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On the monotonicity of the sequence of bivariate Bernstein polynomials

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ABSTRACT. The paper has a methodical content and is addressed to young researchers. Its main goal is to prove how the property of monotonicity can be transferred from the sequence of univariate Bernstein polynomials to those of bivariate Bernstein polynomials.

Let $\mathbb{N}$ be the set of positive integers, $m, n \in \mathbb{N}$, $I = [0,1]$, $I^2 = [0,1] \times [0,1]$, $\mathbb{R}^{I^2} = \{f | f : I^2 \to \mathbb{R}\}$, $C(I^2) = \{f \in \mathbb{R}^{I^2} | f \text{ continuous on } I^2\}$. Denote by $B_{m,n} : C(I^2) \to C(I^2)$ the Bernstein bivariate operator. This operator associates to each function $f \in C(I^2)$ the bivariate Bernstein polynomial $B_{m,n}(f; x, y)$. It is well known that the sequence $\{B_{m,n}(f; x, y)\}_{m,n\in\mathbb{N}}$ converges to f, uniformly on I^2 for each $f \in C(I^2)$.

In the present paper one investigates the monotonicity of the sequence $\{B_{m,n}(f; x, y)\}_{m,n\in\mathbb{N}}$. One proves that if $f \in C(I^2)$ is convex of $(1,1)-$order on I^2 the sequence $\{B_{m,n}(f; x, y)\}_{m,n\in\mathbb{N}}$ is monotonous decreasing and $B_{m,n}(f; x, y) \geq f(x, y)$, $(\forall)\ (x, y) \in I^2$.

1. PRELIMINARIES

In this section one recalls some basic results regarding the convex sets in $\mathbb{R}^2$ and the bivariate convex real valued functions.

Definition 1.1. [15] The set $D \subseteq \mathbb{R}^2$ is convex if and only if for any points $A_1(x_1, y_1)$, $A_2(x_2, y_2) \in D$ the segment $[A_1 A_2]$ is included in D, i.e. $(\forall)\ \lambda \in [0,1]$ it follows

$$((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2) \in D. \tag{1.1}$$

Suppose $m, n \in \mathbb{N}$ and $A_{ij}(x_i, y_j) \in D$ for each $i \in \{1, 2, \dots, m\}$, $j \in \{1, 2, \dots, n\}$. By complete induction with respect m and respectively n, can be proved the following

Lemma 1.1. *The set $D \subseteq \mathbb{R}^2$ is convex if and only if for any distinct points $A_{ij}(x_i, y_j) \in D$, any $\alpha_i \in [0,1]$, $\beta_j \in [0,1]$ such that $\sum\limits_{i=1}^{m} \alpha_i = 1$, $\sum\limits_{j=1}^{n} \beta_j = 1$ it follows*

$$\left(\sum_{i=1}^{m} \alpha_i x_i, \sum_{j=1}^{n} \beta_j y_j \right) \in D. \tag{1.2}$$

Remark 1.1. The set $D \subseteq \mathbb{R}^2$ is convex if and only if it is convex with respect each of the coordinates x, y.

Definition 1.2. [15] Let $D \subseteq \mathbb{R}^2$ be a convex set. The function $f \in \mathbb{R}^D$ is convex of $(1,1)$-order on D if and only if for each distinct points $A_{ij}(x_i, y_j) \in D$ $(i = 1, 2; j = 1, 2)$ and any constants $\alpha_1, \alpha_2 \in [0,1]$, $\beta_1, \beta_2 \in [0,1]$ such that $\alpha_1 + \alpha_2 = 1$, $\beta_1 + \beta_2 = 1$, the following

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inequality holds true

$$f\left(\sum_{i=1}^{2}\alpha_i x_i, \sum_{j=1}^{2}\beta_j y_i\right) \leq \sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\beta_j f(x_i, y_j). \tag{1.3}$$

Remark 1.2. Considering only the distinct points $A_{11}(x_1, y_1)$, $A_{22}(x_2, y_2)$ and choosing $\alpha_1 = 1-\alpha$, $\alpha_2 = \alpha$, $\beta_1 = 1-\alpha$, $\beta_2 = \alpha$ (with $\alpha \in [0,1]$) the inequality (1.3) leads to

$$f((1-\alpha)x_1 + \alpha x_2, (1-\alpha)y_1 + \alpha y_2) \leq (1-\alpha)f(x_1, y_1) + \alpha f(x_2, y_2) \tag{1.4}$$

which is the classical definition of the convexity for bivariate real valued functions [15].

