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Tripled coincidence point theorems for mixed g-R-monotone operators in metric spaces endowed with a reflexive relation

MELÁNIA-IULIA DOBRICAN

ABSTRACT. In this paper we present some results regarding tripled coincidence points of mixed $g-R-$monotone operators in the framework of metric spaces endowed with a reflexive relation. Our results extend and generalize some famous results obtained by Berinde , Borcut, Ćirić and Lakshmikantham.

1. INTRODUCTION

Tripled fixed points were first introduced by Berinde and Borcut in [4], in 2011, for the case of mixed-monotone operators, then extended for monotone operators in [6] in partially ordered metric spaces. Not much later, tripled coincidence points were introduced by Borcut in [7] and [8], extending the results of Berinde, Ćirić and Lakshmikantham. Inspired by the remarks of Samet and Turinici in [11], Ben-el-Mechaiekh in [5], Asgari and Mousavi in [1], who emphasize the fact that not all of the properties of the partial order relation are being used in the proofs of some famous results in the field, our purpose is to extend and generalize the results of Borcut regarding mixed-monotone operators, in the case of metric spaces endowed with a reflexive relation, based on the results obtained for coupled coincidence points in [10]. To prove the utility of the theorems presented in the next section, we will also provide an illustrative example. First, we will recall some of the most important results which lead to the ones we obtained in this paper.

Definition 1.1. [7] Let $(X, \leq)$ a partially ordered space , the operator $F : X \times X \times X \to X$ and the mapping $g : X \to X$. We say that F is mixed-g-monotone if $F(x, y, z)$ is g-monotone increasing in x and z and it is g-monotone decreasing in y, i.e., for any $x, y, z \in X$, we have

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, g(z_2) \leq g(z_1) \Rightarrow F(x, y, z_2) \leq F(x, y, z_1).$$

Definition 1.2. [9] Let X be a nonempty set and let $f : X^2 \to X$ and $g : X \to X$. We say that f and g commute if $g(f(x, y)) = f(g(x), g(y))$.

Definition 1.3. [8] An element $(x, y) \in X \times X$ is called tripled coincidence point for the mixed-g-monotone operator $F : X \times X \times x \to X$ and $g : X \to X$ if

$$F(x, y, z) = g(x), F(y, x, y) = g(y) \text{ and } F(z, y, x) = g(z).$$

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Theorem 1.1. [7] *Let $(X, \leq)$ a partially ordered space and let d be a metric on X such that (X, d) is a complete metric space. Let$g : X \to X$ and $F : X \times X \times X \to X$ be a mixed-g-monotone mapping.*
Suppose there exist $j, k, l \in [0, 1), j + k + l < 1$, such that

$$d(F(x,y,z), F(u,v,w)) \leq j \cdot d(g(x), g(u)) + k \cdot d(g(y), g(v)) + l \cdot d(g(z), g(w)) \tag{1.1}$$

for all $x, y, z, u, v, w \in X$ with $g(x) \leq g(u), g(y) \geq g(v)$ and $g(z) \leq g(w)$.

We suppose that $F(X \times X) \subseteq g(X)$, g is continuous and it commutes with F and also suppose either:

(1) F is continuous or
(2) X has the following properties:
- *if there exists an increasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n;*
- *if there exists a decreasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0, z_0),\ g(y_0) \geq F(y_0, x_0, y_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $\overline{x}, \overline{y} \in X$ such that
$g(\overline{x}) = F(\overline{x}, \overline{y}, \overline{z})$ and $g(\overline{y}) = F(\overline{y}, \overline{x}, \overline{y})$ and $g(\overline{z}) = F(\overline{z}, \overline{y}, \overline{x})$, that is, F and g have a coupled coincidence point.

Theorem 1.2. [7] *In addition to the hypothesis of Theorem 1.1, suppose that for every (x, y, z), $(x^*, y^*, z^*) \in X \times X$ there exists $(u, v, w) \in X \times X \times X$, such that $(F(u, v, w), F(v, u, w), F(w, v, u))$ is comparable to $(g(x), g(y), g(z))$ and to $(g(x^*), g(y^*), g(z^*))$. Then F and g have a unique coincidence point, i.e., there exists a unique point $(x, y) \in X \times X$, such that*

$$x = g(x) = F(x, y, z),\ y = g(y) = F(y, x, y) \text{ and } z = g(z) = F(z, y, x).$$

The author also presents many variations of this result, based on this last theorem, by replacing the contractive condition by weaker ones, using one constant instead of three.(see [8]).
Another important result is provided in [2] by Aydi, Karapinar and Postolache, in the case of mixed$-g-$monotone operators. The improvement they brought to the results of Borcut is the symmetrization of the contractive condition, following the idea of Berinde in [3].

