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Upper bounds of Toeplitz determinants for a subclass of alpha-close-to-convex functions

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions in the unit disc $\mathbb{U}$ which are of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $0 \leq \alpha < 1$, let $\mathcal{C}_\alpha$, be the class of all functions $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}\{\mathrm{f}'(\mathrm{z}) + \alpha \mathrm{z}\mathrm{f}''(\mathrm{z})\} > 0$. We consider the Toeplitz matrices whose elements are the coefficients a_n of the function f in the class $\mathcal{C}_\alpha$. In this paper we obtain upper bounds for the Toeplitz determinants.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions defined on the unit disc $\mathbb{U} = \{z : |z| < 1\}$ with the normalization condition $f(0) = 0 = f'(0) - 1$. Thus $f \in \mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$. Let $\mathcal{P}$ denote the class of functions $p(z)$, has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

which are regular in the open unit disc $\mathbb{U}$ and satisfy $\mathrm{Re}\, p(z) > 0$, for $z \in \mathbb{U}$. Here $p(z)$ is called the Caratheodory function [5].
Let $\mathcal{S}^*$ denote the class of functions $f \in \mathcal{A}$ which maps $\mathbb{U}$ onto a starlike domain with respect to origin. It is well known that $f \in \mathcal{S}^*$ if and only if $\frac{zf'(z)}{f(z)} \in \mathcal{P}$, $z \in \mathbb{U}$. Let $\mathcal{C}$ be the class of functions in $\mathcal{A}$, which maps $\mathbb{U}$ onto a convex domain. So $f \in \mathcal{C}$ if and only if $1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}$, $z \in \mathbb{U}$.

Definition 1.1. ([4]) For $\alpha \geq 0$, a function $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ is said to be an alpha-close-to-convex function if for a starlike function $\phi(z)$, it satisfies the condition

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

We denote $\mathcal{C}_\alpha$ by the class of all alpha-close-to-convex functions. This class was introduced and studied by Chichra [4].
For $\alpha = 0$, $\mathcal{C}_\alpha \equiv \mathcal{K}$, the class of close-to-convex functions. We denote the subclass of $\mathcal{C}_\alpha$ by $\tilde{\mathcal{R}}_\alpha$ for which $\phi(z) = z$.

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Definition 1.2. ([4]) Let $\tilde{\mathcal{R}}_\alpha$ be the class of all functions $f \in \mathcal{A}$ which satisfy

$$\mathrm{Re}(f'(z) + \alpha z f''(z)) > 0, \quad \text{for all } z \in \mathbb{U}.$$

For $\alpha = 0$, $\tilde{\mathcal{R}}_\alpha \equiv \mathcal{R}_0 \equiv \mathcal{R} = \{f(z) \in \mathcal{A} : \mathrm{Re}(f'(z)) > 0, \text{ for all } z \in \mathbb{U}\}$. These classes have been studied by many authors [4, 9, 10, 12, 13] in various viewpoints.

Remark 1.1. Bieberbach's conjecture, which is also known as de Branges Theorem after its proof, has been one of the most popular problems in the theory of univalent functions. From the introduction of the Bieberbach conjecture in 1916, until its proof given by de Branges in 1985, a lot of methods and concepts have been developed. For instance, the problem of estimating bounds for successive coefficients was studied with an idea to solve the Bieberbach conjecture. The successive coefficient problem is still open for the whole class of univalent analytic functions as well as its many subclasses. Such investigations more often lead to new techniques and ideas to deal the main open problem or related problems. In the line of such investigation, later various types of coefficient problems appeared in function theory while considering various transformation of univalent functions such as the Fekete-Szegö coefficient problem which appears from the inversion transformation. Hankel and Toeplitz determinants of Taylor coefficients of analytic functions are also attracted to many researchers in the field of geometric function theory in the sense that those are somehow related to the above coefficient problems. In fact, Toeplitz determinants have many applications in operator theory, linear algebra, physics, etc. (see for instance [16] and references therein).

