On extensions of pseudo-valuations on BCK algebras

DUMITRU BUSNEAG, DANA PICIU and MIHAELA ISTRATA

Abstract. In this paper we define a pseudo-valuation on a BCK algebra \((A, \rightarrow, 1)\) as a real-valued function \(v : A \rightarrow \mathbb{R}\) satisfying \(v(1) = 0\) and \(v(x \rightarrow y) \geq v(y) - v(x)\), for every \(x, y \in A\); \(v\) is called a valuation if \(x = 1\) whenever \(v(x) = 0\). We prove that every pseudo-valuation (valuation) \(v\) induces a pseudo-metric (metric) on \(A\) defined by \(d_v(x, y) = v(x \rightarrow y) + v(y \rightarrow x)\) for every \(x, y \in A\), where \(\rightarrow\) is uniformly continuous in both variables. The aim of this paper is to provide several theorems on extensions of pseudo-valuations (valuations) on BCK algebras.

1. Introduction and basic results

BCK algebras are an important class of logical algebras investigated by many researchers (see [2], [3], [6], [8], [9], [10], [13]). BCK algebras were originally introduced by Iséki in [9]. Further properties of them and their connections with other fuzzy structures were established by Iorgulescu in [8].

In [1], Busneag defined pseudo-valuations on Hilbert algebras and proved that every pseudo-valuation induced a pseudo-metric. Using this model, in [4], [5], [7], [11], [12], [14], [15], [16] is introduced the notion of pseudo-valuation on BCK, BCI, BCC algebras and several properties are discussed.

The main goal of this paper is to introduce the notions of pseudo-valuation and valuation on BCK algebras and to prove theorems on extensions of pseudo-valuations (valuations) on BCK algebras.

The paper is organized as follows: In Section 1 we review some relevant concepts relative to BCK algebras. In Section 2 we introduce the notions of pseudo-valuation and valuation on BCK algebras and we induce a pseudo-metric by using pseudo-valuations on BCK algebras (Theorem 2.1). Also, we show that the binary operation \(\rightarrow\) is uniformly continuous, see Corollary 2.1. Finally, we prove some theorems (2.2 and 2.3) on extensions of pseudo-valuations (valuations) on BCK algebras. Section 3 contains results about pseudo-valuations on the dual BCK algebra, see Theorem 3.4 and the final section contains conclusions, open problems and future work about the presented topics.

A BCK algebra is an algebra \((A, \rightarrow, 1)\) of type \((2,0)\) satisfying:

\((a_1)\) \(x \rightarrow x = 1;\)
\((a_2)\) If \(x \rightarrow y = y \rightarrow x = 1\), then \(x = y;\)
\((B)\) \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z);\)
\((C)\) \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);\)
\((K)\) \(x \leq y \rightarrow x.\)

Example 1.1. ([8]) We give an example of a finite bounded BCK algebra. Let \(A = \{0, a, b, c, 1\}\) with \(0 < a, b < c < 1\), but \(a, b\) are incomparable. \(A\) becomes a BCK algebra relative to the
following operation:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>1</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

If $A$ is a BCK algebra, then the relation $\leq$ defined by $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on $A$ (called the natural order); with respect to this order 1 is the largest element of $A$. A bounded BCK algebra is a BCK algebra with a smallest element 0 relative to the natural order.

In BCK algebras we have the following rules of calculus (see [2] and [10]):

1. $(x \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow (z \rightarrow y))$.
2. If $x \leq y$, then for every $z \in A$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

For a BCK algebra $A$ and $x_1, x_2, \ldots, x_n, x \in A$ ($n \geq 1$) we define $X = x_1 \rightarrow (x_2 \rightarrow \ldots (x_n \rightarrow x) \ldots)$. If $\sigma$ is a permutation of $\{1, \ldots, n-1\}$, $n \geq 2$, then:

1. $(x_1, \ldots, x_{\sigma(n-1)}; x_n) = (x_1, \ldots, x_{n-1}; x_n)$.
2. $x \rightarrow (x_1, \ldots, x_{n-1}; x_n) = (x, x_1, \ldots, x_{n-1}; x_n)$.