Theorem 1.3. [2] *Let $(X, \leq)$ a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Let$g : X \to X$ and $F : X \times X \times X \to X$ be a mixed-g-monotone mapping Suppose there exist $\varphi \in \Phi$, such that*

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) + d(F(y,x,y), F(v,u,v)) + d(F(z,y,x), F(w,v,u)) \\ \leq 3 \cdot \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(w))}{3}\right) \end{aligned} \tag{1.2}$$

for all $x, y, z, u, v, w \in X$ with $g(x) \leq g(u), g(y) \geq g(v)$ and $g(z) \leq g(w)$.

We suppose that $F(X \times X \times X) \subset g(X)$, g is continuous and it commutes with F and F is continuous;

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0, z_0),\ g(y_0) \geq F(y_0, x_0, y_0) \text{ and } g(z_0) \leq F(z_0, y_0, x_0)$$

then there exist $\overline{x}, \overline{y} \in X$ such that
$g(\overline{x}) = F(\overline{x}, \overline{y}, \overline{z})$ and $g(\overline{y}) = F(\overline{y}, \overline{x}, \overline{y})$ and $g(\overline{z}) = F(\overline{z}, \overline{y}, \overline{x})$, that is F and g have a coupled coincidence point.

Further on, we will recall some of the results regarding coupled fixed points and coupled coincidence points in metric spaces endowed with a reflexive relation:

Definition 1.4. [12] Let A and B be two sets. An ordered triple $r = (A, B, R)$ is called a binary relation, where R is a subset of the cartesian product $A \times B$. The set A is called the domain of the relation and B, the codomain of the relation.
If $r = (A, B, R)$ is a relation, we say that $x \in A$ is related to $y \in B$ by R, i.e. $(x, y) \in R$, also written as xRy.

Definition 1.5. [1] *Let X be a nonempty set and let R be a reflexive relation on X, $f : X^2 \to X$. An element $(x, y) \in X^2$ is called $R-$coupled fixed point of f, if $f \times f(x, y) \in X_R(x, y)$, where $X_R(x, y) = \{(z, t) \in X^2 : zRx \wedge yRt\}$, $\forall (x, y) \in X^2$.*

Definition 1.6. [10] Let X be a nonempty set and let R be a reflexive relation on X, $f : X^2 \to X$, $g : X \to X$. An element $(x, y) \in X^2$ is called **lower-$R-$coupled coincidence point** for f and g, if $(f \times g)(x, y) \in X_R(x, y)$.

Definition 1.7. [1] *Let X be a nonempty set and let R be a reflexive relation on X, $f : X^2 \to X$. The mapping f has the mixed $R-$monotone property on X if $(f \times f)(X_R(x, y)) \subseteq X_R(f \times f(x, y))$, for all $(x, y) \in X^2$.*

Definition 1.8. [10] Let X be a nonempty set and let R be a reflexive relation on X, $f : X^2 \to X$, $g : X \to X$. The mapping f has the **mixed $g - R-$monotone property** on X if $(f \times g)(X_R(x, y)) \subseteq X_R((f \times g)(x, y))$, for all $(x, y) \in X^2$.

Definition 1.9. [1] A sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subseteq X^2$ is called $R-$monotone sequence if $(x_n, y_n) \in X_R(x_{n-1}, y_{n-1})$ for all $n \in \mathbb{N}$.

Definition 1.10. [1] Let X be a topological space and let $f : X^2 \to X$ be a mapping. The mapping f is called **orbitally continuous** if $(x, y), (a, b) \in X \times X$ and $f^{n_k}(x, y) \to a, f^{n_k}(y, x) \to b$, when $k \to \infty$, implies $f^{n_k+1}(x, y) \to f(a, b)$ and $f^{n_k+1}(y, x) \to f(b, a)$, when $k \to \infty$.

Definition 1.11. [10] The mapping f is called **orbitally g-continuous** if $(x, y), (a, b) \in X^2$ and $f^{n_k}(x, y) \to a, f^{n_k}(y, x) \to b$, when $k \to \infty$, implies $f^{n_k+1}(x, y) \to g(a)$ and $f^{n_k+1}(y, x) \to g(b)$ when $k \to \infty$.

2. Main results

First, we will extend the mixed $g - R$ monotone property of a mapping, presented in [10] in the case of mapping defined on a metric space endowed with a reflexive relation:

Definition 2.12. Let X be a nonempty set and let R be a reflexive relation on X, $f : X \times X \times X \to X$, $g : X \to X$. The mapping f has the **mixed $g - R-$monotone property** on X if $(f \times g)(X_R(x, y, z)) \subseteq X_R((f \times g)(x, y, z))$, for all $(x, y, z) \in X \times X \times X$, where $X_R(t, u, v) = \{(x, y, z) \in X^3 : xRt \wedge uRy \wedge zRv\}$.