The estimates of the first four Toeplitz determinants are obtained for the class $\mathcal{R}$ in [15]. In [13], Sahoo considered the class $\tilde{\mathcal{R}}_\alpha$ and obtained the bounds of first four Hankel determinants. We here consider the Toeplitz determinant for the class $\tilde{\mathcal{R}}_\alpha$ with an aim to provide generalizations with sharpness of the upper bounds of the results studied in [15] by Radhika et al.

Here we recall Toeplitz symmetric matrices, that have constant entries along the diagonal.

Definition 1.3. The q-th Toeplitz determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ is defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & a_n \end{vmatrix}. \tag{1.3}$$

So

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}; T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}; T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

To prove our main results we need the following lemma.

Lemma 1.1. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ with $c_1 \geq 0$. Then for some complex valued x with $|x| \leq 1$ and some complex valued z with $|z| \leq 1$, we have*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z.$$

2. MAIN RESULTS

In our first theorem we obtain an upper bound for $T_2(2)$.

Theorem 2.1. *Let f given by* (1.1) *be in the class $\tilde{\mathcal{R}}_\alpha$. Then we have the bound*

$$|T_2(2)| \leq \frac{5(1+\alpha)^2 + 9\alpha(2+3\alpha)}{9(1+\alpha)^2(1+2\alpha)^2}. \tag{2.4}$$

The bound is sharp.

Proof. First note that by equating the corresponding coefficient of $f'(z) + \alpha z f''(z) = p(z)$, we have

$$a_2 = \frac{c_1}{2(1+\alpha)}; \quad a_3 = \frac{c_2}{3(1+2\alpha)}, \quad a_4 = \frac{c_3}{4(1+3\alpha)}. \tag{2.5}$$

Then expanding the determinant $T_2(2)$ and writing a_2, a_3 in terms of c_1, c_2 with the help of (2.5), we get

$$|T_2(2)| = |a_3^2 - a_2^2| = \left|\frac{c_2^2}{9(1+2\alpha)^2} - \frac{c_1^2}{4(1+\alpha)^2}\right|.$$

By substituting the value c_2 in terms of c_1 defined in Lemma 1.1, it follows that

$$|a_3^2 - a_2^2| = \frac{(4-c_1^2)x^2}{36(1+2\alpha)^2} + \frac{2c_1^2(4-c_1^2)x}{36(1+2\alpha)^2} - \frac{[(1+\alpha)^2(9-c_1^2) + 9\alpha(2+3\alpha)]c_1^2}{36(1+\alpha)^2(1+2\alpha)^2}.$$

The class $\mathcal{P}$ is invariant under rotations, so we may assume that, $c := c_1 \in [0,2]$ **([3], see also [[6], Vol.I, page 80, Theorem 3])**. By applying triangle inequality, we get

$$|a_3^2 - a_2^2| \leq \frac{(4-c^2)^2|x|^2}{36(1+2\alpha)^2} + \frac{2c^2(4-c^2)|x|}{36(1+2\alpha)^2} + \frac{[(1+\alpha)^2(9-c^2) + 9\alpha(2+3\alpha)]c^2}{36(1+\alpha)^2(1+2\alpha)^2}.$$

Let $|x| = \mu$, then

$$|a_3^2 - a_2^2| \leq \frac{(4-c^2)^2\mu^2}{36(1+2\alpha)^2} + \frac{2c^2(4-c^2)\mu}{36(1+2\alpha)^2} + \frac{[(1+\alpha)^2(9-c^2) + 9\alpha(2+3\alpha)]c^2}{36(1+\alpha)^2(1+2\alpha)} = F_1(c,\mu). \ (let)$$

Differentiating $F_1(c,\mu)$ with respect to μ, we have

$$\frac{\partial F_1(c,\mu)}{\partial \mu} = \frac{(4-c^2)[\mu(4-c^2) + c^2]}{18(1+2\alpha)^2} > 0.$$

