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# Blending type approximation by generalized Szász type operators based on Charlier polynomials

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ABSTRACT. In the present paper, we introduce a generalized Szász type operators based on $\rho(x)$ where $\rho$ is a continuously differentiable function on $[0,\infty)$, $\rho(0)=0$ and $\inf \rho^{'}(x) \geq 1, x \in [0,\infty)$. This function not only characterizes the operators but also characterizes the Korovkin set $\{1,\rho,\rho^2\}$ in a weighted function space. First, we establish approximation in a Lipschitz type space and weighted approximation theorems for these operators. Then we obtain a Voronovskaja type result and the rate of convergence in terms of the weighted modulus of continuity.

## 1. INTRODUCTION

In 1950, Szász [21] considered the following linear positive operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in [0,\infty)$ and $f(x)$ is a continuous function on $[0,\infty)$ whenever the above sum converges uniformly. Many researchers have studied approximation properties of these operators and modified Szász operators by involving different types of polynomials. Jakimovski and Leviatan [12] constructed a generalization of Szász operators based on Appell polynomials and established the approximation properties of these operators. Varma et al. [22] introduced the generalization of Szász operators including the Brenke type polynomials and studied convergence properties with the help of the Korovkin type theorem and the order of convergence by using classical method. Another recent and interesting results concerning the Szász operators (1.1) could be found also in [2, 3, 4, 5, 7, 13, 16, 19, 20].

In [23], Varma and Taşdelen constituted a link between orthogonal polynomials and the positive linear operators. They have considered Szász type operators including Charlier polynomials. These polynomials [11] have the generating functions of the form

$$e^t\left(1-\frac{t}{a}\right)^u = \sum_{k=0}^{\infty} C_k^{(a)}(u)\frac{t^k}{k!}, \quad |t|<a, \tag{1.2}$$

where $C_k^{(a)}(u) = \sum_{r=0}^{k} \binom{k}{r}(-u)_r\left(\frac{1}{a}\right)^r$ and $(m)_0=1, (m)_j = m(m+1)\cdots(m+j-1)$ for $j \geq 1$.

Varma and Taşdelen [23] defined the following Szász type operators based on Charlier

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polynomials

$$\mathcal{L}_{n,a}(f;x) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}f\left(\frac{k}{n}\right), \tag{1.3}$$

where $a>1$ and $x\in[0,\infty)$. For the special case, $a\to\infty$ and $x-\frac{1}{n}$ instead of $x$, these operators reduce to the well-known Szász operators [21]. They studied uniform convergence of these operators with the help of the Korovkin theorem on compact subsets of $[0,\infty)$ and the order of approximation by applying the classical modulus of continuity. Kajla and Agrawal [15] obtained approximation in a Lipschitz type space, weighted approximation and the error in the approximation of functions having derivatives of bounded variation of the operators (1.3).

In 2014, Aral et al. [6] defined a generalization of Szász-Mirakyan operators involving a function $\rho$ and gave the quantitative type theorems in order to obtain the degree of weighted convergence by using the weighted modulus of continuity constructed using the function $\rho$. Olguna et al. [17] considered a generalization of the Jain operators based on $\rho$ function and established Voronovskaya type theorem and the rate of convergence of these operators. Acar et al. [1] defined a new general class of operators which have the classical Szász-Mirakyan ones as a basis, and fix the functions $e^{ax}$ and $e^{2ax}$ with $a>0$ and studied Voronovskaja type theorem of these operators.

We construct the following Szász type operators involving Charlier polynomials

$$\begin{aligned}\mathcal{L}^{\rho}_{n,a}(f;x) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)n\rho(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)n\rho(x))}{k!}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right) \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)n\rho(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)n\rho(x))}{k!}f\left(\rho^{-1}\left(\frac{k}{n}\right)\right),\end{aligned} \tag{1.4}$$

where $\rho$ is a function such that

($b_1$) $\rho$ is a continuously differentiable function on $[0,\infty)$.
($b_2$) $\rho(0)=0,\ \inf\limits_{x\in(0,\infty)}\rho'(x)\geq 1.$

In the present paper, we construct a generalized Szász type operators based on Charlier polynomials. First, we establish approximation in a Lipschitz type space and weighted approximation theorems for these operators. Then we obtain a Voronovskaja type result and the rate of convergence in terms of the weighted modulus of continuity.