The definition for lower-R-tripled coincidence points is the following:

Definition 2.13. An element $(x, y, z) \in X \times X \times X$ is called **lower-$R-$tripled coincidence point** for f and g, if $(f \times g)(x, y, z) \in X_R(x, y, z)$.

Next, starting from the orbital continuity presented in [1], we will define the orbital g-continuity of a mapping f.

Definition 2.14. The mapping f is called **orbitally g-continuous** if $(x, y, z), (a, b, c) \in X \times X \times X$ and $f^{n_k}(x, y, z) \to a, f^{n_k}(y, x, y) \to b$, $f^{n_k}(z, y, x) \to c$ when $k \to \infty$, implies $f^{n_k+1}(x, y, z) \to g(a)$ and $f^{n_k+1}(y, x, y) \to g(b)$ and $f^{n_k+1}(z, y, x) \to g(c)$ when $k \to \infty$.

Our first result follows the general idea in [7] and [9], extending Theorem 1.1 in the framework of metric spaces endowed with a reflexive relation:

Theorem 2.4. *Let (X,d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(1) f is mixed $g - R$-monotone;
(2) f is orbitally g-continuous;
(3) there exist $k, l, m \in [0,1), k+l+m < 1$ such that

$$d(f(x,y,z), f(t,u,v)) \le k \cdot d(g(x), g(t)) + l \cdot d(g(y), g(u)) + m \cdot d(g(z), g(v)), \tag{2.3}$$
$$\forall (x,y,z) \in X_R(t,u,v);$$

(4) f and g have a lower-R-tripled coincidence point;
(5) $f(X^3) \subseteq g(X)$;
(6) g is continuous ;
(7) f and g commute.

Then f and g have a tripled coincidence points, i.e., there exists $(x,y,z) \in X^3$ such that $f(x,y,z) = g(x)$, $f(y,x,y) = g(y)$ and $f(z,y,x) = g(z)$.

Proof. Since f and g have a lower-R-tripled coincidence point, let (x_0, y_0, z_0) be it. Thus, $(f \times g)(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.
From (i) we have that $(f \times g)(X_R(x_0, y_0, z_0)) \subseteq X_R((f \times g)(x_0, y_0, z_0))$.
Further, it can easily be checked that

$$(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(z_0,y_0,x_0))$$
$$\in X_R(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0))g^{n-1}(f(z_0,y_0,x_0))).$$

Since $f(X^3) \subseteq g(X)$, let $x_1, y_1, z_1 \in X$ such that $g(x_1) = f(x_0,y_0,z_0)$, $g(y_1) = f(y_0,x_0,y_0)$, $g(z_1) = f(z_0,y_0,x_0)$ and so on. Step by step, we obtain the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$g(x_{n+1}) = f(x_n,y_n,z_n), g(y_{n+1}) = f(y_n,x_n,y_n), g(z_{n+1}) = f(z_n,y_n,x_n). \tag{2.4}$$

Now, using (iii), we have that

$$d(f(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(z_0,y_0,x_0)),$$
$$f(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0)), g^{n-1}(f(z_0,y_0,x_0))))$$
$$\le k^n \cdot d(g(g^n(f(x_0,y_0,z_0))), g(g^{n-1}(f(x_0,y_0,z_0))))+$$
$$l^n \cdot d(g(g^n(f(y_0,x_0,y_0))), g(g^{n-1}(f(y_0,x_0,y_0))))$$
$$+m^n \cdot d(g(g^n(f(z_0,y_0,x_0))), g(g^{n-1}(f(z_0,y_0,x_0))))$$
$$\Leftrightarrow d(f(g^n(f(x_0,y_0,z_0)), g^n(f(y_0,x_0,y_0)), g^n(f(z_0,y_0,x_0))),$$
$$f(g^{n-1}(f(x_0,y_0,z_0)), g^{n-1}(f(y_0,x_0,y_0)), g^n(f(z_0,y_0,x_0))))$$
$$\le k^n \cdot d(g^{n+1}(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)), g^n(f(x_0,y_0,z_0)))+$$
$$l^n \cdot d(g^{n+1}(f(y_0,x_0,y_0)), g^n(f(y_0,x_0,y_0)), g^n(f(y_0,x_0,y_0)))$$
$$+m^n \cdot d(g^{n+1}(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)), g^n(f(z_0,y_0,x_0)))$$
$$\Leftrightarrow d(f(g^n(g(x_1)), g^n(g(y_1)), g^n(g(z_1))), f(g^{n-1}(g(x_1)), g^{n-1}(g(y_1)), g^{n-1}(g(z_1))))$$
$$\le k^n \cdot d(g^{n+1}(g(x_1)), g^n(g(x_1)))+l^n \cdot d(g^{n+1}(g(y_1)), g^n(g(y_1)))+m^n d(g^{n+1}(g(z_1)), g^n(g(z_1)))$$
$$\Leftrightarrow d(f(g^{n+1}(x_1), g^{n+1}(y_1), g^{n+1}(z_1)), f(g^n(x_1), g^n(y_1)g^n(z_1)))$$
$$\le k^n \cdot d(g^{n+2}(x_1), g^{n+1}(x_1)) + l^n \cdot d(g^{n+2}(y_1), g^{n+1}(y_1)) + m^n \cdot d(g^{n+2}(z_1), g^{n+1}(z_1))$$

