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Computational approach to Transient solution for single unreliable server Retrial Queue under non-preemptive priority

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ABSTRACT. Single server Retrial Queueing model with system repair and breakdown under non-preemptive priority is considered in this paper. The study of the model so far is mostly explored only for steady state solutions with few exceptions to Transient solutions. Though some researchers have attempted transient solution to the model analytically, the solution thus obtained is more complicated that it cannot be solved. In this context, we have found the transient numerical solution to the model using eigenvalues and eigenvectors. Time dependent system performance measures and probability distributions are evaluated and validated with steady state solutions.

1. INTRODUCTION

In many practical queueing models, the occurrence of interruption is inevitable. Interruption arises due to either server break down or in case server goes for vacation. The impact of inclusion of interruption in Queueing models reflects in the characteristics of the queueing system. In most of the daily life queueing models such as Ticket reservation, Banking services, ATM services, Vending machines, Billing counters and Toll collection centres, often we come across the situation wherein either the server goes for vacation or the system breaksdown and the customers need to wait either for the server to return from vacation or the server to be made ready. Thus the study of system interruption become more significant.

Study of Interrupted Queueing models started way back in 1950's. Performance measures of a waiting line with interruptions are explored by Gaver [6]. By applying generating function method, White and Christie [13] investigated Queueing structure with breakdown. Queue length generating function of several breakdown models of queue were obtained by Thiruvengadam [11]. Explicit moment generating function of Queue size for many server queue with interruptions is found by Mitrany and Avi-itzhak [8]. Shengli and Jingbo [10] analyzed M/M/N queue with variable breakdowns.

Study of server breakdown interruption in Retrial Queueing models were initiated only in 1990's. By employing Markov regenerating processes, Kulkarni and Choi [7], derived stability and limiting behavior of Retrial queues with server subject to breakdowns and repairs. Artalejo [3] investigated the asymptotic behavior of Retrial queueing systems with breakdown of the servers. Applying the theory of piecewise Markovian Process, Aissani[1] analyzed Retrial queue with redundancy and unreliable server. Aissani and Artalejo [2] obtained expression of the generating function of the server state for the Single server retrial queue subject to breakdowns. Jinting Wang and Jinhua Cao [12] studied Reliability Analysis of Retrial Queue with server breakdown and repairs. Discrete time Geo/G/1 retrial queue with server Breakdown is investigated by Atencia and Moreno [4].

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Muthu ganapathy Subramanian et al [9] studied unreliability of M/M/c retrial queue. All these efforts provide only steadystate solutions to breakdown models.

Transient analysis of finite capacity single server queueing system with working breakdowns is explored by using fourth order Runge Kutta method by Ezeagu [5]. The purpose of this paper is to find time dependent solution to M/M/1 Retrial queue with server breakdown and repair. The transient solution to the model is found by matrix exponential method. Time dependent probabilities and system performance measures are obtained.

2. Model description

Queueing model of single server retrial system with active breakdown and repair of service under priority is taken up for study. Two types of customers arrive at the system to get the service done. Arrival of customers follow Poisson distribution. Low and high priority customers arrive with rates λ_1 and λ_2 respectively. Time taken by the server to serve follow exponential distribution with parameters μ_1 and μ_2 for low and high priority customers respectively. Service breakdown is distributed exponentially with parameter α and repair of server follows an exponential distribution with parameter β. Active breakdown is considered here, which means that the system does not break while it is idle. The priority taken up is non-preemptive. According to non-preemptive, high priority customer on arrival to the system and finds the server busy with low priority customer and there is no one in the high priority queue, need to wait for the server to complete the service for the low priority customer, to get his service done. The maximum number of waiting spaces for high priority customers in front of the service station is s. When a low priority customer arrive at the system, then

(1) When the server is idle, he get his service done immediately and leaves the system
(2) When the server is busy, he joins the orbit and become a source of repeated customer with the intention of trying for the service independently after some random time.

The retrial rate is exponentially distributed with intensity rate σ.

When a high priority customer arrive at the system, then

(1) When the server is idle, he get his service done immediately and leaves the system
(2) When the server is busy, and the high priority queue is not empty and in case there is a space in the high priority queue, then he joins the queue otherwise he leaves the system.
(3) When the server is busy with low/high priority customer and the high priority queue is empty, due to non-preemptive priority principle, the customer waits for the server to complete the service for the low/high priority customer and then proceed to get his service done.

When the server breakdown while serving for a low priority customer (active breakdown), the customer with his incomplete service goes to orbit and the server goes to the state of breakdown. When the server breakdown while serving for a high priority customer (active breakdown), the customer with incomplete service will stay in the queue in front of the service station and the server goes to the state of breakdown. Access of customer from the orbit to the server is controlled by classical retrial policy. According to this policy the customers from the orbit try for service independently of each other. The probability of retrial in the interval $(t, t + \Delta t)$, when there are n customers in the orbit is $n\sigma\Delta(t) + O(t)$.