Let $A$ be a BCK algebra. A subset $D$ of $A$ is called a deductive system of $A$ if $1 \in D$ and $x, y \in D$ implies $y \in D$.

Let $A$ be a bounded BCK algebra. An element $x \in A$ is called boolean (see [6]) if $< x > \cap < x^* > = \{1\}$. Let $B(A)$ the set of all boolean elements of $A$.

Remark 1.1. If $D \in Ds(A)$, then $D$ is a BCK subalgebra of $A$ (since $1 \in D$ and if $x, y \in D$ from $y \leq x \rightarrow y$ we deduce that $x \rightarrow y \in D$).

2. Pseudo-valuations (valuations) on BCK algebras

Using the model of Hilbert algebras (see [1]), in this section we introduce the notions of pseudo-valuations and valuations on BCK algebras and we prove some theorems of extension for these.

Let $A$ be a BCK algebra. A real-valued function $v : A \rightarrow \mathbb{R}$ is called a pseudo-valuation on $A$ if $v(1) = 0$ and $(*)$: $v(x \rightarrow y) \geq v(y) - v(x)$, for every $x, y \in A$. The pseudo-valuation $v$ is called valuation if $v(x) = 0$ implies $x = 1$.

If we interpret $A$ as an implicative calculus, $x \rightarrow y$ as the proposition $x \Rightarrow y$ and 1 as truth, a pseudo-valuation on $A$ can be interpreted as a “falsity-valuation”.

Example 2.2. $v : A \rightarrow \mathbb{R}$, $v(x) = 0$ for every $x \in A$ is a pseudo-valuation on $A$.  


Example 2.3. If $D \in Ds(A)$ and $0 \leq r \in \mathbb{R}$, then $v_D : A \rightarrow \mathbb{R}$, $v_D(x) = 0$, if $x \in D$ and $r$ otherwise, is a pseudo-valuation on $A$. Indeed, $v_D(1) = 0$ since $1 \in D$. Let $x, y \in A$. If $x, y \in D$, since $y \leq x \rightarrow y$ we obtain $x \rightarrow y \in D$. So, $v_D(x \rightarrow y) = v_D(x) = v_D(y) = 0$ and $v_D(x \rightarrow y) = v_D(y) - v_D(x)$. If $x, y \notin D$, then $v_D(y) - v_D(x) = r - r = 0 \leq v_D(x \rightarrow y)$. If $x \notin D$ and $y \in D$ we deduce that $x \rightarrow y \in D$, so $0 = v_D(x \rightarrow y) \geq v_D(y) - v_D(x) = 0 - r = -r$. If $y \notin D$ and $x \in D$ then $x \rightarrow y \notin D$. We obtain $r = v_D(x \rightarrow y) = v_D(y) - v_D(x) = r - r = 0$.

Remark 2.2. Let $A$ be a non trivial BCK algebra, $D \in Ds(A)$, $r \geq 0$ and $v_D : A \rightarrow \mathbb{R}$ the function given by $v_D(x) = 0$ if $x \in D$ and $r$ otherwise. Then $v_D$ is a valuation if and only if $D = \{1\}$ and $r > 0$.

Example 2.4. Let $M$ be a finite set with $n$ elements and $A = P(M)$ be the power set of $M$ (the set of all subsets of $M$). Then $(P(M), \cap, \cup, C_M, \emptyset, M)$ is a Boolean algebra (where for $X \subseteq M$, $C_M(X) = M \setminus X$). The function $v : P(M) \rightarrow \mathbb{R}$, defined by $v(X) = n - n(X)$ is a valuation on $A$, where $n(X)$ is the number of elements of $X$. Indeed, $v(M) = 0$. Let $X, Y \subseteq M$. We have $v(X \rightarrow Y) = v(C_M X \cup Y) = n - n(C_M X \cup Y) = n - n(C_M X) - n(Y) + n(C_M X \cap Y) = n(X) - n(Y) + n(C_M X \cap Y) \geq n(X) - n(Y) = v(X) - v(Y)$. Obviously, $v(X) = 0$ iff $X = M$.