So

$$\max_{(0\leq\mu\leq1)} F_1(c,\mu) = F_1(c,1) = G_1(c) \text{ (say)},$$

where

$$G_1(c) = \frac{(4-c^2)^2 + 2c^2(4-c^2)}{36(1+2\alpha)^2} + \frac{[(1+\alpha)^2(9-c^2) + 9\alpha(2+3\alpha)]c^2}{36(1+\alpha)^2(1+2\alpha)^2}.$$

Differentiating $G_1(c)$ with respect to c, we have

$$\frac{\partial G_1}{\partial c} = \frac{c[9(1+2\alpha)^2 - 4(1+\alpha)^2c^2]}{18(1+\alpha)^2(1+2\alpha)^2},$$

which implies that the critical point $c_0 = \frac{9(1+2\alpha)^2}{4(1+\alpha)^2}$. As the critical point c_0 is not in the interval $[0,2]$ hence

$$\max_{(0\leq c\leq 2)} G_1(c) = \max\left\{\frac{4}{9(1+2\alpha)^2}, \frac{5(1+\alpha)^2 + 9\alpha(2+3\alpha)}{9(1+\alpha)^2(1+2\alpha)^2}\right\} = \frac{5(1+\alpha)^2 + 9\alpha(2+3\alpha)}{9(1+\alpha)^2(1+2\alpha)^2}.$$

Equality in (2.4) holds for the function g_α that satisfies $f'(z)+\alpha z f''(z) = \dfrac{1+z}{1-z}$. This gives

$$g_\alpha(z) = z + \frac{1}{1+\alpha}z^2 + \frac{2}{3(1+2\alpha)}z^3 + \frac{1}{2(1+3\alpha)}z^4 + \cdots . \tag{2.6}$$

This completes the proof of the Theorem 2.1. □

Remark 2.2. Theorem 2.1 for $\alpha = 0$ gives the bound $|a_2^2 - a_3^2| \le 5/9$ for the class of functions with bounded boundary rotation $\mathcal{R}$ derived by Radhika et.al [15] and in [14] for $\beta = 1$.

Theorem 2.2. *Let f given by* (1.1) *be in the class $\tilde{\mathcal{R}}_\alpha$. Then we have the bound*

$$|T_2(3)| \le \frac{4}{9(1+2\alpha)^2}. \tag{2.7}$$

The bound is sharp.

Proof. Expanding the determinant $T_2(3)$ and writing a_3, a_4 in terms of c_2, c_3 with the help of (2.5), we get

$$|T_2(3)| = |a_4^2 - a_3^2| = \left|\frac{c_3^2}{16(1+3\alpha)^2} - \frac{c_2^2}{9(1+2\alpha)^2}\right|.$$

From Lemma 1.1, we get

$$c_2 = \frac{1}{2}\left[c_1^2 + (4-c_1^2)x\right],\ c_3 = \frac{1}{4}\left[c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\right].$$

The class $\mathcal{P}$ is invariant under rotations, so we may assume that, $c := c_1 \in [0,2]$ ([3], see also [[6], Vol. I, page 80, Theorem 3]. By using triangle inequality with $M = 4 - c^2$, we get