2.1. **Random Process.** The random variables $N(t), P(t)$ and $S(t)$ are respectively stand for number of low priority customers, number of high Priority customers in the queue and the status of the server at time t. The random process is $(N(t), P(t), S(t))$. The different status of the server in this model are either idle or busy with low priority customer or busy with high priority customer or in breakdown.

(1) $S(t) = 0$ if the server is idle at time t
(2) $S(t) = 1$ if the server is busy with low priority customer at time t
(3) $S(t) = 2$ if the server is busy with high priority customer at time t
(4) $S(t) = 3$ if the server is in the status of breakdown at time t.

The possible State spaces are

$$\{(u, v, w)/u = 0, 1, 2, 3 \cdots ; v = 0; w = 0, 1, 2, 3\}\, U$$
$$\{(u, v, w)/u = 0, 1, 2, 3 \cdots ; v = 1, 2, 3 \cdots , s; w = 1, 3\}\, U$$
$$\{(u, v, w)/u = 0, 1, 2, 3 \cdots ; v = 1, 2, 3 \cdots , s - 1; w = 2\}$$

2.2. **Truncation.** In order to carry out the computation, the infinitesimal generator matrix must be of finite order. Since the number of customers in the low and high priority queue takes infinite values, the infinitesimal generator matrix will not be a matrix of finite order. To get a matrix of finite order, the number of customers in the queue of high priority order is restricted to the finite number s and the number of customers in the orbit needs to be truncated to M in such a way that the probability that will be lost is negligible. In this model, open truncation is applied to overcome the difficulty.

2.3. **Transitions.** The transitions can be categorized into three classes

(1) Transitions from the state when there are no customers in the Orbit
(2) Transitions from the state when there is at least one customer in the Orbit
(3) Transitions from the state at the Truncation level.

These transitions are provided in Tables 1, 2 and 3.

2.4. **Chapman-Kolmogorov Governing equations.** The probability at time t for $N(t) = i, P(t) = j$ and $S(t) = k$ is denoted as $P_{ijk}(t)$. The Chapman – Kolmogorov governing Difference Differential equations for the M/M/1 queue with system breakdown and repair under non-preemptive priority services which are obtained using transition tables 1, 2 and 3 are shown in page

3. Transient Solution

3.1. **Matrix form.** Let A_{ij} be the transition matrix corresponding to the state of $N(t) = j$ from the state $N(t) = i$. Then the infinitesimal generator matrix is $Q = (A_{ij})$ The Chapman-Kolmogorov Difference Differential equations can be expressed as

$$\frac{d(X(t))}{dt} = X(t)Q \tag{3.1}$$

where

$$X(t) = ((X_0(t), X_1(t), X_2(t), \cdots , X_M(t))$$

and

$$X_i(t) = ((P_{i00}(t), P_{i01}(t), P_{i02}(t), P_{i03}(t), P_{i11}(t), P_{i12}(t), P_{i13}(t), \cdots ,$$
$$P_{is-11}(t), P_{is-12}(t), P_{is-13}(t), P_{is1}(t), P_{is3}(t))$$

TABLE 1. Transitions from the state when there are no customers in the Orbit.

From the state			Event	To the state		
i	j	k		l	m	n
0	0	0	$-(\lambda_1+\lambda_2+\nu)$	0	0	0
0	0	0	λ_1	0	0	1
0	0	0	λ_2	0	0	2
0	0	1	$-(\lambda_1+\lambda_2+\mu_1+\nu)$	0	0	1
0	0	1	λ_1	1	0	1
0	0	1	λ_2	0	1	1
0	0	1	μ_1	0	0	0
0	0	1	α	1	0	3
0	0	2	$-(\lambda_1+\lambda_2+\mu_2+\nu)$	0	0	2
0	0	2	λ_1	1	0	2
0	0	2	λ_2	0	1	2
0	0	2	μ_2	0	0	0
0	0	2	α	0	1	3
0	0	3	$-(\lambda_1+\lambda_2+\beta)$	0	0	3
0	0	3	λ_1	1	0	3
0	0	3	λ_2	0	1	3
0	0	3	β	0	0	0
0	1,2,3,...,s-1	1	$-(\lambda_1+\lambda_2+\mu_1+\alpha)$	0	j	1
0	1,2,3,...,s-1	1	λ_1	1	j	1
0	1,2,3,...,s-1	1	λ_2	0	j+1	1
0	1,2,3,...,s-1	1	μ_1	0	j-1	2
0	1,2,3,...,s-1	1	α	1	j	3
0	1,2,3,...,s-2	2	$-(\lambda_1+\lambda_2+\mu_2+\alpha)$	0	j	2
0	1,2,3,...,s-2	2	λ_1	1	j	2
0	1,2,3,...,s-2	2	λ_2	0	j+1	2
0	1,2,3,...,s-2	2	μ_2	0	j-1	2
0	1,2,3,...,s-2	2	α	0	j+1	3
0	1,2,3,...,s-1	3	$-(\lambda_1+\lambda_2+\beta)$	0	j	3
0	1,2,3,...,s-1	3	λ_1	1	j	3
0	1,2,3,...,s-1	3	λ_2	0	j+1	3
0	1,2,3,...,s-1	3	β	0	j-1	2
0	s	1	$-(\lambda_1+\mu_1+\alpha)$	0	s	1
0	s	1	λ_1	1	s	1
0	s	1	μ_1	0	s-1	2
0	s	1	α	1	s	3
0	s-1	2	$-(\lambda_1+\mu_2+\alpha)$	0	s-1	2
0	s-1	2	λ_1	1	s-1	2
0	s-1	2	μ_2	0	s-2	2
0	s-1	2	α	0	s	3
0	s	3	$-(\lambda_1+\beta)$	0	s	3
0	s	3	λ_1	1	s	3
0	s	3	β	0	s-1	2

TABLE 2. Transitions from the state when there is \atleast one customer in the Orbit.