## 2. Preliminaries

Let $e_i(x)=\rho^i(x), i=\overline{0,4}$

**Lemma 2.1.** *For the operators $\mathcal{L}^{\rho}_{n,a}(f;x)$, we get*

*(i)* $\mathcal{L}^{\rho}_{n,a}(e_0(t);x)=1;$

*(ii)* $\mathcal{L}^{\rho}_{n,a}(e_1(t);x)=\rho(x)+\dfrac{1}{n};$

*(iii)* $\mathcal{L}^{\rho}_{n,a}(e_2(t);x)=\rho^2(x)+\dfrac{\rho(x)}{n}\left(3+\dfrac{1}{a-1}\right)+\dfrac{2}{n^2}.$

*(iv)* $\mathcal{L}^{\rho}_{n,a}(e_3(t);x)=\rho^3(x)+\dfrac{\rho^2(x)}{n}\left(6+\dfrac{3}{a-1}\right)+\dfrac{2\rho(x)}{n^2}\left(\dfrac{1}{(a-1)^2}+\dfrac{3}{a-1}+5\right)+\dfrac{5}{n^3};$

*(v)* $\mathcal{L}_{n,a}^{\rho}(e_4(t);x) = \rho^4(x) + \frac{\rho^3(x)}{n}\left(10 + \frac{6}{a-1}\right) + \frac{\rho^2(x)}{n^2}\left(31 + \frac{30}{a-1} + \frac{11}{(a-1)^2}\right)$
$+ \frac{\rho(x)}{n^3}\left(67 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}\right) + \frac{15}{n^4}.$

**Lemma 2.2.** *For the operators* $\mathcal{L}_{n,a}^{\rho}(f;x)$*, we have*

*(i)* $\mathcal{L}_{n,a}^{\rho}(\rho(t)-\rho(x);x) = \frac{1}{n}$;

*(ii)* $\mathcal{L}_{n,a}^{\rho}((\rho(t)-\rho(x))^2;x) = \frac{a\rho(x)}{n(a-1)} + \frac{2}{n^2}$;

*(iii)* $\mathcal{L}_{n,a}^{\rho}((\rho(t)-\rho(x))^4;x) = \frac{\rho(x)}{n^3}\left(17 + \frac{49}{(a-1)} - \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}\right)$
$+ \frac{\rho^2(x)}{n^2}\left(19 - \frac{46}{(a-1)} + \frac{3}{(a-1)^2}\right) + \frac{15}{n^4}.$

From Lemma 2.2, for $\rho(x) \in (0,\infty)$ and sufficiently large $n$, we have

$$\mathcal{L}_{n,a}^{\rho}(|\rho(t)-\rho(x)|;x) \leq \left(\mathcal{L}_{n,a}^{\rho}((\rho(t)-\rho(x))^2;x)\right)^{1/2} \leq \sqrt{\frac{\lambda(a)\rho(x)}{n}}, \tag{2.5}$$

where $\lambda(a)$ is a positive constant depending on $a$.
Let $C[0,\infty)$ denote the space of all continuous and real valued functions defined on $[0,\infty)$.

## 3. Main results

3.1. **Degree of Approximation.** Let $\alpha_1 > 0, \alpha_2 \geq 0$ be fixed. We consider the following Lipschitz-type space (see [18]):

$$Lip_M^{(\alpha_1,\alpha_2)}(r) := \left\{ f \in C[0,\infty) : |f(t)-f(x)| \leq M\frac{|t-x|^r}{(t+\alpha_1 x^2+\alpha_2 x)^{\frac{r}{2}}}; x,t \in (0,\infty) \right\},$$

where $M$ is a positive constant and $0 < r \leq 1$.

**Theorem 3.1.** *Let* $f \in Lip_M^{(\alpha_1,\alpha_2)}(r)$ *and* $r \in (0,1]$*. Then, for all* $x \in (0,\infty)$*, we have*

$$| \mathcal{L}_{n,a}^{\rho}(f;x) - f(x) | \leq M\left(\frac{\zeta_{n,a}(\rho(x))}{\alpha_1\rho^2(x)+\alpha_2\rho(x)}\right)^{\frac{r}{2}},$$

*where* $\zeta_{n,a}(\rho(x)) = \mathcal{L}_{n,a}^{\rho}((\rho(t)-\rho(x))^2;x)$.

