CREAT. MATH. INFORM. 27 (2018), No. 1, 15 - 20
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Comments on some fixed point theorems in metric spaces

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ABSTRACT. In a recent paper [Pata, V., *A fixed point theorem in metric spaces*, J. Fixed Point Theory Appl., **10** (2011), No. 2, 299–305], the author stated and proved a fixed point theorem that is intended to generalize the well known Banach's contraction mapping principle. In this note we show that the main result in the above paper does not hold at least in two extremal cases for the parameter ε involved in the contraction condition used there. We also present some illustrative examples and related results.

1. INTRODUCTION

Let (X, d) be a metric space. By selecting an arbitrary point $x_0 \in X$, which we call the zero of the metric space X, we denote, according to the terminology and notations in [19],

$$\|x\| := d(x, x_0), \ \forall x \in X.$$

We also consider an increasing function $\Psi : [0, 1] \to [0, \infty)$ which is vanishing with continuity at zero and the (vanishing) sequence

$$\omega_n(\alpha) = \left(\frac{\alpha}{n}\right) \sum_{k=1}^{n} \Psi\left(\frac{\alpha}{k}\right), \tag{1.1}$$

where $\alpha \geq 1$.

The following theorem is the main result in [19].

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$ a self mapping of X. Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(f(x), f(y)) \leq (1 - \varepsilon)d(x, y) + \Lambda\varepsilon^{\alpha}\Psi(\varepsilon)\,[1 + \|x\| + \|y\|] \tag{1.2}$$

is satisfied for every $\varepsilon \in [0, 1]$ and every $x, y \in X$, then f possesses a unique fixed point $x^* = f(x^*)$. Furthermore, by denoting the nth iterate of f by f^n, we have the estimate

$$d(x^*, f^n(x_0)) \leq C\omega_n(\alpha), \tag{1.3}$$

for some positive constant $C \leq \Lambda(1 + 4\|x^*\|)\beta$.

Our aim in this note is to show that Theorem 1.1 does not hold at least for two extremal cases of the parameter ε involved in the contraction condition (1.2). We also provide a correct version (but not fully in the spirit) of Theorem 1.1 and discuss some other related results.

Received: 01.12.2016. In revised form: 12.07.2017. Accepted: 12.09.2017
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *metric space, contraction mapping principle, almost contraction, nonexpansive mapping, Picard iteration, Ishikawa iteration, generalized Lipschitzian mapping.*

2. Preliminaries on the fixed point theory of non-expansive mappings

We present in this section some notions and results from the fixed point theory of nonexpansive mappings that will be needed for our discussion.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an *α-contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \; \forall\, x, y \in X. \tag{2.4}$$

A point $x \in X$ is called a *fixed point* of T if $Tx = x$. It is well known, see [20], that, under the strict contraction condition (2.4) in a complete metric space X, there exists a unique fixed point of T and, moreover, the Picard iteration determined by an $x_0 \in X$ and the relation

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \ldots, \tag{2.5}$$

converges to that fixed point. In the case where $\alpha = 1$ in (2.4), the mapping T is said to be *non-expansive*.

As the technique of non-expansive mappings applied to functional differential equations appears to be less used in literature, for the sake of completeness we present in the following some basic concepts and results in the fixed point theory of non-expansive operators, most of them taken from [5].

Let K be a nonempty subset of a real normed linear space E and let $T : K \to K$ be a map. In this setting, T is *non-expansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \; \forall\, x, y \in K. \tag{2.6}$$

Although the non-expansive mappings are generalizations of α-contractions, they do not inherit properties of contractive mappings. More precisely, if K is a nonempty closed subset of a Banach space E and $T : K \to K$ is a non-expansive mapping which is not an α-contraction, then, as is shown by the following example, T may not have fixed points.

Example 2.1. ([11], Example 3.3, pp. 30) In the space $c_0(\mathbb{N})$ the isometry T defined by

$$T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$$

maps the unit ball into its boundary but T has not fixed points.

Moreover, as is shown by the next example, even in the cases where T has a fixed point, the Picard iteration associated to T (i.e., the sequence $\{x_n\}$ defined by (2.5) for an $x_0 \in K$), may fail to converge to a fixed point.

Example 2.2. Let $[0, 1]$ be the unit interval with the usual norm. The function $T : [0, 1] \to [0, 1]$ given by $Tx = 1 - x$, for all $x \in [0, 1]$ has a unique fixed point, $x^* = \frac{1}{2}$ but, except for the trivial case $x_0 = \frac{1}{2}$, the Picard iteration starting from x_0 yields an oscillatory sequence.

