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Semi-local convergence of a Newton-like method for solving equations with a singular derivative

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ABSTRACT. We present a semi-local convergence analysis for a Newton-like method to approximate solutions of equations when the derivative is not necessarily non-singular in a Banach space setting. In the special case when the equation is defined on the real line the convergence domain is improved for this method when compared to earlier results. Numerical results where earlier results cannot apply but the new results can apply to solve nonlinear equations are also presented in this study.

1. INTRODUCTION

We are concerned with the problem of approximating a locally unique solution of an equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Newton's method defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

where x_0 is an initial guess is undoubtedly the most popular iterative method for generating a sequence $\{x_n\}$ converging quadratically under certain conditions to x^* [1–15].

In the present paper, we study the semi-local converge analysis of the Newton-like method defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - A_n^{-1}F'(x_n)^{-1}F(x_n), \tag{1.3}$$

where x_0 is an initial point, $A_n = I + \alpha_n F'(x_n)^{-1}F(x_n)$ and $\alpha_n : X \longrightarrow L(X, X)$ is a given sequence chosen in such a way that $\{x_n\}$ is converging to x^*. In the special case when $X = Y = \mathbb{R}$, the results provide a larger convergence domain than in earlier studies such as [8–15]. Moreover, the error bounds on the distances involved are tighter and the information on the location of the solution at least as precise. These advantages hold also for Newton's method, if $\alpha_n = 0$.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence analysis of Newton-like method (1.3). The numerical examples are provided in the concluding Section 3.

2. SEMI-LOCAL CONVERGENCE ANALYSIS

We need an auxiliary result on majorizing sequences for modified Newton method (1.3).

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Lemma 2.1. *Let $L_0 > 0, M > 0, \beta > 0, t_1 > 0, K > 0$ and $\eta_0 > 0$. Suppose that there exists $\alpha \geq 0$ such that*

$$\alpha < \frac{K}{M} \tag{2.4}$$

$$\beta(\alpha\eta_0 + (L_0 + \alpha M)t_1) < 1 \tag{2.5}$$

$$\frac{\beta(Kt_1 + \alpha\eta_0)}{1 - \beta(\alpha\eta_0 + (L_0 + \alpha M)t_1)} \leq \delta := \frac{-K + \sqrt{K^2 + 4\beta(K - \alpha M)(L_0 + \alpha M)}}{2\beta(L_0 + \alpha M)} \tag{2.6}$$

$$g_0(\delta) \leq 0, \tag{2.7}$$

where

$$g_0(t) = (1 - \alpha\beta\eta_0)t^2 + [\beta(L_0 + \alpha M)t_1 - 1]t + \alpha\beta(\eta_0 + Mt_1). \tag{2.8}$$

Then, the scalar sequence $\{t_n\}$ defined by

$$t_0 = 0,\ t_1 \geq 0,\ t_{n+2} = t_{n+1} + \frac{\beta[K(t_{n+1} - t_n) + \alpha(Mt_n + \eta_0)](t_{n+1} - t_n)}{1 - \beta[L_0 t_{n+1} + \alpha(Mt_{n+1} + \eta_0)]} \tag{2.9}$$

is well defined, non-decreasing, bounded from above by

$$t^{**} = \frac{1}{1 - \delta} t_1 \tag{2.10}$$

and converges to each unique least upper bound t^* which satisfies

$$t_1 \leq t^* \leq t^{**}. \tag{2.11}$$

Moreover, the following estimate hold

$$0 \leq t_{n+2} - t_{n+1} \leq \delta(t_{n+1} - t_n) \leq \delta^{n+1}(t_1 - t_0) \tag{2.12}$$

and

$$0 \leq t^* - t_n \leq \frac{\delta^n}{1 - \delta}(t_1 - t_0). \tag{2.13}$$

Proof. We must show using induction that

$$0 \leq \frac{\beta[K(t_{m+1} - t_m) + \alpha(Mt_m + \eta_0)]}{1 - \beta[L_0 t_{m+1} + \alpha(Mt_{m+1} + \eta_0)]} \leq \delta \tag{2.14}$$

and

$$\beta(L_0 t_{m+1} + \alpha M t_{m+1} + \alpha\eta_0) < 1. \tag{2.15}$$

It follows from (2.5) and (2.6) that estimates (2.14) and (2.15) hold, yielding to $t_2 - t_1 \leq \delta(t_1 - t_0)$ and $t_2 \leq \frac{1-\delta^2}{1-\delta}(t_1 - t_0)$ by (2.9). Suppose that (2.14), (2.15)