This implies that $\{g^n(x_1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Now, because X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x,\ \lim_{n\to\infty} g(y_n) = y,\ \lim_{n\to\infty} g(z_n) = z. \tag{2.5}$$

From the continuity of g, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(x),\ \lim_{n\to\infty} g(g(y_n)) = g(y),\ \lim_{n\to\infty} g(g(z_n)) = g(z).$$

Because f and g commute, and from (2.4), we have

$$g(g(x_{n+1})) = g(f(x_n, y_n, z_n)) = f(g(x_n), g(y_n), g(z_n)),$$
$$g(g(y_{n+1})) = g(f(y_n, x_n, y_n)) = f(g(y_n), g(x_n), g(y_n)$$

and

$$g(g(z_{n+1})) = g(f(z_n, y_n, x_n)) = f(g(z_n), g(y_n), g(x_n)$$

From (2.5) and the orbital continuity of f we get

$$g(x) = f(x, y, z), \quad g(y) = f(y, x, y) \quad \text{and} \quad g(z) = f(z, y, x).$$

□

Corollary 2.1. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(1) f is mixed $g - R$-monotone;
(2) f is orbitally g-continuous;
(3) there exist $\alpha \in [0, 1)$ such that

$$d(f(x, y, z), f(t, u, v)) \leq \frac{\alpha}{3} \cdot [d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(t))], \tag{2.6}$$
$$\forall (x, y, z) \in X_R(t, u, v);$$

(4) f and g have a lower-R-tripled coincidence point;
(5) $f(X^3) \subseteq g(X)$;
(6) g is continuous ;
(7) f and g commute.

Then f and g have a tripled coincidence point, i.e., there exists $(x, y, z) \in X^3$ such that $f(x, y, z) = g(x)$, $f(y, x, y) = g(y)$ and $f(z, y, x) = g(z)$.

Proof. From the proof of Theorem 2.4, for $k = l = m = \dfrac{\alpha}{3}, \alpha \in [0, 1)$, there exist $x, y, z \in X$ such that

$$g(x) = f(x, y, z), \quad g(y) = f(y, x, y) \quad \text{and} \quad g(z) = f(z, y, x).$$

□

Next, let's recall the definition of a mapping φ introduced in [9] by Ćirić and Lakshmikantham: Let $\varphi : [0, \infty) \to [0, \infty)$ satisfying :

(1) $\varphi(t) < t, \forall t \in (0, \infty)$;
(2) $\lim_{r\to t_+} \varphi(r) < t, \forall t \in (0, \infty)$;

The set of all these mappings φ is denoted by Φ.

The following result is obtained by replacing the contraction (2.3) with one that uses the mapping φ defined above, following the idea in [9]. Thus, we obtain :

Theorem 2.5. *Let (X, d) be a complete metric space, R be a binary reflexive relation on X such that R and d are compatible. If $f : X^3 \to X$ and $g : X \to X$ are two mappings such that*

(1) f is mixed $g - R$-monotone;

(2) f *is orbitally* g*-continuous;*
(3)
$$d(f(x,y,z),f(t,u,v)) \leq \varphi\left(\frac{d(g(x),g(t))+d(g(y),g(u))+d(g(z),g(v))}{3}\right), \tag{2.7}$$
$$\forall (x,y,z) \in X_R(t,u,v);$$
(4) f *and* g *have a lower-*R*-tripled coincidence point;*
(5) $f(X^3) \subseteq g(X)$*;*
(6) g *is continuous ;*
(7) f *and* g *commute.*