For these and many other reasons, some richer geometrical properties of the ambient space E are needed in order to ensure the existence of a fixed point or/and the convergence of an iterative method (generally a more complex iterative method than Picard iteration) to a fixed point of T. For the sake of completeness, let us recall some concepts and results, taken mainly from [5].

One of the most important important fixed point theorems for non-expansive mappings, due to Browder, Göhde and Kirk, see e.g. [4], is stated as follows.

Theorem 2.2. *If K is a nonempty closed convex and bounded subset of a uniformly convex Banach space E then any non-expansive mapping $T : K \to K$ has a fixed point.*

Remark 2.1. Theorem 2.2 provides no information on the approximation of the fixed point of T is given. From Example 2.2, we see that the Picard iteration does not resolve this situation, in general. Due to this fact, several other fixed point iteration procedures have been considered (see [4], [8]). The most usual ones will be defined in the sequel in view of their use.

Let Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a self-mapping. Given an $x_0 \in K$ and a real number $\lambda \in [0, 1]$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n,\ n = 0, 1, 2, \cdots \tag{2.7}$$

is usually called *Krasnoselskij iteration*, or *Krasnoselskij-Mann iteration*. Clearly, (2.7) reduces to Picard iteration (2.5) for $\lambda = 1$.

For an $x_0 \in K$, the sequence $\{x_n\}$ defined by the formula

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n,\ n = 0, 1, 2, \cdots \tag{2.8}$$

where $\{\lambda_n\} \subset [0, 1]$ is a sequence of real numbers satisfying some appropriate conditions, is called *Mann iteration*.

It was shown by Krasnoselskij [15] in the case $\lambda = 1/2$, and latter by Schaefer [24] for $\lambda \in (0, 1)$ arbitrary, that if E is a uniformly convex Banach space and K is a convex and compact subset of E (and therefore, by Theorem 2.2), containing fixed points of T), then the Krasnoselskij iteration converges to a fixed point of T.

Moreover, Edelstein [9] proved that strict convexity of E suffices for the same conclusion. The question of whether or not strict convexity can be removed has been answered in the affirmative by Ishikawa [12] by the following result.

Theorem 2.3. *Let K be a subset of a Banach space E and let $T : K \to K$ be a non-expansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process $\{x_n\}$ given by (2.8) under the following assumptions*
(a) $x_n \in K$ for all positive integers n;
(b) $0 \leq \lambda_n \leq b < 1$ for all positive integers n;
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.
If $\{x_n\}$ is bounded, then $x_n - T x_n \to 0$ as $n \to \infty$.

3. Some comments on Pata's fixed point theorem

First of all, we note that, if in (1.2) we have $\varepsilon = 0$ (or we let $\varepsilon \to 0$), then this condition becomes

$$d(f(x), f(y)) \leq d(x, y),\ \forall x, y \in X, \tag{3.9}$$

which is exactly the non expansiveness condition (2.6) in the case of a metric space.

So, in view of examples above and Remark 2.1, both conclusions of Theorem 1.1 are no more valid in the context of a general metric space.

If in (1.2) we have $\varepsilon = 1$ (or we let $\varepsilon \to 1$), then this condition becomes

$$d(f(x), f(y)) \leq L \cdot [1 + d(x, x_0) + d(y, x_0)]\,,\ \forall x, y \in X, \tag{3.10}$$

for a fixed element $x_0 \in X$ and a constant $L = \Lambda\Psi(1) \geq 0$. Since

$$d(x, y) \leq d(x, x_0) + d(x_0, y) = d(x, x_0) + d(y, x_0),$$

condition (3.10) is implied by the so-called generalized Lipschitzian condition

$$d(f(x), f(y)) \leq L \cdot [1 + d(x, y)]\,,\ \forall x, y \in X, \tag{3.11}$$

introduced by Zhou [27] and used by various authors: [16], [17], [18], [25], [26], [28], [29] etc.

Note that any nonexpansive mapping is Lipschitzian (with Lipschitz constant equal to 1) and, hence, generalized Lipschitzian. So, there exists a natural relationship between the class of nonexpansive mappings, by the one hand, and the class of generalized Lipschitzian mappings, on the other hand.

In [16], the authors obtained convergence theorems for Ishikawa iterative scheme with errors associated to generalized Lipschitzian and m-accretive operators in uniformly smooth real Banach spaces.

It is then quite obvious that only a condition of the form (3.11) cannot ensure the existence of a fixed point of f in a general metric space. The next example illustrates such a case.