From the state			Event	To the state		
i	j	k		l	m	n
1,2,3,...	0	0	$-(\lambda_1+\lambda_2+i\sigma)$	i	0	0
1,2,3,...	0	0	λ_1	i	0	1
1,2,3,...	0	0	λ_2	i	0	2
1,2,3,...	0	0	$i\sigma$	i-1	0	1
1,2,3,...	0	1	$-(\lambda_1+\lambda_2+\mu_1+\alpha)$	i	0	1
1,2,3,...	0	1	λ_1	i+1	0	1
1,2,3,...	0	1	λ_2	i	1	1
1,2,3,...	0	1	μ_1	i	0	0
1,2,3,...	0	1	α	i+1	0	3
1,2,3,...	0	2	$-(\lambda_1+\lambda_2+\mu_2+\alpha)$	i	0	2
1,2,3,...	0	2	λ_1	i+1	0	2
1,2,3,...	0	2	λ_2	i	1	2
1,2,3,...	0	2	μ_2	i	0	0
1,2,3,...	0	2	α	i	1	3
1,2,3,...	0	3	$-(\lambda_1+\lambda_2+\beta)$	i	0	3
1,2,3,...	0	3	λ_1	i+1	0	3
1,2,3,...	0	3	λ_2	i	1	3
1,2,3,...	0	3	β	i	0	0
1,2,3,...	1,2,3,...,s-1	1	$-(\lambda_1+\lambda_2+\mu_1+\alpha)$	i	j	1
1,2,3,...	1,2,3,...,s-1	1	λ_1	i+1	j	1
1,2,3,...	1,2,3,...,s-1	1	λ_2	i	j+1	1
1,2,3,...	1,2,3,...,s-1	1	μ_1	i	j-1	2
1,2,3,...	1,2,3,...,s-1	1	α	i+1	j	3
1,2,3,...	1,2,3,...,s-2	2	$-(\lambda_1+\lambda_2+\mu_2+\alpha)$	i	j	2
1,2,3,...	1,2,3,...,s-2	2	λ_1	i+1	j	2
1,2,3,...	1,2,3,...,s-2	2	λ_2	i	j+1	2
1,2,3,...	1,2,3,...,s-2	2	μ_2	i	j-1	2
1,2,3,...	1,2,3,...,s-2	2	α	i	j+1	3
1,2,3,...	1,2,3,...,s-1	3	$-(\lambda_1+\lambda_2+\beta)$	i	j	3
1,2,3,...	1,2,3,...,s-1	3	λ_1	i+1	j	3
1,2,3,...	1,2,3,...,s-1	3	λ_2	i	j+1	3
1,2,3,...	1,2,3,...,s-1	3	β	i	j-1	2
1,2,3,...	s	1	$-(\lambda_1+\mu_1+\alpha)$	i	s	1
1,2,3,...	s	1	λ_1	i+1	s	1
1,2,3,...	s	1	μ_1	i	s-1	2
1,2,3,...	s	1	α	i+1	s	3
1,2,3,...	s-1	2	$-(\lambda_1+\mu_2+\alpha)$	i	s-1	2
1,2,3,...	s-1	2	λ_1	i+1	s-1	2
1,2,3,...	s-1	2	μ_2	i	s-2	2
1,2,3,...	s-1	2	α	i	s	3
1,2,3,...	s	3	$-(\lambda_1+\beta)$	i	s	3
1,2,3,...	s	3	λ_1	i+1	s	3
1,2,3,...	s	3	β	i	s-1	2

TABLE 3. Transitions from the state at the \Truncation level.