Then f *and* g *have a tripled coincidence point, i.e., there exists* $(x,y,z) \in X^3$ *such that* $f(x,y,z) = g(x)$*,* $f(y,x,y) = g(y)$ *and* $f(z,y,x) = g(z)$*.*

Proof. From the hypothesis, we know that f and g have a lower-R-triple coincidence point; let (x_0,y_0,z_0) be it. Thus, using the definition of the lower-R-tripled coincidence point, it follows that $(f \times g)(x_0,y_0,z_0) \in X_R(x_0,y_0,z_0)$.
From (i) we know that $(f \times g)(X_R(x_0,y_0,z_0)) \subseteq X_R((f \times g)(x_0,y_0,z_0))$.
Further, it can easily be checked that
$$(g^n(f(x_0,y_0,z_0)),g^n(f(y_0,x_0,y_0)),g^n(f(z_0,y_0,x_0)))$$
$$\in X_R(g^{n-1}(f(x_0,y_0,z_0)),g^{n-1}(f(y_0,x_0,y_0)),g^{n-1}(f(z_0,y_0,x_0))).$$

Since $f(X^3) \subseteq g(X)$, let $x_1,y_1,z_1 \in X$ such that $g(x_1) = f(x_0,y_0,z_0)$, $g(y_1) = f(y_0,x_0,y_0)$, $g(z_0) = f(z_0,y_0,x_0)$ and so on. Thus, we obtain the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ such that
$$g(x_{n+1}) = f(x_n,y_n,z_n), g(y_{n+1}) = f(y_n,x_n,y_n) \text{ and } g(z_{n+1} = f(z_n,y_n,x_n). \tag{2.8}$$

Let's consider the nonnegative sequence $\{\eta_n\}_{n\in\mathbb{N}^*}$ such that $\eta_n = d(g(x_{n+1}),g(x_n)) + d(g(y_{n+1}),g(y_n)) + d(g(z_{n+1},g(z_n)), n \in \mathbb{N}^*$.
Now, using (2.7), (2) and letting $x := x_n$, $y := y_n$ and $z := z_n$, $t := x_{n-1}$, $u := y_{n-1}$ and $v := z_{n-1}$, we obtain
$$d(g(x_{n+1}),g(x_n)) = d(f(x_n,y_n,z_n),f(x_{n-1},y_{n-1},z_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))+d(g(z_n),g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right),$$
$$d(g(y_{n+1}),g(y_n)) = d(f(y_n,x_n,y_n),f(y_{n-1},x_{n-1},y_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))+d(g(z_n),g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$
and
$$d(g(z_{n+1}),g(z_n)) = d(f(z_n,y_n,x_n),f(z_{n-1},y_{n-1},x_{n-1})) \leq$$
$$\varphi\left(\frac{d(g(x_n),g(x_{n-1}))+d(g(y_n),g(y_{n-1}))+d(g(z_n),g(z_{n-1}))}{3}\right) = \varphi\left(\frac{\eta_{n-1}}{3}\right).$$
By summing up the last three relations, we obtain that
$$d(g(x_{n+1}),g(x_n)) + d(g(y_{n+1}),g(y_n)) + d(g(z_{n+1}),g(z_n)) = \eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right).$$

Now, using the properties of φ, we have that

$$\eta_n \leq 3 \cdot \varphi\left(\frac{\eta_{n-1}}{3}\right) < 3 \cdot \frac{\eta_{n-1}}{3} = \eta_{n-1}. \tag{2.9}$$

Thus, $\{\eta_n\}_{n\in\mathbb{N}^*}$ is a decreasing and positive sequence. Therefore, there exists $\varepsilon_0 \geq 0$ such that

$$\lim_{n\to\infty} \eta_n = \varepsilon_0.$$

Now, we will prove that $\varepsilon_0 = 0$. In (2.9), let $n \to \infty$. Using (1), we have

$$\varepsilon_0 = \lim_{n\to\infty} \eta_n \leq 3 \cdot \lim_{n\to\infty} \varphi\left(\frac{\eta_{n-1}}{3}\right) = 3 \cdot \lim_{\eta_n\to\varepsilon_0+} \varphi\left(\frac{\eta_{n-1}}{3}\right) < \varepsilon_0,$$

which is a contradiction. Thus, $\lim\limits_{n\to\infty} \eta_n = 0$ and, consequently, $\lim\limits_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0$, $\lim\limits_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0$ and $\lim\limits_{n\to\infty} d(g(z_{n+1}), g(z_n)) = 0$.