$$\begin{aligned}
|a_4^2 - a_3^2| \le\ & \left(\frac{M^2c^2}{256} + \frac{M^2}{64} + \frac{M^2c}{64}\right)\frac{|x|^4}{(1+3\alpha)^2} + \left(\frac{M^2c^2}{64} + \frac{M^2c}{32}\right)\frac{|x|^3}{(1+3\alpha)^2} \\
+ & \left(\frac{M^2c^2}{64} - \frac{M^2(1-12\alpha-36\alpha^2)}{288(1+2\alpha)^2} + \frac{Mc^4}{128} + \frac{Mc^3}{64} + \frac{M^2c}{64}\right)\frac{|x|^2}{(1+3\alpha)^2} \\
+ & \left(\frac{Mc^4}{64(1+3\alpha)^2} + \frac{M^2c}{32(1+3\alpha)^2} + \frac{Mc^2}{18(1+2\alpha)^2}\right)|x| \\
+ & \left|\frac{Mc^3}{64(1+3\alpha)^2} + \frac{M^2}{64(1+3\alpha)^2} + \frac{c^4}{36(1+2\alpha)^2} - \frac{c^6}{256(1+3\alpha)^2}\right|.
\end{aligned}$$

Let $|x| = \mu$, then

$$\begin{aligned}
|a_4^2 - a_3^2| \le\ & \left(\frac{M^2c^2}{256} + \frac{M^2}{64} + \frac{M^2c}{64}\right)\frac{\mu^4}{(1+3\alpha)^2} + \left(\frac{M^2c^2}{64} + \frac{M^2c}{32}\right)\frac{\mu^3}{(1+3\alpha)^2} \\
+ & \left(\frac{M^2c^2}{64} - \frac{M^2(1-12\alpha-36\alpha^2)}{288(1+2\alpha)^2} + \frac{Mc^4}{128} + \frac{Mc^3}{64} + \frac{M^2c}{64}\right)\frac{\mu^2}{(1+3\alpha)^2} \\
+ & \left(\frac{Mc^4}{64(1+3\alpha)^2} + \frac{M^2c}{32(1+3\alpha)^2} + \frac{Mc^2}{18(1+2\alpha)^2}\right)\mu \\
+ & \left|\frac{Mc^3}{64(1+3\alpha)^2} + \frac{M^2}{64(1+3\alpha)^2} + \frac{c^4}{36(1+2\alpha)^2} - \frac{c^6}{256(1+3\alpha)^2}\right| \\
=\ & F_2(c,\mu).
\end{aligned}$$

First we consider the modulus term in $F_2(c,\mu)$ is positive. Then on differentiating $F_2(c,\mu)$ with respect to μ, we get

$$\begin{aligned}\frac{\partial F_2}{\partial \mu} &= \left(\frac{M^2c^2}{256}+\frac{M^2}{64}+\frac{M^2c}{64}\right)\frac{4\mu^3}{(1+3\alpha)^2}\\ &+ \left(\frac{M^2c^2}{64}+\frac{M^2c}{32}\right)\frac{3\mu^2}{(1+3\alpha)^2}+\left(\frac{M^2c^2}{64}-\frac{M^2(1-12\alpha-36\alpha^2)}{288(1+2\alpha)^2}+\frac{Mc^4}{128}\right.\\ &+ \left.\frac{Mc^3}{64}+\frac{M^2c}{64}\right)\frac{2\mu}{(1+3\alpha)^2}+\left(\frac{Mc^4}{64(1+3\alpha)^2}+\frac{M^2c}{32(1+3\alpha)^2}+\frac{Mc^2}{18(1+2\alpha)^2}\right).\end{aligned}$$

We need to find the maximum value of $F_2(c,\mu)$ in $[0,2]\times[0,1]$. First assume that there is a maximum of $F_2(c,\mu)$ attains at an interior point of $[0,2]\times[0,1]$. But $\dfrac{\partial F_2}{\partial \mu}=0$ gives that $M=4-c^2=0$ which implies that $c=2$, which is a contradiction. Thus for the maximum of $F_2(c,\mu)$, we need to consider the boundary points of $[0,2]\times[0,1]$.
For $c=0$, $c=2$ and $0\le\mu\le 1$, a simple calculation shows that