Lemma 2.1. If $v : A \rightarrow \mathbb{R}$ is a pseudo-valuation on $A$ and $x, x_1, ..., x_n \in A$ such that $(x_1, ..., x_n ; x) = 1$ then

$$(c_7) \quad v(x) \leq \sum_{i=1}^{n} v(x_i).$$

Proof. $0 = v(1) = v((x_1, ..., x_n ; x)) \geq v(x) - \sum_{i=1}^{n} v(x_i)$, so $v(x) \leq \sum_{i=1}^{n} v(x_i).$  

A pseudo-valuation $v : A \rightarrow \mathbb{R}$ is called decreasing if $v(x) \geq v(y)$ for every $x, y \in A$ with $x \leq y$.

Lemma 2.2. A pseudo-valuation $v$ is a positive decreasing function satisfying

$$(c_8) \quad v(x \rightarrow y) + v(y \rightarrow z) \geq v(x \rightarrow z), \text{ for any } x, y, z \in A.$$ 

Proof. If in (*) we put $y = 1$ we obtain $v(x \rightarrow 1) \geq v(1) - v(x)$, so $v(x) \geq 0$, for every $x \in A$. If $x \leq y$, then $x \rightarrow y = 1$, so from (*) we deduce that $0 = v(1) = v(x \rightarrow y) \geq v(y) - v(x)$. We conclude that $v(x) \geq v(y)$ for every $x \leq y$, so, $v$ is a decreasing function. Let now $x, y, z \in A$. Since $v$ is a decreasing function, from (B), we deduce that $v(x \rightarrow y) \geq v(y \rightarrow z) \rightarrow (x \rightarrow z) \geq v(x \rightarrow z) - v(y \rightarrow z)$. Thus, $v(x \rightarrow z) \leq v(x \rightarrow y) + v(y \rightarrow z)$.  

We recall that by a pseudo-metric space we mean an ordered pair $(M, d)$, where $M$ is a non-empty set and $d : M \times M \rightarrow \mathbb{R}$ is a positive function satisfying the following properties: $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in M$. If in the pseudo-metric space $(M, d)$, $d(x, y) = 0$ implies $x = y$, then $(M, d)$ is called a metric space.

Theorem 2.1. Let $v : A \rightarrow \mathbb{R}$ be a pseudo-valuation on $A$. If we define $d_v : A \times A \rightarrow \mathbb{R}$, $d_v(x, y) = v(x \rightarrow y) + v(y \rightarrow x)$, for every $(x, y) \in A \times A$, then

(i) $(A, d_v)$ is a pseudo-metric space satisfying:

$$(c_9) \quad \max\{d_v(x \rightarrow z, y \rightarrow z), d_v(z \rightarrow x, z \rightarrow y)\} \leq d_v(x, y), \text{ for every } x, y, z \in A;$$ 

(ii) $d_v$ is a metric on $A$ iff $v$ is a valuation on $A$.

Proof. (i). Let $x, y, z \in A$. Clearly, $d_v(x, y) = d_v(y, x) \geq 0$ and $d_v(x, x) = v(x \rightarrow x) + v(x \rightarrow x) = v(1) + v(1) = 0 + 0 = 0$. Also, $d_v(x, y) + d_v(y, z) = [v(x \rightarrow y) + v(y \rightarrow x)] + [v(y \rightarrow z) + v(z \rightarrow y)] = [v(x \rightarrow y) + v(y \rightarrow z)] + [v(z \rightarrow y) + v(y \rightarrow x)] \geq v(x \rightarrow z) + v(z \rightarrow y) \geq v(x \rightarrow z) + v(z \rightarrow y) \geq v(x \rightarrow z) + v(z \rightarrow y).$
Let \( v : A \to R \) be a valuation. Then the operation \( \to : A \times A \to A \) is uniformly continuous.