Next, we will prove that $\{g(x_n)\}_{n\in\mathbb{N}}$, $\{g(y_n)\}_{n\in\mathbb{N}}$ and $\{g(z_n)\}_{n\in\mathbb{N}}$ are Cauchy sequences. Suppose that at least one of them is not a Cauchy sequence. Then, there exists a constant $\delta > 0$ and two integer sequences $\{n_1(k)\}$ and $\{n_2(k)\}$, such that

$$s_k := d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \geq \delta, \tag{2.10}$$

where $n_1(k) > n_2(k) \geq k, k \in \mathbb{Z}^*$. We chose $n_1(k)$ to be the smallest integer satisfying $n_1(k) > n_2(k) \geq k$ and (2.10). Then, we have

$$d(g(x_{n_2(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_2(k)}), g(y_{n_1(k)-1})) + d(g(z_{n_2(k)}), g(z_{n_1(k)-1}) < \delta. \tag{2.11}$$

Now, using the triangle inequality and the last two inequalities ((2.10) and (2.11)), we have

$$\begin{aligned}
\delta &\leq d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \\
&\leq d(g(x_{n_1(k)}), g(x_{n_1(k)-1})) + d(g(x_{n_1(k)}), g(x_{n_2(k)})) \\
&\quad + d(g(y_{n_1(k)}), g(y_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_2(k)})) \\
&\quad d(g(z_{n_1(k)}), g(z_{n_1(k)-1})) + d(g(z_{n_1(k)}), g(z_{n_2(k)})) \\
&\leq d(g(x_{n_1(k)}), g(x_{n_1(k)-1})) + d(g(y_{n_1(k)}), g(y_{n_1(k)-1})) + +d(g(z_{n_1(k)}), g(z_{n_1(k)-1})) + \delta.
\end{aligned}$$

For $k \to \infty$ we obtain

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} [d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)}))] = \delta.$$

Next, we will show that $\delta = 0$. Supposing the contrary, we have

$$\begin{aligned}
s_k &= d(g(x_{n_2(k)}), g(x_{n_1(k)})) + d(g(y_{n_2(k)}), g(y_{n_1(k)})) + d(g(z_{n_2(k)}), g(z_{n_1(k)})) \\
&\leq d(g(x_{n_1(k)}), g(x_{n_1(k)+1})) + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) \\
&\quad + d(g(y_{n_1(k)}), g(y_{n_1(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) \\
&\quad + d(g(z_{n_1(k)}), g(z_{n_1(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)})) \\
&= \eta_{n_1(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)})) \\
&= 2\eta_{n_1(k)} + 2\eta_{n_2(k)} + d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) \\
&\quad + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1})) + d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})).
\end{aligned} \tag{2.12}$$

But

$$\begin{aligned}
&d(g(x_{n_1(k)+1}), g(x_{n_2(k)+1})) + d(g(y_{n_1(k)+1}), g(y_{n_2(k)+1}))d(g(z_{n_1(k)+1}), g(z_{n_2(k)+1})) \\
&\quad = d(f(x_{n_1(k)}, y_{n_1(k)}, z_{n_1(k)}), f(x_{n_2(k)}, y_{n_2(k)}, z_{n_2(k)})) \\
&\quad + d(f(y_{n_1(k)}, x_{n_1(k)}, y_{n_1(k)}), f(y_{n_2(k)}, x_{n_2(k)}, y_{n_2(k)}))
\end{aligned}$$

$$+d(f(z_{n_1(k)}, y_{n_1(k)}, x_{n_1(k)}), f(z_{n_2(k)}, y_{n_2(k)}, x_{n_2(k)}))$$

$$\leq 2 \cdot \varphi \left(\frac{d(g(x_{n_1(k)}, g(x_{n_2(k)})) + d(g(y_{n_1(k)}), g(y_{n_2(k)}) + d(g(z_{n_1(k)}), g(z_{n_2(k)})}{3} \right) +$$

$$\leq 2 \cdot \varphi \left(\frac{s_k}{3} \right).$$

Now, returning to (2), we have $s_k \leq 2\eta_{n_1(k)} + 2\eta_{n_2(k)} + 2 \cdot \varphi \left(\frac{s_k}{3} \right)$. Let $k \to \infty$. We obtain

$$\delta \leq 3 \cdot \lim_{k \to \infty} \varphi \left(\frac{s_k}{3} \right) < \delta,$$