$$F_2(c,\mu)\le\frac{4}{9(1+2\alpha)^2}. \tag{2.8}$$

If we consider the modulus term in $F_2(c,\mu)$ is negative, then we have

$$\begin{aligned}F_2(c,\mu) &= \left(\frac{M^2c^2}{256}+\frac{M^2}{64}+\frac{M^2c}{64}\right)\frac{\mu^4}{(1+3\alpha)^2}+\left(\frac{M^2c^2}{64}+\frac{M^2c}{32}\right)\frac{\mu^3}{(1+3\alpha)^2}\\ &+ \left(\frac{M^2c^2}{64}-\frac{M^2(1-12\alpha-36\alpha^2)}{288(1+2\alpha)^2}+\frac{Mc^4}{128}+\frac{Mc^3}{64}+\frac{M^2c}{64}\right)\frac{\mu^2}{(1+3\alpha)^2}\\ &+ \left(\frac{Mc^4}{64(1+3\alpha)^2}+\frac{M^2c}{32(1+3\alpha)^2}+\frac{Mc^2}{18(1+2\alpha)^2}\right)\mu\\ &- \left[\frac{Mc^3}{64(1+3\alpha)^2}+\frac{M^2}{64(1+3\alpha)^2}+\frac{c^4}{36(1+2\alpha)^2}-\frac{c^6}{256(1+3\alpha)^2}\right].\end{aligned}$$

Following the steps as in the above case and a simple calculation shows

$$\max_{(c,\mu)\in[0,2]\times[0,1]} F_2(c,\mu)=\frac{-1+12\alpha+36\alpha^2}{18(1+2\alpha)^2(1+3\alpha)^2}, \tag{2.9}$$

Thus the result follows from (2.8), (2.9). Equality in (2.7) holds for the function f_α, that satisfies $f'(z)+\alpha z f''(z)=\dfrac{1+z^2}{1-z^2}$.
This completes the proof of Theorem 2.2. $\square$

Remark 2.3. Theorem 2.2 for $\alpha=0$ gives the bound $|T_2(3)|\le 4/9$ for the class of functions with bounded boundary rotation $\mathcal{R}$ derived by Radhika et.al [15] and in [14] for $\beta=1$.

Theorem 2.3. *Let f given by (1.1) be in the class $\tilde{\mathcal{R}}_\alpha$. Then we have the bound*

$$|T_3(1)|\le\frac{13+36\alpha+36\alpha^2}{9(1+2\alpha)^2}. \tag{2.10}$$

The bound is sharp.

Proof. From the expansion the determinant $T_3(1)$ and (2.5) with $M = 4 - c^2$, we get

$$|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right|$$

$$= \left|1 + \frac{c_1^2}{2(1+\alpha)^2}(\frac{c_2}{3(1+2\alpha)} - 1) - \frac{c_2^2}{9(1+2\alpha)^2}\right|$$

$$= \left|1 + \frac{c_1^4(2+4\alpha-\alpha^2)}{36(1+\alpha)^2(1+2\alpha)^2} - \frac{c_1^2}{2(1+\alpha)^2} + \frac{c_1^2 xM(1+2\alpha-2\alpha^2)}{36(1+\alpha)^2(1+2\alpha)^2} - \frac{x^2M^2}{36(1+2\alpha)^2}\right|.$$

The class $\mathcal{P}$ is invariant under rotations, so we may assume that, $c := c_1 \in [0,2]$ ([3], see also [[6], Vol.I, page 80, Theorem 3]). By using the triangle inequality and the fact that $|x| \leq 1$, we obtain

$$\begin{aligned} T_3(1) &\leq \left|1 + \frac{c^4(2+4\alpha-\alpha^2)}{36(1+\alpha)^2(1+2\alpha)^2} - \frac{c^2}{2(1+\alpha)^2}\right| + \frac{c^2(1+2\alpha-2\alpha^2)(4-c^2)}{36(1+\alpha)^2(1+2\alpha)^2} + \frac{(4-c^2)^2}{36(1+2\alpha)^2} \\ &= G_2(c). \end{aligned} \tag{2.11}$$