$$t_{m+1} - t_m \leq \delta^m(t_1 - t_0) \ \text{ and } \ t_{m+1} \leq \frac{1 - \delta^{m+1}}{1 - \delta}(t_1 - t_0) < t^{**} \tag{2.16}$$

hold for all positive integers $m \leq n$. Evidently, (2.14) and (2.15) hold, if

$$\begin{aligned} &\beta[K\delta^m(t_1 - t_0) + \alpha\eta_0 + \alpha M \frac{1 - \delta^m}{1 - \delta}(t_1 - t_0)] \\ &+(\beta(L_0 + \alpha M)\frac{1 - \delta^{m+1}}{1 - \delta}(t_1 - t_0) + \alpha\beta\eta_0)\delta \leq \delta. \end{aligned} \tag{2.17}$$

Estimate (2.17) motivates us to introduce recurrent polynomials f_m defined on the interval $[0, 1)$ by

$$\begin{aligned} f_m(t) \;=\; &\beta[K(t_1 - t_0)t^m + \alpha M(1 + t + t^2 + \cdots + t^{m-1})(t_1 - t_0)] \\ &+\beta(L_0 + \alpha M)(1 + t + t^2 + \cdots + t^m)(t_1 - t_0)t \\ &+\alpha\beta\eta_0 t - t + \beta\alpha\eta_0. \end{aligned} \tag{2.18}$$

We need a relationship between two consecutive polynomials f_m. Using (2.18) we can write

$$f_{m+1}(t) = f_m(t) + g(t)t^m(t_1 - t_0), \tag{2.19}$$

where

$$g(t) = \beta(L_0 + \alpha M)t^2 + Kt + \alpha M - K. \tag{2.20}$$

Notice that $\delta > 0$ by (2.5) and $g(\delta) = 0$ by (2.6) and consequently

$$f_{m+1}(\delta) = f_m(\delta) \tag{2.21}$$

holds. But, we have

$$f_\infty(\delta) = \lim_{m \longrightarrow \infty} f_m(\delta) = f_m(\delta), \tag{2.22}$$

so

$$f_\infty(\delta) = f_m(\delta). \tag{2.23}$$

Then, estimate (2.17) holds, if

$$f_\infty(\delta) \leq 0. \tag{2.24}$$

But, in view of (2.18), we get that

$$\begin{aligned} f_\infty(\delta) &= \frac{\alpha\beta M}{1-\delta}(t_1 - t_0) + \frac{\beta(L_0 + \alpha M)}{1-\delta}(t_1 - t_0)t \\ &\quad + \alpha\beta\eta_0 + \alpha\beta\eta_0\delta - \delta \leq 0, \end{aligned} \tag{2.25}$$

which is true by the definition of δ and (2.1). Hence, the induction for (2.14) and (2.15) is completed. It follows that sequence $\{t_n\}$ is non-decreasingly convergent to t^* and satisfying (2.11). Moreover, estimate (2.13) follows from (2.12) by using standard majorization techniques [3,5,12–15]. □

Next, we present the semi-local convergence result for modified Newton's method given in [8]:

Theorem 2.1. *Let $F : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Suppose that there exist $\beta > 0$, $\eta \geq 0$, $L > 0$, $\alpha \geq 0$, $\{\alpha_n\} \in \mathbb{R}$ such that*

$$|F'(x_0)^{-1}| \leq \beta \tag{2.26}$$

$$|F'(x_0)^{-1}F(x_0)| \leq \eta \tag{2.27}$$

$$|F'(x) - F'(y)| \leq L|x - y| \tag{2.28}$$

$$h = \beta L \eta \leq \frac{1}{2} \tag{2.29}$$

$$|\alpha_n| = |\alpha| \leq \frac{1}{2}\beta L \tag{2.30}$$

and

$$I_0 := [x_0 - 2\eta, x_0 + 2\eta] \subset I. \tag{2.31}$$

Then, the sequence $\{x_n\}$ generated by Newton's method (1.2) is well defined in I_0, remains in I_0 for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^* \in I$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$|x_n - x^*| \leq \left(\frac{1 + 2\alpha\eta}{2 + 2\alpha\eta}\right)(2h)^{2^n - 1}2\eta. \tag{2.32}$$