a contradiction. Consequently, $\{g(x_n)\}_{n \in \mathbb{N}}$, $\{g(y_n)\}_{n \in \mathbb{N}}$ and $\{g(z_n)\}_{n \in \mathbb{N}}$ are Cauchy sequences in the complete metric space (X, d). Since X is complete, there exist $\overline{x}$,$\overline{y}$ and $\overline{z}$ such that $g^n(x_n) \to \overline{x}$, $g^n(y_n) \to \overline{y}$ and $g^n(z_n) \to \overline{y}$ as $n \to \infty$. Which means that $f^{n-1}(x_n, y_n, z_n) \to \overline{x}$, $f^{n-1}(y_n, x_n, y_n) \to \overline{y}$ and $f^{n-1}(z_n, y_n, x_n) \to \overline{z}$, as $n \to \infty$. Using the orbital g-continuity of f, we get that $f^n(x_n, y_n, z_n) \to g(\overline{x})$, $f^n(y_n, x_n, y_n) \to g(\overline{y})$ and $f^n(z_n, y_n, x_n) \to g(\overline{z})$, as $n \to \infty$, i.e., $(\overline{x}, \overline{y}, \overline{z})$ is a tripled coincidence point of f and g. □

In order to obtain the uniqueness of the tripled coincidence point, the following assumption has to be added to the hypotheses of Theorem 2.5:

Theorem 2.6. *In addition to the hypotheses of Theorem 2.5, suppose that for every* (x^*, y^*, z^*), $(\overline{x}, \overline{y}, \overline{z}) \in X \times X \times X$, *there exists* $(t, u, v) \in X \times X \times X$ *such that* $(g(x^*), g(y^*), g(z^*)), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t))$. *Then* f *and* g *have a unique tripled coincidence point.*

Proof. From Theorem 2.5, there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $f(\overline{x}, \overline{y}, \overline{z}) = g(\overline{x})$, $f(\overline{y}, \overline{x}, \overline{y}) = g(\overline{y})$ and $f(, \overline{z}, \overline{y}, \overline{x}) = g(\overline{z})$. We have to show that, if (x^*, y^*, z^*) is another coincidence point for f and g,

$$d((g(\overline{x}), g(\overline{y}), g(\overline{z})), (g(x^*), g(y^*), g(z^*))) = 0.$$

Since (x^*, y^*, z^*) and $(\overline{x}, \overline{y}, \overline{z})$ are both tripled coincidence points, it follows that

$$g(x^*) = f(x^*, y^*, z^*), g(y^*) = f(y^*, x^*, y^*), g(z^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}) = f(\overline{z}, \overline{y}, \overline{x}).$$

Now, using the hypothesis of Theorem 2.5, from $f(X^3) \subseteq g(x)$, there exist $t_1, u_1, v_1 \in X^3$ such that $g(t_1) = f(t_0, u_0, v_0)$, $g(u_1) = f(u_0, t_0, u_0)$, $g(v_1) = f(v_0, u_0, t_0)$. Using the same procedure as in the proof of Theorem 2.4, we build the sequences $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$, where

$$g(t_{n+1}) = f(t_n, u_n, v_n), g(u_{n+1}) = f(u_n, t_n, u_n) \text{ and } g(v_{n+1}) = f(v_n, u_n, t_n).$$

Next, let $x_0 = x^*, y_0 = y^*, z_0 = z^*$ and $\overline{x_0} = \overline{x}, \overline{y_0} = \overline{y}, \overline{z_0} = \overline{y}$. Thus, we obtain the sequences $\{x_n^*\}_{n \in \mathbb{N}}, \{y_n^*\}_{n \in \mathbb{N}}, \{z_n^*\}_{n \in \mathbb{N}}$ $\{\overline{x}_n\}_{n \in \mathbb{N}}$, $\{\overline{y}_n\}_{n \in \mathbb{N}}$ and $\{\overline{z}_n\}_{n \in \mathbb{N}}$ such that

$$g(x_n^*) = f(x^*, y^*, z^*), g(y_n^*) = f(y^*, x^*, y^*), g(z_n^*) = f(z^*, y^*, x^*)$$

and

$$g(\overline{x}_n) = f(\overline{x}, \overline{y}, \overline{z}), g(\overline{y}_n) = f(\overline{y}, \overline{x}, \overline{y}), g(\overline{z}_n) = f(\overline{z}, \overline{y}, \overline{x}).$$

From the hypothesis, we know that there exists $(t, u, v) \in X \times X \times X$ such that

$$(g(x^*), g(y^*), g(z^*), (g(\overline{x}), g(\overline{y}), g(\overline{z})) \in X_R(f(t, u, v), f(u, t, u), f(v, u, t)).$$

From $(g(x_0), g(y_0), g(z_0)) \in X_R(f(t,u,v), f(u,t,u), f(v,u,t))$ and the completeness of the metric space it follows that

$$(f^n(g(x_0), g(y_0), g(z_0)), f^n(g(y_0), g(x_0), g(y_0)), f^n(g(z_0), g(y_0), g(x_0)))$$
$$\in X_R(f^{n+1}(t,u,v), f^{n+1}(u,t,u), f^{n+1}(v,u,t))$$