Proof. Let \( x, x', y, y' \in A \) and \( 0 < \varepsilon \in R \). Then \( \overline{d}_v : A \times A \to R, \overline{d}_v((x, y), (x', y')) = \max\{d_v(x, x'), d_v(y, y')\} \), for every \((x, y), (x', y') \in A \times A\) is a metric on \( A \times A \). Obviously, by definition, \( \overline{d}_v \) is a positive function. Since \( v \) is a valuation on \( A \), using Theorem 2.1, we deduce that \( d_v \) is a metric on \( A \). Thus, \( \overline{d}_v((x, y), (x', y')) = \max\{d_v(x, x'), d_v(y, y')\} = \max\{d_v(x, y), d_v(y, x)\} \). For every \((x, y), (x', y') \in A \times A \) we have:\n
\[
\overline{d}_v((x, y), (x', y')) = \max\{d_v(x, x'), d_v(y, y')\} = \max\{d_v(x, y), d_v(x, y') + d_v(y', y)\} = \max\{d_v(x, x'), d_v(y, y') + d_v(y', y)\} = \max\{d_v(x, x'), d_v(y, y')\} = \max\{d_v(x, x'), d_v(y, y')\} \]

We conclude that \( (x, y) = (x', y') \). Thus, \( d_v \) is a metric on \( A \times A \). If \( \overline{d}_v((x, y), (x', y')) < \varepsilon/2 \) then \( d_v(x, x') < \varepsilon/2 \). We have \( d_v(x, x') \leq d_v(x, y, x', y') + d_v(y', y) \leq d_v(x, x') \leq d_v(x, x') + d_v(y, y') \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \), that is, \( \to \) is uniformly continuous. \( \square \)

We have the following theorems of extension:

**Theorem 2.2.** Let \( A \) and \( B \) two BCK algebras such that \( A \) is a subalgebra of \( B \) and \( v : A \to R \) is a pseudo-valuation on \( A \). Then there exists a pseudo-valuation \( v' : B \to R \) such that \( v'|_A = v \).

Proof. For \( x \in B \) we define \( v'(x) = \inf\{n \sum v(x_i) : x_1, \ldots, x_n \in A \) and \( (x_1, \ldots, x_n; x) = 1\} \). Since \( 1 \in A \) and \( 1 \to 1 = 1 \) we deduce that \( v'(1) = v(1) = 0 \). For \( x, y \in B \), let \( x_1, \ldots, x_n, z_1, \ldots, z_m \in A \) such that \((x_1, \ldots, x_n; x) = (z_1, \ldots, z_m; x \to y) = 1\). We deduce that \((x_1, \ldots, x_n, z_1, \ldots, z_m; y) = 1\), hence, by the definition of \( v' \) we have \( v'(y) \leq \sum v(z_i) + \sum v(x_i) \). So, \( v'(y) \leq \inf\{\sum v(z_i) : z_1, \ldots, z_m \in A \) and \( (z_1, \ldots, z_m; x \to y) = 1\} + \inf\{\sum v(x_i) : x_1, \ldots, x_n \in A \) and \( (x_1, \ldots, x_n; x) = 1\} \).

Thus, \( v'(y) \leq v'(x \to y) + v'(x) \) so, \( v'(y) - v'(x) \leq v'(x \to y) \), for every \( x, y \in B \). We conclude that \( v' \) is a pseudo-valuation on \( B \).
If $x \in A$, since $x \to x = 1$, we deduce that $v'(x) \leq v(x)$. Let $x_1, \ldots, x_n \in A$ such that $(x_1, \ldots, x_n; x) = 1$. From Lemma 2.1, $v(x) \leq \sum v(x_i)$, hence $v(x) \leq \inf \{ \sum v(x_i) : x_1, \ldots, x_n \in A \text{ and } (x_1, \ldots, x_n; x) = 1 \} = v'(x)$, that is, $v'_A = v$.

**Remark 2.3.** If $A$ and $B$ are two BCK algebras such that $A$ is a subalgebra of $B$, $v : A \to \mathbb{R}$ is a pseudo-valuation on $A$ and $v' : B \to \mathbb{R}$ is a real-valued function such that $v'_A = v$, then $v'$ is not necessarily a pseudo-valuation on $B$. Indeed, let $B = \{0, a, b, c, 1\}$ be BCK algebra from Example 1.1. Obviously, $A = \{1\}$ is a sub-BCK-algebra of $B$ and $v : A \to \mathbb{R}$, $v(1) = 0$ is a pseudo-valuation on $A$, see Example 2.2. Let $v' : B \to \mathbb{R}$ be a real-valued function on $B$ defined by $v' = \left( \begin{array}{ccccc} 0 & a & b & c & 1 \\ 7 & 2 & 2 & 2 & 0 \end{array} \right)$. Then $v'_A = v$, but $v'$ is not a pseudo-valuation on $B$ since $v'(b \to 0) = v'(a) = 2 < v'(0) - v'(b) = 7 - 2 = 5$.