From the state i	j	k	Event	To the state l	m	n
M	0	0	$-(\lambda_2 + M\sigma)$	M	0	0
M	0	0	λ_2	M	0	2
M	0	0	$M\sigma$	M-1	0	1
M	0	1	$-(\lambda_2 + \mu_1 + \alpha)$	i	0	1
M	0	1	λ_2	M	1	1
M	0	1	μ_1	M	0	0
M	0	2	$-(\lambda_2 + \mu_2 + \alpha)$	M	0	2
M	0	2	λ_2	M	1	2
M	0	2	μ_2	M	0	0
M	0	2	α	M	1	3
M	0	3	$-(\lambda_2 + \beta)$	M	0	3
M	0	3	λ_2	M	1	3
M	0	3	β	M	0	0
M	1,2,3,...,s-1	1	$-(\lambda_2 + \mu_1)$	M	j	1
M	1,2,3,...,s-1	1	λ_2	M	j+1	1
M	1,2,3,...,s-1	1	μ_1	M	j-1	2
M	1,2,3,...,s-2	2	$-(\lambda_2 + \mu_2 + \alpha)$	M	j	2
M	1,2,3,...,s-2	2	λ_2	M	j+1	2
M	1,2,3,...,s-2	2	μ_2	M	j-1	2
M	1,2,3,...,s-2	2	α	M	j+1	3
M	1,2,3,...,s-1	3	$-(\lambda_2 + \beta)$	M	j	3
M	1,2,3,...,s-1	3	λ_2	M	j+1	3
M	1,2,3,...,s-1	3	β	M	j-1	2
M	s	1	$-\mu_1$	M	s	1
M	s	1	μ_1	M	s-1	2
M	s-1	2	$-\mu_2$	M	s-1	2
M	s-1	2	μ_2	M	s-2	2
M	s-1	2	α	M	s	3
M	s	3	$-\beta$	M	s	3
M	s	3	β	M	s-1	2

3.2. **Generator Matrix.** The truncated generator matrix is

$$Q = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0M-1} & A_{0M} \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1M-1} & A_{1M} \\ A_{20} & A_{21} & A_{22} & \cdots & A_{2M-1} & A_{2M} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{M-1,0} & A_{M-1,1} & A_{M-1,2} & ... & A_{M-1M-1} & A_{M-1M} \\ A_{M0} & A_{M1} & A_{M2} & ... & A_{MM-1} & A_{MM} \end{pmatrix}$$