Example 3.3. Let $X = \mathbb{R}$ be the real line with the usual norm. The function $f : \mathbb{R} \to \mathbb{R}$ given by $Tx = x + 1$, for all $x \in \mathbb{R}$ satisfies the inequality (3.10) with $L = 1$:

$$d(f(x), f(y)) \leq 1 \cdot [1 + d(x, x_0) + d(y, x_0)], \forall x, y \in X, \tag{3.12}$$

but f is fixed point free. Moreover, the Picard iteration associated to f does not converge.

However, a generalized Lipschitzian mapping may have fixed points provided some additional conditions are also satisfied, as shown by the next example taken from [16].

Example 3.4. Let $X = \mathbb{R}$ be the real line with the usual norm. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x - 1, \ x < -1 \\ x - \sqrt{-x}, \ x \in [-1, 0) \\ x + \sqrt{x}, \ x \in [0, 1] \\ x + 1, \ x \in (1, +\infty). \end{cases} \tag{3.13}$$

Then f is not Lipschitzian, f is generalized Lipschitzian and m-accretive and, in view of Theorem 3.1 in [16], f has a unique fixed point. Moreover, the Picard iteration associated to f does not converge for all initial approximation $x_0 \in X$ but the Ishikawa iteration with errors (see [16]) converges to the fixed point of f.

By summarizing the comments above, we conclude that Theorem 1.1 does not hold if we have $\varepsilon = 0$ or $\varepsilon = 1$ in (1.2).

The same remarks work for the fixed point theorems obtained in [14] in the case of Chatterjea type contraction condition and in [7] for the case of Kannan type contraction condition. Note that the authors of [14] actually excluded the value $\varepsilon = 0$ in the proof of uniqueness in their main result (Theorem 2.1).

In 2004, the first author [3] introduced the concept of weak (almost) contraction and established a general fixed point theorem, i.e., Theorem 3.4 below.

Definition 3.1. ([3]) Let (X, d) be a metric space. A map $T : X \to X$ is called *an almost contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in X. \tag{3.14}$$

In order to be more precise, we shall also call T as *a (δ, L)-almost contraction*.

Theorem 3.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a (δ, L)-almost contraction.*

Then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) *For any $x_0 \in X$, Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots; \, i = 1, 2, \ldots \tag{3.15}$$

If we relate the contraction conditions involved in Theorems 1.1 and 3.4, we obtain the following interesting existence result.

Theorem 3.5. *Let (X, d) be a complete metric space and $f : X \to X$ a self mapping of X. Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(f(x), f(y)) \leq (1-\varepsilon)d(x, y) + \Lambda\varepsilon^\alpha \Psi(\varepsilon) d(y, f(x)) \tag{3.16}$$

is satisfied for some $\varepsilon \in (0, 1)$ and every $x, y \in X$, then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) *For any $x_0 \in X$, Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{(1-\varepsilon)^i}{\varepsilon}\, d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots; \, i = 1, 2, \ldots \tag{3.17}$$

Proof. We denote $L = \Lambda\varepsilon^\alpha\Psi(\varepsilon) \geq 0$ and apply Theorem 2.1 in [3]. □

At the end of this note, let us also mention the fact that the contraction condition (1.2) used in Theorem 1.1 is similar to the contraction conditions that appear in the theory of best proximity points. The first paper that tackled this topic, due to Eldred and Veeramani [10], introduced and studied the following concept.

Let (X, d) be a complete metric space, A, B nonempty subsets of X, and $T : A \cup B \to X$ such that $T(A) \subset B$ and $T(B) \subset A$. The mapping T is called a cyclic contraction if there exists $0 < k < 1$, such that

$$d(Tx, Ty) \leq kd(x, y) + (1-k)dist(A, B), \tag{3.18}$$

for all $x \in A, y \in B$, where $dist(A, B)$ denotes a sort of "distance" between the sets A and B, defined by

$$dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

A point $\overline{x} \in A \cup B$ such that $d(\overline{x}, T\overline{x}) = dist(A, B)$ is called a *best proximity point* of T. The similarity mainly comes from the fact that in both contraction conditions we have a sort of convex combination (in (3.18), a true convex combination) and that $d(x, x_0) + d(y, x_0)$ could be viewed as the distance in X^2 between the points (x, x_0) and (y, x_0).

Acknowledgments The first author would like to acknowledge the support provided by the Deanship of Scientific Research at King Fahd University of Petroleum and Minerals for funding this work through the project IN151014. It is the custom that a corrigendum paper is accepted by the journal in which the corrected paper has been originally published. However, the journal that published [19] did not accept this corrigendum paper.