We consider $D \in Ds(A)$ and the relation $\delta_D$ on $A$ defined by $(x, y) \in \delta_D$ iff $x \to y \in D$ and $y \to x \in D$. Hence $\delta_D$ is a congruence on $A$, see [3] and [13]. For $x \in A$ we denote by $x/D$ the congruence class of $x$ modulo $\delta_D$ and by $A/D = \{ x/D : x \in A \}$ the quotient algebra. Then $A/D$ is a BCK algebra, where for $x, y \in A$, $x/D \to y/D = (x \to y)/D$. Also, we denote by $p_D : A \to A/D$ the canonical surjective morphism of BCK algebras, $p_D(x) = x/D$, for every $x \in A$. For $x \in D$, we have $x/D = 1/D = 1$.

**Theorem 2.3.** If $D \in Ds(A)$ and $v : A \to \mathbb{R}$ is a pseudo-valuation (valuation) on $A$, then the following assertions are equivalent:

(i) There exists a pseudo-valuation (valuation) $v' : A/D \to \mathbb{R}$ such that $v' \circ p_D = v$;

(ii) $v(a) = 0$, for every $a \in D$.

**Proof.** (i) $\Rightarrow$ (ii). Let $v' : A/D \to \mathbb{R}$ be a pseudo-valuation on $A/D$ such that $v' \circ p_D = v$ and let $a \in D$. Then $v(a) = (v' \circ p_D)(a) = v'(p_D(a)) = v'(1/D) = 0$.

(ii) $\Rightarrow$ (i). For $x \in A$ we define $v'(x/D) = v(x)$. Let $x, y \in A$ such that $x/D = y/D$. Then $x \to y \in D$ and $y \to x \in D$. We obtain $0 = v(x \to y) \geq v(y) - v(x)$ and $0 = v(y \to x) \geq v(x) - v(y)$, so, $v(x) = v(y)$, hence $v'$ is correctly defined. Also, we have $v'(1/D) = 0$ and for $x, y \in A$, $v'(x/D \to y/D) = v'((x \to y)/D) = v(x \to y) \geq v(y) - v(x) = v'(y/D) - v'(x/D)$, hence $v'$ is a pseudo-valuation on $A$. Clearly, $v' \circ p_D = v$.

If $v$ is a valuation on $A$ and $x \in A$ such that $v'(x/D) = 0$, then $v(x) = 0$, hence $x = 1$. Thus, $x/D = 1/D = 1$. We conclude that $v'$ is a valuation on $A/D$ such that $v' \circ p_D = v$. □

3. The dual BCK algebra

In this section we introduce the notion of dual BCK algebra and taking as guide line [1], we obtain results for BCK algebras.

Let $A \in Ds(A)$, $D_1 \land D_2 = D_1 \cap D_2$, $D_1 \lor D_2 = D_1 \cup D_2 = \cup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\}$ and $D_1 \to D_2 = \vee\{D \in Ds(A) : D_1 \cap D \subseteq D_2\} = \inf\{D \in A^\circ : D_1 \cap D \leq D_2\}$.

**Definition 3.1.** The dual BCK algebra of $A$, denoted by $A^\circ$, is the Heyting algebra $Ds(A)$ with the order $D_1 \leq D_2$ iff $D_2 \subseteq D_1$.

In $(A^\circ, \leq)$, $0 = A, 1 = \{1\}$ and for $D_1, D_2 \in A^\circ$, $D_1 \cap D_2 = \leq D_1 \cup D_2$, $D_1 \cup D_2 = D_1 \lor D_2$ and $D_1 \to D_2 = \cup\{D \in A^\circ : D_1 \cap D \leq D_2\} = \cap\{D \in A^\circ : D_2 \subseteq D_1 \lor D\}$.

**Example 3.5.** Let $A$ be the BCK algebra from Example 1.1. It is immediate to prove that $Ds(A) = \{\{1\}, \{1, c\}, \{1, a, c\}, \{1, b, c\}, A\}$ and $A^\circ$ is the Heyting algebra $Ds(A)$ with the order $D_1 \leq D_2$ iff $D_2 \subseteq D_1$, for $D_1, D_2 \in Ds(A)$. 