If one considers the distinct points $A_{11}(x_1, y_1)$ and $A_{21}(x_2, y_1)$, then with $\alpha_1 = 1-\alpha$, $\alpha_2 = \alpha$, $\beta_1 = 1$, $\beta_2 = 0$, the inequality (1.3) becomes

$$f(1-\alpha)x_1 + \alpha x_2, y_1) \leq (1-\alpha)f(x_1, y_1) + \alpha(x_2, y_1) \tag{1.5}$$

which proves that f is convex of first order with respect x.

Considering the distinct points $A_{11}(x_1, y_1)$, $A_{12}(x_1, y_2)$, with $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1-\alpha$, $\beta_2 = \alpha \in [0,1]$, the inequality (1.3) becomes

$$f(x_1, (1-\alpha)y_1 + \alpha y_2) \leq (1-\alpha)f(x_1, y_1) + \alpha f(x_2, y_2) \tag{1.6}$$

which proves that f is convex of first order with respect y.
We can conclude that if $f \in \mathbb{R}^D$ is convex of $(1,1)$-order on D, then f is convex of first order with respect each on the variables x, y on D.

Reciprocally, if $f \in \mathbb{R}^D$ is convex of first order on D with respect each of the variables x, y, then f is convex of $(1,1)$-order on D.

Taking the above into account, we can state

Lemma 1.2. *Let $D \subseteq \mathbb{R}^2$ be a convex set. The function $f \in \mathbb{R}^D$ is convex of $(1,1)$-order on D if and only if it is convex of first order on D with respect each of the variables x, y.*

In the following one presents the Jensen's inequality for bivariate functions, convex of $(1,1)$-order.

Lemma 1.3. *Let $D \subseteq \mathbb{R}^2$ be a convex set. The function $f \in \mathbb{R}^D$ is convex of $(1,1)$-order on D if and only if for each distinct points $A_{ij}(x_i, y_j) \in D$ $(1 = \overline{1,m}, j = \overline{1,n})$ and each constants $\alpha_i \in [0,1]$, $\beta_j \in [0,1]$ such that $\sum\limits_{i=1}^{m}\alpha_i = 1$, $\sum\limits_{j=1}^{n}\beta_j = 1$, the following inequality holds true*

$$f\left(\sum_{i=1}^{m}\alpha_i x_i, \sum_{j=1}^{n}\beta_j y_j\right) \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j f(x_i, y_j). \tag{1.7}$$

Proof. Let be $n = 1$, $m \in \mathbb{N}$. Because $f \in \mathbb{R}^D$ is convex of $(1,1)$-order on D, by virtue of Lemma 1.1 f is convex of first order on D with respect x. Then for each distinct points $A_{i1}(x_i, y_1) \in D$ $(i = \overline{1,m})$ and each constants $\alpha_i \in [0,1]$ such that $\sum\limits_{i=1}^{m}\alpha_i = 1$, using the Jensen's inequality for univariate convex functions one obtains

$$f\left(\sum_{i=1}^{m}\alpha_i x_i, y_1\right) \leq \sum_{i=1}^{m}\alpha_i f(x_i, y_1). \tag{1.8}$$

But f is also convex of first order with respect y. From (1.8) yields (1.7), applying again the Jensen's inequality for univariate convex functions of first order. $\square$

2. MAIN RESULTS

Let I be the unity interval of real axis, i.e. $I = [0,1]$. The Bernstein operators $B_m : C[0,1] \to C[0,1]$, introduced in [9] are defined by

$$B_m(f;x) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}\right) \tag{2.9}$$

where $p_{m,i}(x) = \binom{m}{i} x^i (1-x)^{m-i}$, $m \in \mathbb{N}$, $i - \overline{0,m}$, $x \in [0,1]$ are the Bernstein fundamental polynomials. The polynomial from the right side of (2.9) is the Bernstein polynomial. Recall that the images of the test functions $e_i(x) = x^i$ $(i = \overline{0,2})$ by the operator B_m are expressed by

$$B_m(1;x) = 1,\ B_m(t;x) = x,\ B_m(t^2;x) = x^2 + \frac{x(1-x)}{m}\,. \tag{2.10}$$