*Proof.* By the Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we obtain

$$\begin{aligned} &| \mathcal{L}_{n,a}^{\rho}(f;x) - f(x) | \\ &\leq \left\{ e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)n\rho(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)n\rho(x))}{k!} \left| f\left(\rho^{-1}\left(\frac{k}{n}\right)\right) - f(x)\right|^{\frac{2}{r}} \right\}^{r/2} \\ &\leq M\left\{ e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)n\rho(x)} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)n\rho(x))}{k!} \frac{\left(\frac{k}{n}-\rho(x)\right)^2}{\left(\frac{k}{n}+\alpha_1\rho^2(x)+\alpha_2\rho(x)\right)} \right\}^{r/2}. \end{aligned}$$

Since $f \in Lip_M^{(\alpha_1,\alpha_2)}(r)$ and $\frac{1}{\sqrt{\frac{k}{n}+\alpha_1\rho^2(x)+\alpha_2\rho(x)}} < \frac{1}{\sqrt{\alpha_1\rho^2(x)+\alpha_2\rho(x)}}, \forall x \in (0,\infty)$,

we have

$$
\begin{aligned}
&\mid \mathcal{L}_{n,a}^{\rho}(f;x)-f(x)\mid \\
&\leq \frac{M}{(\alpha_1\rho^2(x)+\alpha_2\rho(x))^{\frac{r}{2}}}\left\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)n\rho(x)}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)n\rho(x))}{k!}\left(\frac{k}{n}-\rho(x)\right)^2\right\}^{r/2} \\
&\leq M\left(\frac{\zeta_{n,a}(\rho(x))}{\alpha_1\rho^2(x)+\alpha_2\rho(x)}\right)^{\frac{r}{2}}.
\end{aligned}
$$

This completes the proof of the theorem. □

Let $B_\eta[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)|\leq M_f\eta(x)$, where $M_f$ is a positive constant depending only on $f$ and $\eta(x)=1+\rho^2(x)$ is a weight function and $\rho(x)\to\infty$ as $x\to\infty$. Let $C_\eta[0,\infty)$ be the space of all continuous functions in $B_\eta[0,\infty)$ with the norm

$$
\|f\|_\eta := \sup_{x\in[0,\infty)}\frac{|f(x)|}{\eta(x)} \quad \text{and} \quad C_\eta^*[0,\infty):=\left\{f\in C_\eta[0,\infty): \lim_{x\to\infty}\frac{|f(x)|}{\eta(x)} \text{ is finite}\right\}.
$$

Let $U_\eta[0,\infty)$ be the space of functions $f\in C_\eta[0,\infty)$ such that $\frac{f(x)}{\eta(x)}$ is uniformly continuous. It is obvious that $C_\eta^*[0,\infty)\subset U_\eta[0,\infty)\subset C_\eta[0,\infty)\subset B_\eta[0,\infty)$.

The usual modulus of continuity of $f$ on $[0,b]$ is defined as

$$
\omega_b(f,\delta)=\sup_{|t-x|\leq\delta}\sup_{x,t\in[0,b]}|f(t)-f(x)|.
$$

**Theorem 3.2.** *Let $f\in C_\eta[0,\infty)$. Then, we have*

$$
\|\mathcal{L}_{n,a}^{\rho}(f;\cdot)-f\|_{C[0,b]}\leq 4M_f(1+b^2)\zeta_{n,a}(b)+2\omega_{b+1}(f,\sqrt{\zeta_{n,a}(b)}),
$$

*where* $\zeta_{n,a}(b)=\dfrac{ab}{n(a-1)}+\dfrac{2}{n^2}$.