Also, by using the contractivity condition, we have

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t,u,v)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,u)) + d(g(z_0), f(v,u,t))}{3}\right),$$
$$d(f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u,t,u)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,u)) + d(g(z_0), f(v,u,t))}{3}\right)$$

and

$$d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v,u,t)) \leq$$
$$\varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,u)) + d(g(z_0), f(v,u,t))}{3}\right).$$

By summing up, we obtain that

$$d(f^n(g(x_0), g(y_0), g(z_0)), f^{n+1}(t,u,v)) + d(f^n(g(y_0), g(x_0), g(y_0)), f^{n+1}(u,t,u))$$
$$+d(f^n(g(z_0), g(y_0), g(x_0)), f^{n+1}(v,u,t))$$
$$\leq 3 \cdot \varphi\left(\frac{d(g(x_0), f(t,u,v)) + d(g(y_0), f(u,t,u)) + d(g(z_0), f(v,u,t))}{3}\right).$$

But $x_0 = x^*$, $y_0 = y^*$ and $z_0 = z^*$. We obtain

$$d(f^n(g(x^*), g(y^*), g(z^*)), f^{n+1}(t,u,v)) + d(f^n(g(y^*), g(x^*), g(y^*)), f^{n+1}(u,t,u))$$
$$+d(f^n(g(z^*), g(y^*), g(x^*)), f^{n+1}(v,u,t)) \leq$$
$$3 \cdot \varphi\left(\frac{d(g(x^*), f(t,u,v)) + d(g(y^*), f(u,t,u)) + d(g(z^*), f(v,u,t))}{3}\right).$$

Letting $n \to \infty$ we obtain that

$$\lim_{n\to\infty} d(g(x^*), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(y^*), f(u,t,u)) = 0 \text{ and } \lim_{n\to\infty} d(g(z^*), f(v,u,t)) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} d(g(\overline{x}), f(t,u,v)) = 0, \lim_{n\to\infty} d(g(\overline{z}), f(u,t,u)) = 0 \text{ and } \lim_{n\to\infty} d(g(\overline{z}), f(v,u,t)) = 0.$$

Now, using the triangle inequality, we have

$$d(g(x^*), g(\overline{x})) \leq d(g(x^*), f(t,u,v)) + d(f(t,u,v), g(\overline{x})) \to 0, \text{ when } n \to \infty,$$

$$d(g(y^*), g(\overline{y})) \leq d(g(y^*), f(u,t,u)) + d(f(u,t,u), g(\overline{y})) \to 0, \text{ when } n \to \infty,$$

and

$$d(g(z^*), g(\overline{z})) \leq d(g(z^*), f(v,u,t)) + d(f(v,u,t), g(\overline{z})) \to 0, \text{ when } n \to \infty,$$

so the proof of the theorem is complete. □

Example 2.1. *Let $X = \mathbb{R}$, $d = |x - y|$, the relation R on X given by*

$$(x, y, z)R(t, u, v) \Leftrightarrow xRt \wedge yRu \wedge zRv,$$

where $xRt \Leftrightarrow x^2 + x = t^2 + t$.

Let $f : X \times X \times X \to X$, $g : X \to X$ be defined by

$$f(x, y, z) = \frac{x - y + 3z - 2}{6}, \quad g(x) = x - 1,$$

So, $\forall (x, y, z) \in X^3$, we have :

$$X_R(x, y, z) = \{(x, y, z), (x, -y-1, z), (-x-1, y, z), (-x-1, -x-1, z), (x, y, -z-1),$$
$$(-x-1, -y-1, -z-1), (-x-1, y, -z-1), (x, -y-1, -z-1)\}.$$
$$f \times g(X_R(x, y, z)) \subseteq X_R(f \times g(x, y, z))$$

So, f has the mixed $g - R-$monotone property. It can easily be checked that f and g satisfy all the other conditions of Theorem 2.5, whereas Theorems 1.3, 1.1, 1.2 cannot be applied because the relation R considered is not antisymmetric. The contraction also holds for $\varphi(t) = \dfrac{kt}{3}, k \in [0, 1)$. Note that the additional assumption in Theorem 2.6 also holds. Thus, f and g have a unique tripled coincidence point, $\left(\dfrac{4}{3}, \dfrac{4}{3}, \dfrac{4}{3}\right)$, obtained by solving the following system

$$f(x, y, z) = g(x),\ f(y, x, y) = g(y),\ f(z, y, x) = g(z).$$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
NORTH UNIVERSITY CENTER OF BAIA MARE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
E-mail address: melania.cozma@yahoo.com