One of the concerns in [8–15] was that the convergence domain provided in Theorem 2.1 is small in general. In the present study we address this problem by showing that the convergence domain can be extended, if a different approach than the Newton-Kantorovich technique employed in the proof of Theorem 2.1 is used. Indeed, in view of (2.28) we have that there exists $L_0 > 0$ such that

$$|F'(x) - F'(x_0)| \le L_0|x - x_0| \text{ for each } x \in I. \tag{2.33}$$

Set $I_1 = I \cap [x_0 - \frac{1}{\beta L_0}, x_0 + \frac{1}{\beta L_0}]$. Then, there exists $L_1 > 0$ such that

$$|F'(x) - F'(y)| \le L_1|x - y| \text{ for each } x \in I_1. \tag{2.34}$$

In view of (2.28), (2.33) and (2.34) we have that

$$L_0 \le L \tag{2.35}$$

$$L_1 \le L \tag{2.36}$$

hold in general and $\frac{L}{L_0}$ can be arbitrarily large [1–7]. The definition of Lipschitz constant L_1 was not possible in [8] since only (2.28) is used. Moreover, the definition of L_1 depends on (2.33). It is worth noticing that the introduction of (2.33) or (2.34) does not imply additional hypotheses, since in practice the computation of L requires the computation of L_0 or L_1 as special cases. If one follows the proof of Theorem 2.1, then it can be seen that constant L_1 can replace constant L in the proof. Hence, we arrive at the following improvement of Theorem 2.1:

Theorem 2.2. *Let $F : I \longrightarrow \mathbb{R}$ be a differentiable function. Suppose that there exist $\beta > 0$, $\eta \ge 0$, $L_0 > 0$, $L_1 > 0$, $\alpha \ge 0$, $\{\alpha_n\} \in \mathbb{R}$ such that (2.26), (2.27), (2.31), (2.33)*

$$h_1 = \beta L_1 \eta \le \frac{1}{2} \tag{2.37}$$

and

$$|\alpha_n| = |\alpha| \le \frac{1}{2}\beta L_1. \tag{2.38}$$

hold. Then, the sequence $\{x_n\}$ converges to a solution $x^* \in I_1$ of equation $F(x) = 0$. Moreover, the following estimates hold:

$$|x_{n+1} - x_n| \le \left(\frac{1 + 2\alpha\eta}{2 + 2\alpha\eta}\right)(2h_1)^{2^n - 1}2\eta. \tag{2.39}$$

Furthermore, the point x^* is the only solution of equation $F(x) = 0$ in $I_2 := I_1 \cap U(x_0, \frac{2}{\beta L_0})$.

Proof. Simply replace L by L_1 in the proof of Theorem 2.1 and notice that the iterates x_n lie in I_1 which is a more accurate domain than I used in [14] (see also [11–15]). Hence, we arrive at (2.39). Concerning the uniqueness part (not studied in [14]), let $y^* \in I_2$ be such that $F(y^*) = 0$. Set $T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Using (2.27) and (2.33), we get in turn that

$$|F'(x_0)^{-1}||F'(x_0) - T| \le \beta \int_0^1 L_0((1-\theta)|x^* - x_0) + \theta(y^* - x_0|)d\theta \le \frac{\beta L_0}{2} < 1. \tag{2.40}$$

It follows from (2.40) and the Banach Lemma on invertible functions [5, 7, 13] that T is invertible. Then, from the identity $0 = F(y^*) = F(x^*) = T(y^* - x^*)$, we conclude that $y^* = x^*$. □

Remark 2.1. If strict inequality hold in (2.36), then the results of Theorem 2.2 improve the results of Theorem 2.1, since

$$h \le \frac{1}{2} \Longrightarrow h_1 \le \frac{1}{2} \tag{2.41}$$

and the new error bounds (2.39) are more precise than the old ones (2.32), since $h_1 < h$. Concerning the uniqueness, notice that if a uniqueness result was given in [14], then it would have been less accurate, since L would have been used and $L_0 \leq L$.

Next, using Lemma 2.1, we present another semi-local convergence result that is not based on the Newton-Kantorovich theorem, which can apply in cases Theorem 2.1 or Theorem 2.2 cannot apply. We shall present this result in the more general setting of a Banach space setting. Let K denote $\frac{L}{2}$ or $\frac{L_1}{2}$ in the rest of this paper. Let X, Y be Banach spaces, $D \subseteq X$ be an open convex subset of X. Let also $L(X,Y)$ denote the space of bounded linear operators from X into Y. Moreover, let $U(v,\rho), \bar{U}(v,\rho)$ stand for the open and closed balls in X with center $v \in X$ and of radius $\rho > 0$.