3.3. **Chapman Kolmogorov Equations.**

$$
\begin{aligned}
&For\ \ i = 1,2,3,\cdots,M-1;\\
&P_{000}^{'}(t) = -(\lambda_1+\lambda_2)P_{000}(t) + \mu_1 P_{001}(t) + \mu_2 P_{002}(t) + \beta P_{003}(t)\\
&P_{001}^{'}(t) = -(\lambda_1+\lambda_2+\mu_1+\alpha)P_{001}(t) + \lambda_1 P_{000}(t) + \sigma P_{100}(t)\\
&P_{002}^{'}(t) = -(\lambda_1+\lambda_2+\mu_2+\alpha)P_{002}(t) + \lambda_2 P_{000}(t) + \mu_1 P_{011}(t) + \mu_2 P_{012}(t) + \beta P_{013}(t)\\
&P_{003}^{'}(t) = -(\lambda_1+\lambda_2+\beta))P_{003}(t)\\
&P_{0j1}^{'}(t) = -(\lambda_1+\lambda_2+\mu_1+\alpha)P_{0j1}(t) + \lambda_2 P_{0j-11}(t)\ ,\ j=1,2,3,\cdots,s-1\\
&P_{0s1}^{'}(t) = -(\lambda_1+\mu_1+\alpha)P_{0s1}(t) + \lambda_2 P_{0s-11}(t)\\
&P_{0j2}^{'}(t) = -(\lambda_1+\lambda_2+\mu_2+\alpha)P_{0j2}(t) + \lambda_2 P_{0j-12}(t) + \mu_1 P_{0j+11}(t) + \mu_2 P_{0j+12}(t)\\
&\qquad\qquad +\beta P_{0j+13}(t), j=1,2,3,\cdots,s-2\\
&P_{0s-12}^{'}(t) = -(\lambda_1+\mu_2)P_{0s-12}(t) + \lambda_2 P_{0s-22}(t) + \mu_1 P_{0s1}(t) + \beta P_{0s3}(t)\\
&P_{0j3}^{'}(t) = -(\lambda_1+\lambda_2+\beta)P_{0j3}(t) + \lambda_2 P_{0j-13}(t) + \alpha P_{0j-12}(t)\ \ ,\ j=1,2,3,\cdots,s-1\\
&P_{0s3}^{'}(t) = -(\lambda_1+\beta)P_{0s3}(t) + \lambda_2 P_{0s-13}(t) + \alpha P_{0s-12}(t)\\
&P_{i00}^{'}(t) = -(\lambda_1+\lambda_2+i\sigma)P_{i00}(t) + \mu_1 P_{i01}(t) + \mu_2 P_{i02}(t) + \beta P_{i03}(t)\\
&P_{i01}^{'}(t) = -(\lambda_1+\lambda_2+\mu_1+\nu)P_{i01}(t) + \lambda_1 P_{i-101}(t) + \lambda_1 P_{i00}(t) + (i+1)\sigma P_{i+100}(t)\\
&P_{ij1}^{'}(t) = -(\lambda_1+\lambda_2+\mu_1+\nu)P_{ij1}(t) + \lambda_1 P_{i-1j1}(t) + \lambda_2 P_{ij-11}(t)\ \ ,\ j=1,2,3,\cdots,s-1\\
&P_{is1}^{'}(t) = -(\lambda_1+\mu_1+\nu)P_{is1}(t) + \lambda_1 P_{i-1s1}(t) + \lambda_2 P_{is-11}(t)\\
&P_{i02}^{'}(t) = -(\lambda_1+\lambda_2+\mu_2+\nu)P_{i02}(t) + \lambda_1 P_{i-102}(t) + \lambda_2 P_{i00}(t) + \mu_1 P_{i11}(t) + \mu_2 P_{i12}(t)\\
&P_{ij2}^{'}(t) = -(\lambda_1+\lambda_2+\mu_2+\nu)P_{ij2}(t) + \lambda_1 P_{i-1j2}(t) + \lambda_2 P_{ij-12}(t) + \mu_1 P_{ij+11}(t)\\
&\qquad\qquad +\mu_2 P_{ij+12}(t), j=1,2,3,\cdots,s-2\\
&P_{is-12}^{'}(t) = -(\lambda_1+\mu_2+\nu)P_{is-12}(t) + \lambda_1 P_{i-1s-12}(t) + \lambda_2 P_{is-22}(t)\\
&P_{i03}^{'}(t) = -(\lambda_1+\lambda_2+\beta)P_{i03}(t) + \lambda_1 P_{i-103}(t) + \alpha P_{i-101}(t)\\
&P_{ij3}^{'}(t) = -(\lambda_1+\lambda_2+\beta)P_{ij3}(t) + \lambda_1 P_{i-1j3}(t) + \lambda_2 P_{ij-13}(t) + \alpha P_{i-1j1}(t) + \alpha P_{ij-12}(t),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2,3,\cdots,s-1\\
&P_{is3}^{'}(t) = -(\lambda_1+\beta)P_{is3}(t) + \lambda_1 P_{i-1s3}(t) + \lambda_2 P_{is-13}(t) + \alpha P_{i-1s1}(t) + \alpha P_{is-12}(t)\\
&P_{M00}^{'}(t) = -(\lambda_2+M\sigma)P_{M00}(t) + \mu_1 P_{M01}(t) + \mu_2 P_{M02}(t) + \beta P_{M03}(t)\\
&P_{M01}^{'}(t) = -(\lambda_2+\mu_1+\alpha)P_{M01}(t) + \lambda_1 P_{M-101}(t) + \lambda_1 P_{M00}(t)\\
&P_{Mj1}^{'}(t) = -(\lambda_2+\mu_1)P_{Mj1}(t) + \lambda_1 P_{M-1j1}(t) + \lambda_2 P_{Mj-11}(t)\ ,\ j=1,2,3,\cdots,s-1\\
&P_{Ms1}^{'}(t) = -\mu_1 P_{Ms1}(t) + \lambda_1 P_{M-1s1}(t) + \lambda_2 P_{Ms-11}(t)\\
&P_{M02}^{'}(t) = -(\lambda_2+\mu_2+\alpha)P_{M02}(t) + \lambda_1 P_{M-102}(t) + \lambda_2 P_{M00}(t) + \mu_1 P_{M11}(t)\\
&\qquad\qquad +\mu_2 P_{M12}(t) + \beta P_{M13}\\
&P_{Mj2}^{'}(t) = -(\lambda_2+\mu_2+\nu)P_{Mj2}(t) + \lambda_1 P_{M-1j2}(t) + \lambda_2 P_{Mj-12}(t) + \mu_1 P_{Mj+11}(t)+\\
&\qquad\qquad \mu_2 P_{Mj+12}(t) + \beta P_{Mj+13}\ ,\qquad\qquad\qquad\qquad j=1,2,3,\cdots,s-2\\
&P_{Ms-12}^{'}(t) = -\mu_2 P_{Ms-12}(t) + \lambda_1 P_{M-1s-12}(t) + \lambda_2 P_{Ms-22}(t) + \beta P_{Ms3}(t)\\
&P_{M03}^{'}(t) = -(\lambda_2+\beta)P_{M03}(t) + \lambda_1 P_{M-103}(t) + \alpha P_{M-101}(t)\\
&P_{Mj3}^{'}(t) = -(\lambda_2+\beta)P_{Mj3}(t) + \lambda_1 P_{M-1j3}(t) + \lambda_2 P_{Mj-13} + \alpha P_{Mj-12}(t) + \alpha P_{M-1j1}(t),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2,3,\cdots,s-1\\
&P_{Ms3}^{'}(t) = \beta P_{Ms3}(t) + \lambda_1 P_{M-1s3}(t) + \lambda_2 P_{Ms-13}(t) + \alpha P_{Ms-12}(t)
\end{aligned}
$$

The matrices A_{ij} can be obtained from the system of Chapman - Kolmogrov equations.Under Markovian process $A_{01} = A_{12} = A_{23} = \cdots = A_{M-1M}$ and$A_{ij} = 0$ for

$|i-j|>1$.Thus the generator matrix is

$$Q = \begin{pmatrix} A_{00} & A_0 & 0 & \ldots & 0 & 0 \\ A_{10} & A_{11} & A_0 & \ldots & 0 & 0 \\ 0 & A_{21} & A_{22} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & A_{M-1M-1} & A_0 \\ 0 & 0 & 0 & \ldots & A_{MM-1} & A_{MM} \end{pmatrix}$$

The solution of the equation is

$X(t) = X(0)e^{tQ}$ where $[X(0)] = (1, 0, 0, ..., 0)$

4. Description of the computational method

The procedure of computational method to find transient solution to the M/M/1 Retrial model combined with system breakdown and repair under non-preemptive priority.