Let I^2 be the unity square and $f \in C(I^2)$. The parametric extensions $B_m^x; B_n^y : C(I^2) \to C(I^2)$ of the operator (2.9) are defined respectively by

$$B_m^x(f;x,y) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}, y\right),\ y \in I; \tag{2.11}$$

$$B_n^y(f;x,y) = \sum_{j=0}^{n} p_{n,j}(y) f\left(x, \frac{j}{n}\right),\ x \in I. \tag{2.12}$$

It is well known [4] that the operators B_m^x, B_n^y commute on $C(I^2)$, their product being the bivariate Bernstein operator $B_{m,n} = B_m^x B_n^y$, defined for any $m, n \in \mathbb{N}$, $f \in C(I^2)$ by

$$B_{m,n}(f;x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{i}{n}\right). \tag{2.13}$$

It is well known [4], [17], [21] that the sequence $\{B_{m,n}(f;x,y)\}_{m,n}$ converges to f, uniformly on I^2 for each $f \in C(I^2)$.

We investigate some monotonicity properties of the sequence of bivariate Bernstein polynomials.
For start, let us to recall two results related to the Bernstein univariate polynomials, which can be found in Agratini's monograph [1]. They were established by Aramă [3].

Theorem 2.1. *If $f \in C(I)$ is convex of first order on $I = [0,1]$, the following inequality holds true*

$$B_m(f;x) \geq f(x),\ (\forall)\, x \in I. \tag{2.14}$$

Theorem 2.2. *If $f \in C(I)$ is convex of first order on $I = [0,1]$ the sequence $\{B_m(f;x)\}_{m \in \mathbb{N}}$ of Bernstein polynomials is monotonous decreasing, i.e.*

$$B_{m+1}(f;x) \leq B_m(f;x),\ (\forall)\, x \in I. \tag{2.15}$$

Next, we shall prove

Theorem 2.3. *Suppose $f \in C(I^2)$ is convex of $(1,1)-$order on I^2. The following inequality holds true*

$$B_{m,n}(f;x,y) \geq f(x,y) \tag{2.16}$$

for each $m, n \in \mathbb{N}$ and each $(x,y) \in I^2$.

Proof. The function $f \in C(I^2)$ being convex of $(1,1)-$order on I^2, we can apply the Jensen's inequality for bivariate convex of $(1,1)-$order functions (the inequality (1.7)), with

$\alpha_i = p_{m,i}(x) \sum\limits_{i=0}^{m} \binom{m}{i} x^i (1-x)^{m-i}$, $x_i = \dfrac{i}{m}\,(i = \overline{0,m})$ and respectively

$\beta_j = p_{n,j}(y) = \sum\limits_{j=0}^{n} \binom{n}{j} y^j (1-y)^{n-j}$, $y_j = \dfrac{j}{n}\,(j = \overline{0,n})$.

Because $\sum\limits_{i=0}^{m} p_{m,i}(x)\dfrac{i}{m} = B_m(t;x) = x$ and $\sum\limits_{j=0}^{n} p_{n,j}(y)\dfrac{j}{n} = B_n(s;y) = y$ we can write

$$
\begin{aligned}
f(x,y) &= f\left(\sum_{i=0}^{m} p_{m,i}(x)\frac{i}{m}, \sum_{j=0}^{n} p_{n,j}(y)\frac{i}{n}\right)\\
&\le \sum_{i=0}^{m}\sum_{k=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right) = B_{m,n}(f;x,y).
\end{aligned}
$$

□

In order to obtain the monotonicity of the sequence $\{B_{m,n}(f;x,y)\}$, we need two results related to Bernstein's univariate polynomials.
The first one can be found in the monograph [1] (Theorem 2.17, p. 83) and it is contained in the following

Theorem 2.4. *The derivatives of the Bernstein univariate operator $B_m(f;x)$ are expressed as*

$$
B_m^{(j)}(f;x) = m(m-1)\dots(m-j+1)\sum_{i=0}^{m-i} p_{m-j,i}(x)\Delta^j_{\frac{1}{m}} f\left(\frac{i}{m}\right), j \le m \tag{2.17}
$$

where

$$
\Delta^i_{\frac{1}{m}} f\left(\frac{i}{m}\right) = f\left(\frac{i+1}{m}\right) - f\left(\frac{i}{m}\right)
$$

and

$$
\Delta^j_{\frac{1}{m}} f\left(\frac{i}{m}\right) = \Delta^{j-1}_{\frac{1}{m}} f\left(\frac{i+1}{m}\right) - \Delta^{j-1}_{\frac{1}{m}} f\left(\frac{i}{m}\right)
$$

is the finite difference of f with starting point $\dfrac{i}{m}$ and step $h = \dfrac{1}{m}$.