Theorem 2.3. *Let $F : D \subset X \longrightarrow Y$ be a Fréchet-differentiable operator. Suppose that there exist $x_0 \in D, \beta > 0, \eta_0 > 0, t_1 \geq 0, L_0 > 0, K > 0, M > 0, \alpha \geq 0, \{\alpha_n\} : D \longrightarrow L(X,X)$ such that*

$$F'(x_0)^{-1} \in L(Y,X),\ [I + \alpha_0 F'(x_0)^{-1}F(x_0)]^{-1} \in L(Y,X),\ \|F'(x_0)^{-1}\| \leq \beta, \tag{2.42}$$

$$\|F'(x_0)\| \leq \eta_0,\ \|\alpha_n\| \leq \alpha, \tag{2.43}$$

$$\|[I + \alpha_0 F'(x_0)^{-1}F(x_0)]^{-1}F'(x_0)^{-1}F(x_0)\| \leq t_1 \tag{2.44}$$

$$\|F'(x) - F'(x_0)\| \leq L_0\|x - x_0\| \text{ for each } x \in D \tag{2.45}$$

$$\|F'(x) - F'(y)\| \leq 2K\|x - y\| \text{ for each } x, y \in D_1 := D \cap U(x_0, \frac{1}{\beta L_0}), \tag{2.46}$$

$$\|F'(x)\| \leq M \text{ for each } x \in D_1 \tag{2.47}$$

and

$$\bar{U}(x_0, t^*) \subseteq D_1, \tag{2.48}$$

where t^* is defined in Lemma 2.1. Then, sequence $\{x_n\}$ generated by modified Newton's method (1.3) is well defined in D_1 remains in D_1 for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^* \in \bar{U}(x_0, t^*)$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{2.49}$$

and

$$\|x_n - x^*\| \leq t^* - t_n, \tag{2.50}$$

where sequence $\{t_n\}$ is given by (2.9). Furthermore, the point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x_0, t^*)$.

Proof. We shall show using mathematical induction that

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{2.51}$$

and

$$\bar{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \bar{U}(x_k, t^* - t_k) \text{ for each } k = 0, 1, 2, \cdots. \tag{2.52}$$

Let $z \in \bar{U}(x_0, t^* - t_0)$. Notice that by (2.44)

$$\|x_1 - x_0\| = \|A_0^{-1}F'(x_0)^{-1}F(x_0)\| \leq t_1 - t_0,$$

since $t_0 = 0$. Hence, estimates (2.51) and (2.52) hold for $k = 0$. Suppose these estimates hold for all $n \leq k$. Then, we get that

$$\begin{aligned} \|x_{n+1} - x_0\| &\leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) \\ &= t_{k+1} - t_0 = t_{k+1} \end{aligned}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leq t_k + \theta(t_{k+1} - t_k) \leq t^*$$

for each $\theta \in [0,1]$. Using (2.45), Lemma 2.1 and the induction hypotheses, we get

$$\begin{aligned} \|F'(x_0)^{-1}\|\|F'(x_{k+1}) - F'(x_0)\| &\leq \beta\|x_{k+1} - x_k\| \leq \beta L_0(t_{k+1} - t_0) \\ &= \beta L_0 t_{k+1} < 1 \text{ by (2.14).} \end{aligned} \tag{2.53}$$

Hence, $F'(x_{k+1})^{-1} \in L(Y, X)$ and

$$\|F'(x_{k+1})^{-1}\| \leq \frac{\beta}{1 - \beta L_0\|x_{k+1} - x_0\|}. \tag{2.54}$$

Next, we shall show $A_{k+1}^{-1} \in L(Y, X)$. We can write

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_0) + F(x_0) \\ &= \int_0^1 F'(x_0 + \theta(x_{k+1} - x_0))(x_{k+1} - x_0)d\theta + F(x_0). \end{aligned} \tag{2.55}$$

Using (2.43), (2.47) and (2.55), we get that

$$\|F(x_{k+1})\| \leq M\|x_{k+1} - x_0\| + \eta_0. \tag{2.56}$$

Then, we have by (2.15), (2.43), (2.54) and (2.56) that

$$\begin{aligned} &\|\alpha_{k+1}\|\|F'(x_{k+1})^{-1}\|\|F(x_{k+1})\| \\ \leq\ & \frac{\alpha\beta[M\|x_{k+1} - x_0\| + \|F(x_0)\|]}{1 - \beta L_0\|x_{k+1} - x_0\|} \\ \leq\ & \frac{\alpha\beta(Mt_{k+1} + \eta_0)}{1 - \beta L_0 t_{k+1}} := \gamma_{k+1} < 1 \end{aligned} \tag{2.57}$$

so

$$\|A_{k+1}^{-1}\| \leq \frac{1}{1 - \gamma_{k+1}}. \tag{2.58}$$

Using modified Newton's method (1.3) we can write in turn that

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(I + \alpha_k F'(x_k)^{-1}F(x_k))(x_{k+1} - x_k) \\ &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) - \alpha_k F(x_k)(x_{k+1} - x_k) \\ &= \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)]d\theta(x_{k+1} - x_k) \\ &\quad -\alpha_k F(x_k)(x_{k+1} - x_k). \end{aligned} \tag{2.59}$$