*Proof.* Let $\rho(x)\in[0,b]$ and $\rho(t)>b+1$. Then, $\rho(t)-\rho(x)>1$, hence

$$
\begin{aligned}
&|f(t)-f(x)| \\
&\leq M_f(2+\rho^2(t)+\rho^2(x))=M_f\left\{2+2\rho^2(x)+(\rho(t)-\rho(x))^2+2\rho(x)(\rho(t)-\rho(x))\right\} \\
&\leq M_f(\rho(t)-\rho(x))^2\left(3+2\rho(x)+2\rho^2(x)\right) \\
&\leq 4M_f(1+\rho^2(x))(\rho(t)-\rho(x))^2. \qquad (3.6)
\end{aligned}
$$

For $\rho(x)\in[0,b]$ and $\rho(t)\in[0,b+1]$ we have

$$
|f(t)-f(x)|\leq\omega_{b+1}(f;|\rho(t)-\rho(x)|)\leq\left(1+\frac{|\rho(t)-\rho(x)|}{\delta}\right)\omega_{b+1}(f;\delta). \qquad (3.7)
$$

Thus, from (3.6) and (3.7) for all $\rho(x)\in[0,b]$ and $\rho(t)\geq 0$, we have

$$
|f(t)-f(x)|\leq 4M_f(1+\rho^2(x))(\rho(t)-\rho(x))^2+\left(1+\frac{|\rho(t)-\rho(x)|}{\delta}\right)\omega_{b+1}(f,\delta),\delta>0.
$$

Hence applying Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&|\mathcal{L}_{n,a}^{\rho}(f;x)-f(x)| \\
&\leq 4M_f(1+\rho^2(x))\mathcal{L}_{n,a}^{\rho}((\rho(t)-\rho(x))^2;x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\mathcal{L}_{n,a}^{\rho}(|\rho(t)-\rho(x)|;x)\right) \\
&\leq 4M_f(1+\rho^2(x))\zeta_{n,a}(\rho(x))+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\zeta_{n,a}(\rho(x))}\right) \\
&\leq 4M_f(1+b^2)\zeta_{n,a}(b)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\zeta_{n,a}(b)}\right).
\end{aligned}
$$

Choosing $\delta = \sqrt{\zeta_{n,a}(b)}$, we get the desired result. □

## 4. WEIGHTED APPROXIMATION

**Theorem 4.3.** *Let $f \in C^*_\eta[0,\infty)$. Then, we have*

$$\lim_{n\to\infty} \|\mathcal{L}^\rho_{n,a}(f;\cdot) - f\|_\eta = 0. \tag{4.8}$$

*Proof.* From [9], we know that it is sufficient to verify the following three equations

$$\lim_{n\to\infty} \|\mathcal{L}^\rho_{n,a}(e_m;\cdot) - e_m\|_\eta = 0,\ \ m = 0, 1, 2. \tag{4.9}$$

Since $\mathcal{L}^\rho_{n,a}(e_0;x) = 1$, the condition in (4.9) holds true for $m = 0$.

By Lemma 2.1, we have

$$\|\mathcal{L}^\rho_{n,a}(e_1;\cdot) - e_1\|_\eta = \sup_{\rho(x)\geq 0} \frac{1}{1+\rho^2(x)}\left|\rho(x) + \frac{1}{n} - \rho(x)\right| \leq \frac{1}{n}.$$

Thus, $\lim\limits_{n\to\infty} \|\mathcal{L}^\rho_{n,a}(e_1;\cdot) - e_1\|_\eta = 0$. Similarly, we get

$$\begin{aligned}\|\mathcal{L}^\rho_{n,a}(e_2;\cdot) - e_2\|_\eta &= \sup_{\rho(x)\geq 0} \frac{1}{1+\rho^2(x)}\left|\rho^2(x) + \frac{\rho(x)}{n}\left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2} - \rho^2(x)\right| \\ &\leq \frac{1}{n}\left(3 + \frac{1}{(a-1)}\right) + \frac{2}{n^2},\end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \|\mathcal{L}^\rho_{n,a}(e_2;\cdot) - e_2\|_\eta = 0$. This completes the proof.

□

Let $f \in C_\eta[0,\infty)$. We will consider the weighted modulus of continuity defined by Holhoş [10] as follows:

$$\Omega_\rho(f;\delta) = \sup_{x\in[0,\infty), |\rho(t)-\rho(x)|\leq\delta} \frac{|f(t) - f(x)|}{\eta(t) + \eta(x)},$$

and proved that it has the following properties:

(i) for every $f \in U_\eta[0,\infty)$, $\lim\limits_{\delta\to 0} = \Omega_\rho(f;\delta)$

(ii) for every $\delta \geq 0$ and $\lambda \geq 0$, $\Omega_\rho(f;\lambda\delta) \leq (2+\lambda)\Omega_\rho(f;\delta)$