First we consider the modulus term of $G_2(c)$ in (2.11) positive. Then we write $G_2(c)$ as $G_3(c)$, where

$$G_3(c) = \frac{\left[(1+\alpha)^2c^4 - c^2(11+40\alpha+44\alpha^2) + 2(1+\alpha)^2(13+36\alpha+36\alpha^2)\right]}{18(1+\alpha)^2(1+2\alpha)^2}.$$

Differentiating $G_3(c)$ with respect to c we get

$$G_3'(c) = -\frac{c\left[2((1+\alpha)^2(4-c^2) + 3(1+8\alpha+12\alpha^2\right])}{9(1+\alpha)^2(1+2\alpha)^2} < 0 \text{ for } c \in [0,2].$$

So

$$T_3(1) \leq \max_{0\leq c\leq 2} G_3(c) = G_3(0) = M_1(\alpha),$$

where

$$M_1(\alpha) = \frac{13+36\alpha+36\alpha^2}{9(1+2\alpha)^2}. \tag{2.12}$$

Similarly, considering the modulus term of $G_2(c)$ in (2.11) as negative, writting $G_2(c)$ as $G_4(c)$, where

$$G_4(c) = \frac{-\left[c^4(1+2\alpha-2\alpha^2) - (7+32\alpha+28\alpha^2)c^2 + 2(1+\alpha)^2(5+36\alpha+36\alpha^2)\right]}{18(1+\alpha)^2(1+2\alpha)^2}.$$

Differentiating $G_4(c)$ with respect to c we get

$$G_4'(c) = \frac{-2c\left[2c^2(1+2\alpha-2\alpha^2) - (7+32\alpha+28\alpha^2)\right]}{18(1+\alpha)^2(1+2\alpha)^2}.$$

The critical points of $G_4'(c)$ are $c = 0$ and c_0, where $c_0^2 = \dfrac{7+32\alpha+28\alpha^2}{2(1+2\alpha-2\alpha^2)}$.

A simple calculation shows that $\max_{c\in[0,2]} G_4(c) = G_4(c_0)$ when $\alpha > \beta$, where $\beta = \frac{3\sqrt{3}-4}{22}$. Thus for $\alpha > \beta$,

$$|T_3(1)| \leq G_4(c_0) = M_2(\alpha),$$

where

$$M_2(\alpha) = \frac{9 - 144\alpha^2 - 144\alpha^3 + 576\alpha^4 + 1152\alpha^5 + 576\alpha^6}{72(1+\alpha)^2(1+2\alpha)^2(1+2\alpha-2\alpha^2)}. \tag{2.13}$$

For

$$|T_3(1)| \leq \begin{cases} M_1(\alpha), & 0 < \alpha < \beta \\ \max\{M_1(\alpha),\ M_2(\alpha)\} = M_1(\alpha), & \beta \leq \alpha < 1. \end{cases}$$

Hence $|T_3(1)| \leq M_1(\alpha)$ for all α, $M_1(\alpha)$ defined by (2.12). Equality in (2.10) holds for the function f_α, that satisfies $f'(z) + \alpha z f''(z) = \dfrac{1+iz^2}{1-iz^2}$. This completes the proof of Theorem 2.3. □

Remark 2.4. Theorem 2.3 for $\alpha = 0$ gives the bound $|T_3(1)| \leq 13/9$ for the class of functions with bounded boundary rotation $\mathcal{R}$ derived by Radhika et.al [15] and in [14] for $\beta = 1$.