(1) Determination of the generator matrix Q which is square matrix of order $3(M+1)(s+1)$
(2) Diagonalization of the diagonal matrix of tQ using eigenvalues and eigenvectors for given time instance t.
(3) Estimatation of e^{tQ} using diagonalization matrix of tQ
(4) First row of $e^{\,tQ}$ fetches $X(t)$
(5) Repeat the above steps for different values of t

5. Time dependent system performance measures

5.1. The probability mass function of Server state.

(1) Probability that the server is idle at time t

$$P_0(t) = \sum_{i=0}^{\infty} P_{i00}(t)$$

(2) Probability that the server is busy with a low priority customer at time t is

$$P_1(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{s} P_{ij1}(t)$$

(3) Probability that the server is busy with a high priority customer at time t is

$$P_2(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{s-1} P_{ij2}(t)$$

(4) Probability that the server is inactive at time t is

$$P_3(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{s} P_{ij3}(t)$$

5.2. **The probability mass function of number of customers in the orbit.**

Let $PL_i(t)$ represent the probability of i customers in the Orbit at time t.

(1) Probability of no customers in the orbit at t is

$$PL_0(t) = \sum_{l=0}^{3} P_{00l}(t) + \sum_{j=1}^{s-1}\sum_{l=1}^{3} P_{0jl}(t) + P_{0s1}(t) + P_{0s3}(t)$$

(2) Probability of n customers in the orbit at time t is

$$PL_n(t) = \sum_{j=1}^{s-1}\sum_{l=1}^{2} P_{njl}(t) + P_{ns1}(t) + \sum_{l=0}^{2} P_{n0l}(t)$$

5.3. **The Probability mass function of number of high priority customers.**

Let $PH_i(t)$ represent the probability of i customers in the Orbit at time t.

(1) Probability of no customer in the high priority queue at time t

$$PH_0(t) = \sum_{i=0}^{\infty}\sum_{l=0}^{3} P_{i0l}(t)$$

(2) Probability of $n(< S)$ customers in the high priority queue at time t is

$$PH_n(t) = \sum_{i=0}^{\infty}\sum_{l=1}^{3} P_{inl}(t), \quad n = 1, 2, 3, \cdots, s-1$$

(3) Probability of S customers in the high priority queue at time t is

$$PH_s(t) = \sum_{i=0}^{\infty}\sum_{l=1,3} P_{isl}(t),$$

(4) The Mean Priority Queue Length at time t is

$$MPQL(t) = \sum_{j=1}^{s} j\, PH_j(t)$$

(5) The Mean Number of Customers in the Orbit at time t is

$$MNCO(t) = \sum_{i=0}^{\infty} iPL_i(t)$$

(6) The probability that the orbiting customer is blocked at time t is

$$\sum_{i=1}^{\infty}\sum_{j=0}^{s-1}\sum_{l=1}^{3} P_{ijl}(t) + \sum_{i=1}^{\infty}\sum_{l=1,3} P_{isl}(t)$$

(7) The probability that an arriving customer(high or low) enters the service station immediately at time $t = \sum_{i=0}^{\infty} P_{i00}(t)$

6. TIME DEPENDENT NUMERICAL STUDY

The time dependent probability mass distributions and system performance measures of the model were estimated for μ_1= 10, μ_2= 20 σ = 10, s = 4, α = 100 and ν=5 and for various values of λ_1, λ_2.

Tables 4, 5 and 6 provide time dependent probabilities of number of customers in the orbit at time t for different values of λ_1, λ_2. It is observed that as t increases, $PL_n(t) \rightarrow PL_n$
where PL_n is the Steady state probability that there are n customers in the orbit. These results coincides with Single server retrial queueing system with system breakdown and repair under non-preemptive priority services.

The sequence $\{PL_n(t)\} \rightarrow 0$ as $n \rightarrow \infty$ for all values of t.

Tables 7, 8 and 9 provide time dependent probabilities of number of customers in the high priority queue in front of the service station at time t. It is observed that as the value of t increases, the Transient Probabilities

$$PH_n(t) \rightarrow PH_n$$

where PH_n is the Steady state probability that there are n customers in the high priority queue. These results coincides with Single server retrial queueing system with non-preemptive priority services.