(iii) for every $f \in U_\eta[0,\infty)$, for $\delta > 0$ and for all $x, t \geq 0$,

$$|f(t) - f(x)| \leq (\eta(t) + \eta(x))\left(2 + \frac{|\rho(t) - \rho(x)|}{\delta}\right)\Omega_\rho(f;\delta).$$

**Theorem 4.4.** [10]*: Let $\{K_n\}_{n\geq 1}$ be a sequence of positive linear operators acting on $C_\eta[0,\infty)$ to $B_\eta[0,\infty)$ with*

$$\begin{aligned}\|K_n(\rho^0) - \rho^0\|_{\eta^0} &= a_n \\ \|K_n(\rho) - \rho\|_{\eta^{\frac{1}{2}}} &= b_n \\ \|K_n(\rho^2) - \rho^2\|_\eta &= c_n \\ \|K_n(\rho^3) - \rho^3\|_{\eta^{\frac{3}{2}}} &= d_n,\end{aligned}$$

*where $a_n, b_n, c_n, d_n \to 0$ as $n \to \infty$. Then*

$$\|K_n(f) - f\|_{\eta^{\frac{3}{2}}} \leq (7 + 4a_n + 2c_n)\Omega_\rho(f;\delta) + \|f\|_\eta a_n$$

*for all $f \in C_\eta[0,\infty)$, where*

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n.$$

**Remark 4.1.** In the condition of Theorem 4.3, using the fact that $\lim_{\delta\to 0}\Omega_\rho(f;\delta)=0$ we have

$$\lim_{n\to\infty}\|K_n(f)-f\|_{\eta^{\frac{3}{2}}}=0,$$

for all $f\in U_\eta[0,\infty)$.

**Theorem 4.5.** *For every $f\in C_\eta[0,\infty)$. Then, we have*

$$\|\mathcal{L}^\rho_{n,a}(f)-f\|_{\eta^{\frac{3}{2}}}\leq\left(7+\frac{2}{n}\left(3+\frac{1}{(a-1)}\right)+\frac{4}{n^2}\right)\Omega_\rho(f;\gamma_{n,a}),$$

*where* $\gamma_{n,a}=2\sqrt{\frac{1}{n}\left(5+\frac{1}{a-1}\right)+\frac{6}{n}\left(3+\frac{1}{a-1}\right)+\frac{2(6+a(8a-13))}{n^2(a-1)^2}+\frac{5}{n^3}}$.

*Proof.* From Lemma 2.1, we may write

$$\begin{aligned}a_n &= \|\mathcal{L}^\rho_{n,a}(\rho^0)-\rho^0\|_{\eta^0}=0,\\ b_n &= \|\mathcal{L}^\rho_{n,a}(\rho)-\rho\|_{\eta^{\frac{1}{2}}}\leq\frac{1}{n},\\ c_n &= \|\mathcal{L}^\rho_{n,a}(\rho^2)-\rho^2\|_\eta\leq\frac{1}{n}\left(3+\frac{1}{(a-1)}\right)+\frac{2}{n^2}\end{aligned}$$

and

$$d_n=\|\mathcal{L}^\rho_{n,a}(\rho^3)-\rho^3\|_{\eta^{\frac{3}{2}}}\leq\frac{1}{n}\left(6+\frac{3}{a-1}\right)+\frac{2}{n^2}\left(\frac{1}{(a-1)^2}+\frac{3}{a-1}+5\right)+\frac{5}{n^3}.$$

In view of $a_n,b_n,c_n,d_n\to 0$ as $n\to\infty$ and from Theorem 4.4, we get

$$\|\mathcal{L}^\rho_{n,a}(f)-f\|_{\eta^{\frac{3}{2}}}\leq\left(7+\frac{2}{n}\left(3+\frac{1}{(a-1)}\right)+\frac{4}{n^2}\right)\Omega_\rho(f;\gamma_{n,a}).$$

□

**Remark 4.2.** For $f\in U_\eta[0,\infty)$. Then, we have

$$\lim_{n\to\infty}\|\mathcal{L}^\rho_{n,a}(f)-f\|_{\eta^{\frac{3}{2}}}=0.$$

## 5. Voronovskaja type theorem

In this section we prove a Voronvoskaja type result for the $\mathcal{L}^\rho_{n,a}$ operators by applying the same approach as in [6], [8] and [17] .