Theorem 2.4. *Let f given by* (1.1) *be in the class $\tilde{\mathcal{R}}_\alpha$. Then we have the bound*

$$|T_3(2)| \leq \begin{cases} \dfrac{4(1+5\alpha)}{9(1+\alpha)(1+2\alpha)^2(1+3\alpha)} & \text{for } \alpha \leq \alpha_0 \\[2ex] \dfrac{(1+5\alpha)(11+91\alpha+248\alpha^2+204\alpha^3)}{36(1+\alpha)^3(1+2\alpha)^2(1+3\alpha)^2} & \text{for } \alpha > \alpha_0, \end{cases} \tag{2.14}$$

where $\alpha_0 \approx 0.146157$ is the positive root of the equation $156\alpha^3 + 136\alpha^2 + 11\alpha - 5 = 0$. For $\alpha > \alpha_0$, equality attained by the function $g_\alpha \in \tilde{\mathcal{R}}_\alpha$ defined by (2.6).

Proof. Expanding the determinant $T_3(2)$, we get

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)|.$$

Following the same techniques as in Theorem 2.1 and with the help of (2.5), we get

$$|a_2 - a_4| \leq \frac{1+5\alpha}{2(1+\alpha)(1+3\alpha)}. \tag{2.15}$$

Writing a_2, a_3 and a_4 in terms of c_1, c_2 and c_3 with the help of (2.5), we obtain

$$a_2^2 - 2a_3^2 + a_2 a_4 = \frac{c_1^2}{4(1+\alpha)^2} - \frac{2c_2^2}{9(1+2\alpha)^2} + \frac{c_1 c_3}{8(1+\alpha)(1+3\alpha)}.$$

Expressing c_2 and c_3 in terms of c_1 by using Lemma 1.1, we get

$$\begin{aligned} a_2^2 - 2a_3^2 + a_2 a_4 \quad = \quad & \frac{c_1^2}{4(1+\alpha)^2} - \frac{c_1^4}{18(1+2\alpha)^2} - \frac{(4-c_1^2)^2 x^2}{18(1+2\alpha)^2} \\ & - \frac{(7+28\alpha+12\alpha^2)c_1^2(4-c_1^2)x}{144(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c_1^4}{32(1+\alpha)(1+3\alpha)} \\ & - \frac{c_1^2(4-c_1^2)x^2}{32(1+\alpha)(1+3\alpha)} + \frac{c_1(4-c_1^2)(1-|x|^2)z}{16(1+\alpha)(1+3\alpha)}. \end{aligned}$$

The class $\mathcal{P}$ is invariant under rotations, so we may assume that, $c := c_1 \in [0, 2]$ ([3], see also [[6], Vol. I, page 80, Theorem 3]). Then by using the triangle inequality, we have

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| &\leq \left| \frac{c^2}{4(1+\alpha)^2} - \frac{(7+28\alpha+12\alpha^2)c^4}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \right| \\ &+ \frac{(4-c^2)^2|x|^2}{18(1+2\alpha)^2} + \frac{(7+28\alpha+12\alpha^2)c^2(4-c^2)|x|}{144(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\ &+ \frac{c^2(4-c^2)|x|^2}{32(1+\alpha)(1+3\alpha)} + \frac{c(4-c^2)(1-|x|^2)}{16(1+\alpha)(1+3\alpha)}. \end{aligned}$$

Let $|x| = \mu$, then

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| &\leq \left| \frac{c^2}{4(1+\alpha)^2} - \frac{(7+28\alpha+12\alpha^2)c^4}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \right| \\ &+ \frac{(4-c^2)^2\mu^2}{18(1+2\alpha)^2} + \frac{(7+28\alpha+12\alpha^2)c^2(4-c^2)\mu}{144(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\ &+ \frac{c^2(4-c^2)\mu^2}{32(1+\alpha)(1+3\alpha)} + \frac{c(4-c^2)(1-\mu^2)}{16(1+\alpha)(1+3\alpha)} = F_3(c,\mu).(let) \end{aligned}$$

Differentiating $F_3(c, \mu)$ with respect to μ, we have

$$\frac{\partial F_3}{\partial \mu} = \frac{(4-c^2)^2\mu}{9(1+2\alpha)^2} + \frac{(7+28\alpha+12\alpha^2)c^2(4-c^2)}{144(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c^2(4-c^2)\mu}{16(1+\alpha)(1+3\alpha)} - \frac{c(4-c^2)\mu}{8(1+\alpha)(1+3\alpha)}.$$