References

[1] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19** (2003), No. 1, 7–22

[2] Berinde, V., *Approximating fixed points of weak ϕ-contractions using the Picard iteration*, Fixed Point Theory, **4** (2003), No. 2, 131–142

[3] Berinde, V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9** (2004), No. 1, 43–53

[4] Berinde, V., *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[5] Berinde, V., *Existence and approximation of solutions of some first order iterative differential equations*, Miskolc Math. Notes, **11** (2010), No. 1, 13–26

[6] Berinde, V. and Păcurar, M., *Iterative Approximation of Fixed Points of Single-Valued Almost Contractions*, in Fixed Point Theory and Graph Theory, 1st Edition, Foundations and Integrative Approaches, M. R. Alfuraidan and Q. H. Ansari (Eds), Academic Press, 2016; pp. 29-98
[7] Chakraborty, M. and Samanta, S. K., *A fixed point theorem for Kannan-type maps in metric spaces*, 7331v2 [math. GN] 16 December 2012, arXiv:1211
[8] Chidume, C. E., *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Springer Verlag, Berlin Heidelberg New York, 2009
[9] Edelstein, M., *A remark on a theorem of M. A. Krasnoselski*, Amer. Math. Monthly, **73** (1966), 509–510
[10] Eldred, A. A. and Veeramani, P., *Existence and convergence of best proximity points*. J. Math. Anal. Appl., **323** (2006), No. 2, 1001–1006
[11] Goebel, K. and Kirk, W. A., *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990
[12] Ishikawa, S., *Fixed points and iteration of a non-expansive mapping in a Banach space*, Proc. Amer. Math. Soc., **59** (1976), No. 1, 65–71
[13] Kadelburg, Z. and Radenović, S., *A note on Pata-type cyclic contractions*, Sarajevo J. Math., **11 (24)** (2015), No. 2, 235–245
[14] Kadelburg, Z. and Radenović, S., *Fixed point theorems under Pata-type conditions in metric spaces*, J. Egyptian Math. Soc., **24** (2016), 77–82
[15] Krasnoselskij, M. A., *Two remarks on the method of successive approximations* (Russian), Uspehi Mat. Nauk., **10** (1955), No. 1 (63), 123–127
[16] Liu, Z., Kang, S. M., Cho, Y. J., *Convergence and almost stability of Ishikawa iterative scheme with errors for m-accretive operators*, Comput. Math. Appl., **47** (2004), No. 4-5, 767–778
[17] Ni, R. X., *Ishikawa iteration procedures with errors for certain generalized Lipschitzian nonlinear operators* (Chinese), Acta Math. Sinica (Chin. Ser.), **44** (2001), No. 4, 701–712
[18] Ni, R. X., *Some convergence theorems for iterative sequence of certain generalized Lipschitzian nonlinear operators in Banach spaces* (Chinese), Numer. Math. J. Chinese Univ., **24** (2002), No. 1, 87–96
[19] Pata, V., *A fixed point theorem in metric spaces*, J. Fixed Point Theory Appl., **10** (2011), No. 2, 299–305
[20] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001
[21] Rus, I. A., *A class of nonlinear functional-integral equations, via weakly Picard operators*, Anal. Univ. Craiova, ser. Mat-Inf., **28** (2001), 10–15
[22] Rus, I. A., *Picard operators and applications*, Sci. Math. Japon., **58** (2003), 191–219
[23] Rus, I. A. and Egri, E., *Boundary value problems for iterative functional-differential equations*, Stud. Univ. Babes,-Bolyai Math., **51** (2006), No. 2, 109–126
[24] Schaefer, H., *Uber die Methode sukzessiver Approximationen*, Jahresber. Deutsch. Math. Verein., **59** (1957), 131–140
[25] Xue, Z. and Wang, Z. , *Ishikawa iterative sequence for the generalized Lipschitzian and Φ-strongly accretive mappings in Banach spaces*, Kragujevac J. Math., **29** (2006), 203–213
[26] Xue, Z., *On the convergence between the Mann iteration and Ishikawa iteration for the generalized Lipschitzian and Φ-strongly pseudocontractive mappings*, Bull. Korean Math. Soc., **45** (2008), No. 4, 635–644
[27] Zhou, H. Y. and Chen, D. Q., *Iterative processes for certain nonlinear mappings in uniformly smooth Banach spaces* (Chinese) Math. Appl. (Wuhan), **11** (1998), No. 4, 7073
[28] Zhou, H. Y., Cho, Y. J. and Kang, S. M., *Approximating the zeros of accretive operators by the Ishikawa iterative scheme with mixed errors*, Comm. Appl. Nonlinear Anal., **8** (2001), No. 3, 27–35
[29] Zhou, H. Y., Chang, S. S., Agarwal, R. P. and Cho, Y. J., *Stability results for the Ishikawa iteration procedures*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **9** (2002), No. 4, 477–486

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