Also, we define \( j_A : A \to A^o \), \( j_A(a) = \langle a, > \), for every \( a \in A \). Hence \( j_A(1) = 1 \) and \( j_A(x \to y) \supseteq j_A(x) \to j_A(y) \), for every \( x, y \in A \). Indeed, \( j_A(1) = \langle 1 \rangle = \{1\} = 1 \). Also, \( j_A(x) \to j_A(y) = \langle x \to y \rangle \subseteq \langle x \to y \rangle = \cap \{D \in Ds(A) : \langle y \rangle \subseteq \langle x \rangle \vee D\} \). Since from \((c_1)\), \( x \rightarrow ((x \to y) \to y) = 1 \), we deduce (using \((c_6)\)) that \( y \subseteq \langle x \rangle \vee \langle x \to y \rangle \), so \( y \subseteq \langle x \rangle \vee \langle x \to y \rangle \). Thus, \( j_A(x \to y) = \langle x \to y \rangle \supseteq j_A(x) \to j_A(y) \), for every \( x, y \in A \).

Lemma 3.3. For every \( x, y \in A \), there is a natural number \( m \geq 1 \) such that

\[
j_A(x \to_m y) \subseteq j_A(x) \to j_A(y).
\]

Proof. We have that \( j_A(x) \to j_A(y) = \cap \{D \in Ds(A) : \langle y \rangle \subseteq \langle x \rangle \vee D\} \). So let \( D \in Ds(A) \) such that \( \langle y \rangle \subseteq \langle x \rangle \vee D \). Since \( \langle x \rangle \subseteq \langle y \rangle \vee D = \{z \in A : t \rightarrow (d \to z) = 1 \}, \) for some \( d \in D \) and \( t \in \langle x \rangle \) and \( y \in \langle y \rangle \subseteq \langle x \rangle \vee D \), we deduce that \( t \rightarrow (d \to y) = 1 \), for some \( d \in D \) and \( t \in \langle x \rangle \). But \( t \rightarrow (d \to y) = d \rightarrow (t \to y) \), so \( d \rightarrow (t \to y) = 1 \) and \( d \leq t \rightarrow y \). We deduce that \( t \rightarrow y \in D \) for some \( t \in \langle x \rangle \) (i.e., \( x \rightarrow_n t = 1 \), for some \( n \geq 1 \)).

Finally, \( x \rightarrow_n y \in D \), for some \( n \geq 1 \). Hence, there is a natural number \( m \geq 1 \) such that \( x \rightarrow_m y \in D \), for every \( D \in Ds(A) \). We conclude that there is \( m \geq 1 \) such that \( j_A(x \rightarrow_m y) \subseteq j_A(x) \to j_A(y) \), for every \( x, y \in A \). □

We recall that if \( A \) and \( B \) are two BCK algebras, a function \( f : A \to B \) is a morphism of BCK algebras if \( f(x \to y) = f(x) \rightarrow f(y) \) for every \( x, y \in A \).

Lemma 3.4. \( j_{B(A)} \) is an injective morphism of BCK algebras.

Proof. We recall that (see [6]) if \( a \in B(A) \) then \( a \rightarrow (a \rightarrow x) = a \rightarrow x \) for every \( x \in A \), so \( \langle a \rangle = \{x \in A : a \leq x \} \). Using Lemma 3.3, if we consider \( x, y \in B(A) \) we deduce that \( j_A(x) \to j_A(y) = j_A(x) \to y \). Also, if \( j_A(x) = j_A(y) \), then \( \langle x \rangle = \langle y \rangle \), so \( x \leq y \) and \( y \leq x \). Thus, \( x = y \). We conclude that \( j_{B(A)} \) is an injective morphism of BCK algebras. □

Definition 3.2. We say that a BCK algebra \( A \) has property \( F \) if for every \( D \in A^o \) there exist \( x_1, ..., x_n \in A \) such that \( D \subseteq \langle x_1, ..., x_n \rangle \).