We need to find the maximum value of $F_3(c, \mu)$ on $[0, 2] \times [0, 1]$. First assume that there is a maximum at an interior point of $[0, 2] \times [0, 1]$. Then $\frac{\partial F_3}{\partial \mu} = 0$ implies that $c = 2$, which is a contradiction. Thus for the maximum of $F_3(c, \mu)$, we need only to consider the end points of $[0, 2] \times [0, 1]$.

When $c = 0$, $F_2(0, \mu) \leq \dfrac{8}{9(1+2\alpha)^2}$.
When $c = 2$, $F_3(2, \mu) \leq \dfrac{11+91\alpha+248\alpha^2+204\alpha^3}{18(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}$.
When $\mu = 0$, $F_3(c, 0) = \dfrac{c^2}{4(1+\alpha)^2} - \dfrac{(7+28\alpha+12\alpha^2)c^4}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \dfrac{c(4-c^2)}{16(1+\alpha)(1+3\alpha)}$,
which has maximum value $\dfrac{11+91\alpha+248\alpha^2+204\alpha^3}{18(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}$ on $[0, 2]$.
When $\mu = 1$,

$$\begin{aligned} F_3(c,1) &= \frac{c^2}{4(1+\alpha)^2} - \frac{(7+28\alpha+12\alpha^2)c^4}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{(4-c^2)^2}{18(1+2\alpha)^2} \\ &+ \frac{(7+28\alpha+12\alpha^2)c^2(4-c^2)}{144(1+\alpha)(1+2\alpha)^2(1+3\alpha)} + \frac{c^2(4-c^2)}{32(1+\alpha)(1+3\alpha)}. \end{aligned}$$

which has maximum value $\dfrac{8}{9(1+2\alpha)^2}$ on $[0,2]$.
Thus

$$\begin{aligned} |a_2^2-2a_3^2+a_2a_4| &\leq \max\left\{\frac{8}{9(1+2\alpha)^2},\frac{11+91\alpha+248\alpha^2+204\alpha^3}{18(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}\right\} \\ &= \begin{cases} \dfrac{8}{9(1+2\alpha)^2}, & \text{for } \alpha\leq\alpha_0 \\ \dfrac{11+91\alpha+248\alpha^2+204\alpha^3}{18(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}, & \text{for } \alpha>\alpha_0, \end{cases} \end{aligned} \tag{2.16}$$

where $\alpha_0 \approx 0.146157$ is the positive root of the equation $156\alpha^3+136\alpha^2+11\alpha-5=0$.
Now from (2.15) and (2.16) we get the required bound of $|T_3(2)|$ given in (2.14).
Equality holds for the function $g_\alpha(z)$, defined by(2.6) when $\alpha>\alpha_0$.
This completes the Theorem 2.4. □

Remark 2.5. Theorem 2.4 for $\alpha=0$ gives the bound $|T_3(2)|\leq 4/9$ for the class of functions with bounded boundary rotation $\mathcal{R}$ derived by Radhika et.al [15] and in [14] for $\beta=1$. It would be interesting to know the function $f_\alpha\in\tilde{\mathcal{R}}_\alpha$ so that the equality holds in Theorem 2.4 for $\alpha\leq\alpha_0$.

3. Conclusions

1) In this article we found bounds of Toeplitz determinants $T_q(n)$ for $q=2,3:\ n=1,2,3$ whose entries are the coefficients of functions in the class $\tilde{\mathcal{R}}_\alpha$.

2) Our bounds are sharp and generalizes the results in [15] and in [14].

3) Further, one can obtain the sharp bounds of the fourth Toeplitz determinants $T_4(1)$ for this class.

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