Tables 10, 11 and 12 provide the time dependent probabilities that, the server is idle, busy with low priority customer, busy with high priority customer, $MNCO$ and $MPQL$ for different rate of arrivals times at time t. It is observed that
as the value of t increases and for various values of λ_1, λ_2,μ_1, μ_2, and σ

$$P_0(t) \rightarrow P_0, \qquad P_1(t) \rightarrow P_1, \qquad MNCO(t) \rightarrow MNCO, \qquad MPQL(t) \rightarrow MPQL$$

where $MNCO$ is the mean number of low priority customers in the Orbit and $MPQL$ is the mean number of high priority customers in the steady state. Moreover it is observed from the tables that

(1) $P_0(t)$ decreases as arrival rates λ_1, λ_2 increases for all values of t.
(2) $P_1(t)$ increases as primary arrival rate $\lambda_1$increases for all values t.
(3) $P_2(t)$ increases as arrival rate λ_2 increases for all values$t$.
(4) $MNCO\,(t)$ increases as arrival rate $\lambda_1$increases for all values of t.
(5) $MPQL_(t)$ increases as arrival rate λ_2 increases for all values of t.

7. CONCLUSION

The transient solution of Single server Retrial Queue model with catastrophe under non-preemptive priority is found out using matrix exponential method. The time dependent probabilities and system performance measures are estimated. The scope of the paper is catastrophe can be combined with other restrictions such as reliability, balking and reneging etc. The solution can also be extended to preemptive priority.

TABLE 4. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 1$ and $\lambda_2 = 4$.

t	$PL_0(t)$	$PL_1(t)$	$PL_2(t)$	$PL_3(t)$	$PL_4(t)$	$PL_5(t)$	$PL_6(t)$
11	0.628121	0.238119	0.086820	0.030750	0.010677	0.003654	0.001237
12	0.628111	0.238119	0.086822	0.030753	0.010679	0.003656	0.001238
13	0.628106	0.238119	0.086823	0.030754	0.010680	0.003656	0.001238
14	0.628104	0.238119	0.086824	0.030755	0.010680	0.003656	0.001238
15	0.628103	0.238119	0.086824	0.030755	0.010680	0.003657	0.001239
16	0.628102	0.238119	0.086824	0.030755	0.010680	0.003657	0.001239
17	0.628102	0.238119	0.086824	0.030755	0.010681	0.003657	0.001239
18	0.628102	0.238119	0.086825	0.030755	0.010681	0.003657	0.001239
19	0.628102	0.238119	0.086825	0.030755	0.010681	0.003657	0.001239
20	0.628102	0.238119	0.086825	0.030755	0.010681	0.003657	0.001239

TABLE 5. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 3$ and $\lambda_2 = 4$.

t	$PL_0(t)$	$PL_1(t)$	$PL_2(t)$	$PL_3(t)$	$PL_4(t)$	$PL_5(t)$	$PL_6(t)$
41	0.126224	0.136642	0.129101	0.114513	0.097819	0.081478	0.066649
42	0.126210	0.136627	0.129089	0.114504	0.097812	0.081474	0.066647
43	0.126197	0.136615	0.129078	0.114495	0.097806	0.081471	0.066645
44	0.126186	0.136604	0.129068	0.114488	0.097801	0.081468	0.066644
45	0.126177	0.136594	0.129060	0.114481	0.097796	0.081465	0.066643
46	0.126169	0.136585	0.129053	0.114475	0.097792	0.081462	0.066642
47	0.126161	0.136578	0.129046	0.114470	0.097789	0.081460	0.066641
48	0.126155	0.136571	0.129040	0.114466	0.097785	0.081458	0.066640
49	0.126149	0.136566	0.129035	0.114462	0.097783	0.081457	0.066639
50	0.126144	0.136560	0.129031	0.114458	0.097780	0.081455	0.066639

TABLE 6. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 5$ and $\lambda_2 = 4$.

t	$PL_0(t)$	$PL_1(t)$	$PL_2(t)$	$PL_3(t)$	$PL_4(t)$	$PL_5(t)$	$PL_6(t)$
111	0.000170	0.000307	0.000451	0.000603	0.000766	0.000938	0.001120
112	0.000167	0.000302	0.000443	0.000592	0.000752	0.000921	0.001100
113	0.000164	0.000296	0.000435	0.000582	0.000738	0.000905	0.001080
114	0.000161	0.000291	0.000427	0.000571	0.000725	0.000888	0.001061
115	0.000159	0.000286	0.000419	0.000561	0.000712	0.000873	0.001042
116	0.000156	0.000281	0.000412	0.000551	0.000700	0.000858	0.001024
117	0.000153	0.000276	0.000405	0.000542	0.000688	0.000843	0.001006
118	0.000150	0.000271	0.000398	0.000532	0.000676	0.000828	0.000989
119	0.000148	0.000266	0.000391	0.000523	0.000664	0.000814	0.000972
120	0.000145	0.000262	0.000384	0.000514	0.000653	0.000800	0.000955

TABLE 7. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 1$ and $\lambda_2 = 4$.

t	$PH_0(t)$	$PH_1(t)$	$PH_2(t)$	$PH_3(t)$	$PH_4(t)$
11	0.571719	0.158846	0.067837	0.028718	0.012284
12	0.571718	0.158846	0.067838	0.028718	0.012284
13	0.571718	0.158847	0.067838	0.028718	0.012284
14	0.571717	0.158847	0.067838	0.028718	0.012285
15	0.571717	0.158847	0.067838	0.028718	0.012285
16	0.571717	0.158847	0.067838	0.028718	0.012285
17	0.571717	0.158847	0.067838	0.028718	0.012285
18	0.571717	0.158847	0.067838	0.028718	0.012285
19	0.571717	0.158847	0.067838	0.028718	0.012285
20	0.571717	0.158847	0.067838	0.028718	0.012285