Then, using (2.43), (2.46), (2.56), (2.59) and the induction hypotheses, we get in turn that

$$\begin{aligned} \|F(x_{k+1})\| &= \frac{L}{2}\|x_{k+1} - x_k\|^2 \\ &\quad \alpha(M\|x_{k+1} - x_0\| + \|F(x_0)\|)\|x_{k+1} - x_k\| \\ &\leq K(t_{k+1} - t_k)^2 + \alpha(Mt_{k+1} + \eta_0)(t_{k+1} - t_k). \end{aligned} \tag{2.60}$$

Then, in view of the modified Newton's method (1.3), (2.9), (2.58) and (2.60), we get that

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &\leq \|A_{k+1}^{-1}\|\|F'(x_{k+1})^{-1}\|\|F(x_{k+1})\| \\ &\leq \frac{\beta[K(t_{k+1} - t_k)^2 + \alpha(Mt_{k+1} + \eta_0)(t_{k+1} - t_k)]}{(1 - \gamma_{k+1})(1 - \beta L_0 t_{k+1})} \\ &= t_{k+2} - t_{k+1}, \end{aligned} \tag{2.61}$$

by (2.9), which completes the induction for (2.51). Moreover, let $w \in \bar{U}(x_{k+1}, t^* - t_{k+2})$. Then, we have that

$$\begin{aligned} \|w - x_{k+1}\| &\leq \|w - x_{k+2}\| + \|x_{k+2} - x_{k+1}\| \\ &\leq t^* - t_{k+2} + t_{k+2} - t_{k+1} = t^* - t_{k+1}, \end{aligned}$$

so, $w \in \bar{U}(x_{k+1}, t^* - t_{k+1})$, which completes the induction for (2.52). Lemma 2.1 implies that sequence $\{x_n\}$ is complete in a Banach space X and as such it converges to some $x^* \in \bar{U}(x_0, t^*)$ (since $\bar{U}(x_0, t^*)$ is a closed set). By letting $k \longrightarrow \infty$ in (2.60), we obtain $F(x^*) = 0$. Estimate (2.50) follows from (2.49) by using standard majorization techniques [3,5,8–15]. The uniqueness part has been shown in Theorem 2.2 with "$\|.\|$" replacing "$|.|$". □

Remark 2.2. (a) The limit point t^* can be replaced by t^{**} given in closed form (by (2.10)) in (2.48).

(b) If (2.46) is replaced by

$$\|F'(x) - F'(y)\| \leq L\|x - y\|\| \text{ for each } x, y \in D \tag{2.62}$$

and K by $\frac{L}{2}$ then, the conclusions of Theorem 2.3 hold with (2.62) replacing (2.46).

3. Numerical examples

We present numerical examples in this section to show that the old convergence criteria in [8–15] are not satisfied but the new convergence criteria are satisfied.

Example 3.1. Let $X = Y = \mathbb{R}, x_0 = 1, I = [x_0 - (1-p), x_0 + (1-p)], p \in (0, 0.5)$ and define function F on I by

$$F(x) = x^3 - p.$$

Then, we have that $\beta = \frac{1}{3}, L_0 = 3(3-p), L = 6(2-p), L_1 = 6(1 + \frac{1}{\beta L_0}), \eta_0 = 1 - p, t_1 = \frac{1-p}{4-p}, \eta = \frac{1}{3}(1-p)$. Choose $p = 0.49$ so we can check the convergence criteria. We have by (2.29)

$$h = \frac{1}{3}6(2-p)\frac{1}{3}(1-p) = 0.513399996 > 0.5.$$

Hence, under the old criteria [14] there is no guarantee that Newton's method (1.2) or modified Newton's method (1.3) starting from $x_0 = 1$ converge to $x^* = \sqrt[3]{p}$. However, our condition (2.37) gives

$$h_1 = 0.475458167 < 0.5.$$

In view of (2.38) we must also choose

$$\alpha \leq \frac{1}{2}\beta L_1 = 1.398406344.$$

Then, Theorem 2.2 guarantees the convergence of Newton's method (1.2) and the modified Newton's method (1.3) to x^*.

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