Example 3.6. If we consider BCK algebra from Example 1.1, then \( A \) has property \( F \) since \( A^o = Ds(A) = \{\{1\}, \{1, c\}, \{a, c\}, \{1, b, c\}, A\} \) and \( \{1\} = \langle 1 \rangle = \langle 1, c \rangle = \langle 1, a, c \rangle = \langle 1, b, c \rangle = \langle 1, b, c \rangle \) and \( A = \langle 0 \rangle \).

Remark 3.4. Examples of BCK algebras with property \( F \) are bounded BCK algebras (since \( A = \langle 0 \rangle \)) and finite BCK algebras.

Theorem 3.4. Let \( A \) be a BCK algebra with property \( F \) and \( v : A \to R \) a pseudo-valuation on \( A \). Then there exists a pseudo-valuation on \( A^o \), \( v' : A^o \to R \) such that \( v' \circ j_A = v \).

Proof. For \( D \in A^o \) we define \( v'(D) = \inf \{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A \) and \( D \subseteq \langle x_1, ..., x_n \rangle \} \).

First, we prove that \( v' \) is a pseudo-valuation on \( A^o \).

Clearly, \( v'(1) = \inf \{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A \) and \( \{1\} \subseteq \langle x_1, ..., x_n \rangle \} = v(1) = 0 \). To verify \((*)\), let \( D_1, D_2 \in A^o \) and \( x_1, ..., x_n, z_1, ..., z_m \in A \) such that \( D_1 \subseteq \langle x_1, ..., x_n \rangle \) and \( D_1 \rightarrow D_2 \subseteq \langle z_1, ..., z_m \rangle \). Then \( D_2 \subseteq D_1 \vee (D_1 \rightarrow D_2) \subseteq \langle x_1, ..., x_n \rangle \vee \langle z_1, ..., z_m \rangle \) \( \subseteq \langle x_1, ..., x_n, z_1, ..., z_m \rangle \). Thus \( v'(D_2) \leq \sum_{i=1}^{n} v(x_i) + \sum_{j=1}^{m} v(z_j) \), so \( v'(D_2) \leq \inf \{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A \) and \( D_1 \subseteq \langle x_1, ..., x_n \rangle \} + \inf \{\sum_{j=1}^{m} v(z_j) : z_1, ..., z_m \in A \) and \( D_1 \rightarrow D_2 \subseteq \langle z_1, ..., z_m \rangle \} = \).
On extensions of pseudo-valuations on BCK algebras

49

Let $v'(D_1) + v'(D_1 \rightarrow D_2)$. We obtain that $v'(D_1 \rightarrow D_2) \geq v'(D_2) - v'(D_1)$.

If $a, x_1, \ldots, x_n \in A$ such that $< a > \subseteq < x_1, \ldots, x_n >$ then $(x_1, \ldots, x_n; a) = 1$, for some $x_i, \ldots, x_k \in \{x_1, \ldots, x_n\}$. From Lemma 2.1 we deduce that $v(a) \leq \sum_{i=1}^{n} v(x_i)$, so $v(a) \leq \inf\{\sum_{i=1}^{n} v(x_i) : x_1, \ldots, x_n \in A$ and $< a > \subseteq < x_1, \ldots, x_n >\} = v'(< a >)$. Since $< a > \subseteq \{a\}$ it follows that $v'(< a >) = v(a)$. We conclude that $v' \circ j_A = v$. \hfill \Box

4. Conclusions and future work

In [1], is defined a pseudo-valuation on a Hilbert algebra. In this paper, we generalize this concept for BCK algebras and prove theorems on extensions of pseudo-valuations (valuations) on BCK algebras. Since the power set of a non-empty set is a BCK algebra, using of pseudo-valuations can be useful in the study of theory of sets. As another direction of research one could define and study the concept of free Hilbert algebra with infimum over a BCK algebra. Specifically, the questions are the following: Are there the free Hilbert algebras with infimum over BCK algebras? and if the answer is positive, Which is the relation of these algebras with pseudo-valuations? It is interesting to note that, if the BCK algebra is a Hilbert algebra, then an explicit construction of the free semilattice extension of a Hilbert algebra is not immediate. Also, [1] contains results about pseudo-valuations on free Hertz algebra over a Hilbert algebra.

References