TABLE 8. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 3$ and $\lambda_2 = 4$.

t	$PH_0(t)$	$PH_1(t)$	$PH_2(t)$	$PH_3(t)$	$PH_4(t)$
41	0.519331	0.200879	0.083718	0.035164	0.015029
42	0.519327	0.200882	0.083719	0.035164	0.015029
43	0.519325	0.200884	0.083720	0.035165	0.015029
44	0.519322	0.200886	0.083721	0.035165	0.015029
45	0.519320	0.200888	0.083721	0.035165	0.015029
46	0.519318	0.200890	0.083722	0.035166	0.015029
47	0.519316	0.200891	0.083722	0.035166	0.015029
48	0.519315	0.200892	0.083723	0.035166	0.015029
49	0.519313	0.200893	0.083723	0.035166	0.015029
50	0.519312	0.200894	0.083724	0.035166	0.015030

TABLE 9. Time dependent Probability distribution of number of customers in the orbit for $\lambda_1 = 5$ and $\lambda_2 = 4$.

t	$PH_0(t)$	$PH_1(t)$	$PH_2(t)$	$PH_3(t)$	$PH_4(t)$
111	0.477782	0.234246	0.096323	0.040280	0.017206
112	0.477761	0.234265	0.096330	0.040283	0.017208
113	0.477740	0.234283	0.096337	0.040286	0.017209
114	0.477720	0.234301	0.096344	0.040289	0.017210
115	0.477700	0.234319	0.096351	0.040291	0.017211
116	0.477680	0.234337	0.096358	0.040294	0.017212
117	0.477661	0.234355	0.096364	0.040297	0.017214
118	0.477642	0.234372	0.096371	0.040299	0.017215
119	0.477624	0.234389	0.096377	0.040302	0.017216
120	0.477605	0.234406	0.096384	0.040305	0.017217

TABLE 10. Time dependent System Performance Measures for $\lambda_1 = 5$ and $\lambda_2 = 4$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$MNCO(t)$	$MPQL(t)$
11	0.370724	0.099994	0.195949	0.333333	0.577072	0.429813
12	0.370721	0.099997	0.195949	0.333333	0.577111	0.429815
13	0.370719	0.099999	0.195949	0.333333	0.577129	0.429816
14	0.370718	0.099999	0.195949	0.333333	0.577138	0.429816
15	0.370718	0.100000	0.195949	0.333333	0.577142	0.429816
16	0.370718	0.100000	0.195949	0.333333	0.577144	0.429817
17	0.370718	0.100000	0.195949	0.333333	0.577145	0.429817
18	0.370718	0.100000	0.195949	0.333333	0.577146	0.429817
19	0.370718	0.100000	0.195949	0.333333	0.577146	0.429817
20	0.370718	0.100000	0.195949	0.333333	0.577146	0.429817

TABLE 11. Time dependent System Performance Measures for $\lambda_1 = 5$ and $\lambda_2 = 4$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$MNCO(t)$	$MPQL(t)$
41	0.171701	0.299898	0.195068	0.333333	4.511876	0.533922
42	0.171689	0.299910	0.195068	0.333333	4.512823	0.533928
43	0.171678	0.299921	0.195068	0.333333	4.513659	0.533934
44	0.171669	0.299930	0.195068	0.333333	4.514396	0.533939
45	0.171661	0.299938	0.195068	0.333333	4.515046	0.533943
46	0.171653	0.299945	0.195068	0.333333	4.515620	0.533947
47	0.171647	0.299952	0.195068	0.333333	4.516128	0.533950
48	0.171641	0.299957	0.195068	0.333333	4.516576	0.533953
49	0.171637	0.299962	0.195068	0.333333	4.516972	0.533956
50	0.171632	0.299967	0.195068	0.333333	4.517323	0.533958

TABLE 12. Time dependent System Performance Measures for $\lambda_1 = 5$ and $\lambda_2 = 4$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$MNCO(t)$	$MPQL(t)$
111	0.013713	0.458668	0.194367	0.333251	66.84400	0.616558
112	0.013624	0.458763	0.194367	0.333247	67.24963	0.616605
113	0.013536	0.458856	0.194367	0.333242	67.65390	0.616651
114	0.013450	0.458947	0.194366	0.333236	68.05681	0.616696
115	0.013365	0.459038	0.194366	0.333231	68.45837	0.616741
116	0.013281	0.459128	0.194366	0.333225	68.85857	0.616785
117	0.013198	0.459217	0.194365	0.333220	69.25739	0.616828
118	0.013117	0.459305	0.194365	0.333214	69.65484	0.616871
119	0.013036	0.459392	0.194365	0.333208	70.05090	0.616913
120	0.012957	0.459478	0.194364	0.333201	70.44558	0.616955

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