Using the above results, we shall prove

Theorem 2.5. *If $f \in C(I)$ is convex of first order on I, then $B_m(f;x)$ is also convex of first order on I.*

Proof. It is sufficient to prove that the second order derivative $B_m^{(2)}(f;x) > 0$, $(\forall)\, x \in I$. From (2.17) yields:

$$
B_m^{(2)}(f;x) = m(m-1)\sum_{i=0}^{m-2} p_{m-j,i}(x)\,\Delta^2_{\frac{1}{m}} f\left(\frac{i}{m}\right). \tag{2.18}
$$

But

$$\begin{aligned}
\Delta^2_{\frac{1}{m}} f\left(\frac{i}{m}\right) &= \Delta^1_{\frac{1}{m}} f\left(\frac{i+1}{m}\right) - \Delta^1_{\frac{1}{m}} f\left(\frac{i}{m}\right) \\
&= f\left(\frac{i+2}{m}\right) - f\left(\frac{i+1}{m}\right) - f\left(\frac{i+1}{m}\right) + f\left(\frac{i}{m}\right) \\
&= \frac{1}{m}\cdot\frac{f\left(\frac{i+2}{m}\right) - f\left(\frac{i+1}{m}\right)}{\frac{1}{m}} - \frac{1}{m}\cdot\frac{f\left(\frac{i+1}{m}\right) - f\left(\frac{i}{m}\right)}{\frac{1}{m}} \\
&= \frac{1}{m}\left(\left[\frac{i+1}{m}, \frac{i+2}{m}; f\right] - \left[\frac{i}{m}, \frac{i+1}{m}; f\right]\right) \\
&= \frac{1}{m}\cdot\frac{2}{m}\left[\frac{i}{m}, \frac{i+1}{m}, \frac{i+2}{m}; f\right] \\
&= \frac{2}{m^2}\left[\frac{i}{m}, \frac{i+1}{m}, \frac{i+2}{m}; f\right],
\end{aligned}$$

where the brackets denotes divided differences. In the above calculus we used the definition of divided difference, the relationship between finite and divided differences and the recurrence formula for divided differences.
Because f is convex of first order on I, it follows [13] that

$$\left[\frac{i}{m}, \frac{i+1}{m}, \frac{i+2}{m}; f\right] > 0.$$

Coming back in (2.17), we get $B^{(2)}_m(f;x) > 0$, $(\forall)\, x \in Int\,(I)$, which proves that $B_m(f;x)$ is a convex function of first order on I. □

Now we can prove

Theorem 2.6. *If $f \in C(I^2)$ is convex of first order on I^2 the sequence $\{B_{m,n}(f;x,y)\}_{m,n\in\mathbb{N}}$ is monotone decreasing on I^2, i.e.*

$$B_{m+1,n+1}(f;x,y) \le B_{m,n}(f;x,y) \tag{2.19}$$

for each $m,n \in \mathbb{N}$ and each $(x,y) \in I^2$.

Proof. Recall that $B_{m,n} = B^x_m B^y_n$. The function $f \in C(I^2)$ being convex of $(1,1)-$th order on I^2, it is convex of first order with respect y (by virtue of Lemma 1.2). Applying then the Theorem 2.2 to the operator B^y_n, we get

$$B^y_{n+1}(f;x,y) \le B^y_n(f;x,y). \tag{2.20}$$

By virtue of Theorem 2.5, $B^y_{n+1}(f;x,y)$ is convex of first order with respect x. From (2.19), via Theorem 2.2, it follows

$$B^x_{m+1}B^y_{n+1}(f;x,y) \le B^x_m B^y_n(f;x,y) \tag{2.21}$$

which is in fact the desired inequality (2.19). □

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