma S\Gamma x) \subseteq (B\Gamma S\Gamma B)\Gamma(J\Gamma S\Gamma S) \subseteq B\Gamma J\Gamma S \subseteq B\Gamma J$. Thus for any $x \in J \cap B$ implies that $x \in x\Gamma S\Gamma x$ and $x\Gamma S\Gamma x \subseteq J\Gamma B \cap B\Gamma J$, that is $J \cap B \subseteq (J\Gamma B) \cap (B\Gamma J)$.

$(7) \Longrightarrow (8)$

Assume that (7) holds in a Γ-semihypergroup S and let J, Q be the right Γ-hyperideal and quasi Γ-hyperideal of S respectively. By Proposition 2.1, we know that every quasi Γ-hyperideal is a bi-Γ-hyperideal, hence Q is a bi-Γ-hyperideal of S. Therefore by (7), we have $J \cap Q \subseteq (J\Gamma Q) \cap (Q\Gamma J)$.

$(8) \Longrightarrow (9)$

Assume that (8) holds in a Γ-semihypergroup S and let I, J be the left and right Γ-hyperideals of S respectively. By Proposition 3.5 every right Γ-hyperideal of S is quasi Γ-hyperideal hence by (8) we have $I \cap J \subseteq (I\Gamma J) \cap (J\Gamma I)$.

$(9) \Longrightarrow (1)$

Assume that (9) holds in a Γ-semihypergroup S and let I, J be the left and right Γ-hyperideals of S respectively. From (9) we have $I\cap J \subseteq (I\Gamma J)\cap(J\Gamma I) \subseteq I\Gamma J$ that is $I\cap J \subseteq I\Gamma J$ therefore by Theorem 3.6 we conclude that S is an intra-regular Γ-semihypergroup. Similarly $I\cap J \subseteq (I\Gamma J)\cap(J\Gamma I) \subseteq J\Gamma I$ and $J\Gamma I \subseteq I\cap J$ always hold, we have $I\cap J = J\Gamma I$ so by Theorem 4.9 we see that S is a regular Γ-semihypergroup. Thus it is proved that S is both regular and intra-regular Γ-semihypergroup.

We have proved $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$, $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)$ and $(1) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9) \Rightarrow (1)$, thus establishing the equivalence of all the statements. □

Theorem 4.13. *In a Γ-semihypergroup S following statements are equivalent.*

(1) S is both regular as well as intra-regular Γ-semihypergroup.

(2) For any two bi-Γ-hyperideals B_1 and B_2 of S,
$B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.

(3) *For a bi-*Γ*-hyperideal* B *and quasi* Γ*-hyperideal* Q *of* S,
$B \cap Q \subseteq (B\Gamma Q\Gamma B) \cap (Q\Gamma B\Gamma Q)$.

(4) *For quasi* Γ*-hyperideals* Q_1 *and* Q_2 *of* S,
$Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2\Gamma Q_1) \cap (Q_2\Gamma Q_1\Gamma Q_2)$.

(5) *For a left* Γ*-hyperideal* I *and bi-*Γ*-hyperideal* B *of* S,
$I \cap B \subseteq (I\Gamma B\Gamma I) \cap (B\Gamma I\Gamma B)$.

(6) *For a left* Γ*-hyperideal* I *and quasi* Γ*-hyperideal* Q *of* S,
$I \cap Q \subseteq (I\Gamma Q\Gamma I) \cap (Q\Gamma I\Gamma Q)$.

(7) *For a right* Γ*-hyperideal* J *and bi-*Γ*-hyperideal* B *of* S,
$J \cap B \subseteq (J\Gamma B\Gamma J) \cap (B\Gamma J\Gamma B)$.

(8) *For a right* Γ*-hyperideal* J *and quasi* Γ*-hyperideal* Q *of* S,
$J \cap Q \subseteq (J\Gamma Q\Gamma J) \cap (Q\Gamma J\Gamma Q)$.

Proof. Can be proved now as per earlier proofs. □

5. Conclusion

Algebraic hyperstructures is being studied widely and is a very interesting field to work on for future research. In this paper we have studied the properties of intra-regular Γ- semihypergroup and presented it's characterizations in terms of left and right Γ- hyperideals, bi-Γ-hyperideal and quasi Γ-hyperideal. We also gave an example where a Γ-hyperideal of a Γ-hyperideal of a Γ-semihypergroup S need not be a Γ-hyperideal of S and found that when S is an intra-regular Γ-semihypergroup, a Γ-hyperideal of a Γ-semihypergroup S is indeed a Γ-hyperideal of S. We have also investigated the Γ-semihypergroups which are both regular and intra-regular and presented few characterizations of the same. We have given examples of Γ-semihypergroup which is regular but not intra-regular, a Γ-semihypergroup which is neither regular nor intra-regular and a Γ-semihypergroup which is both regular as well as intra-regular. Example of Γ-semihypergroup which is intra-regular but not regular is left open for readers to find.

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