**Theorem 5.6.** *Let $f\in C[0,\infty)$, $x\in[0,\infty)$. If $\left(f\circ\rho^{-1}\right)''$ exists at $\rho(x)$ and $\left(f\circ\rho^{-1}\right)''$ is bounded on $[0,\infty)$, then we have*

$$\lim_{n\to\infty}n\left[\mathcal{L}^\rho_{n,a}(f;x)-f(x)\right]=\left(f\circ\rho^{-1}\right)(\rho(x))+\frac{a\rho(x)}{2(a-1)}\left(f\circ\rho^{-1}\right)''(\rho(x)).$$

*Proof.* Applying the Taylor's expansion of $\left(f\circ\rho^{-1}\right)$ at the point $\rho(x)\in[0,\infty)$, there exists $\chi$ lying between $x$ and $t$ such that

$$\begin{aligned}f(t) &= \left(f\circ\rho^{-1}\right)(\rho(t))=\left(f\circ\rho^{-1}\right)(\rho(x))+\left(f\circ\rho^{-1}\right)'(\rho(x))\left(\rho(t)-\rho(x)\right)\\ &\quad+\frac{1}{2}\left(f\circ\rho^{-1}\right)''(\rho(x))\left(\rho(t)-\rho(x)\right)^2+j_x(t)\left(\rho(t)-\rho(x)\right)^2,\end{aligned}$$

where

$$j_x(t):=\frac{\left(f\circ\rho^{-1}\right)''(\rho(\chi))-\left(f\circ\rho^{-1}\right)''(\rho(x))}{2}.$$

Hence,

$$\begin{aligned} n\left(\mathcal{L}^{\rho}_{n,a}(f;x)-f(x)\right) &= \left(f\circ\rho^{-1}\right)'(\rho(x))n\mathcal{L}^{\rho}_{n,a}\left(\rho(t)-\rho(x);x\right) \\ &+\frac{1}{2}\left(f\circ\rho^{-1}\right)''(\rho(x))n\mathcal{L}^{\rho}_{n,a}\left((\rho(t)-\rho(x))^2;x\right) \\ &+n\mathcal{L}^{\rho}_{n,a}\left(j_x(t)\left(\rho(t)-\rho(x)\right)^2;x\right). \end{aligned}$$

From Lemma 2.2, we may write

$$\begin{aligned} \lim_{n\to\infty} n\left(\mathcal{L}^{\rho}_{n,a}(f;x)-f(x)\right) &= \left(f\circ\rho^{-1}\right)(\rho(x))+\frac{a\rho(x)}{2(a-1)}\left(f\circ\rho^{-1}\right)''(\rho(x)) \\ &+\lim_{n\to\infty} n\mathcal{L}^{\rho}_{n,a}\left(j_x(t)\left(\rho(t)-\rho(x)\right)^2;x\right). \end{aligned}$$

From the hypothesis of the theorem we have $|j_x(t)|\le C$ and $\lim_{t\to x} j_x(t)=0$. Thus for any $\epsilon>0$ there exists $\delta>0$ such that $|j_x(t)|<\epsilon$ for $|t-x|<\delta$. On the other hand, from the condition $(b_2)$ we have $|\rho(t)-\rho(x)|\ge|t-x|$. Therefore, if $|\rho(t)-\rho(x)|<\delta$, then $|j_x(t)\left(\rho(t)-\rho(x)\right)^2|<\epsilon\left(\rho(t)-\rho(x)\right)^2$ and if $|\rho(t)-\rho(x)|\ge\delta$, then since $|j_x(t)|\le C$ we get $|j_x(t)\left(\rho(t)-\rho(x)\right)^2|<\frac{C}{\delta^2}\left(\rho(t)-\rho(x)\right)^4$. Hence we may write

$$\mathcal{L}^{\rho}_{n,a}\left(j_x(t)\left(\rho(t)-\rho(x)\right)^2;x\right)<\epsilon\mathcal{L}^{\rho}_{n,a}\left(\rho(t)-\rho(x)\right)^2;x)+\frac{C}{\delta^2}\mathcal{L}^{\rho}_{n,a}\left(\left(\rho(t)-\rho(x)\right)^4;x\right).$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} n\mathcal{L}^{\rho}_{n,a}\left(j_x(t)\left(\rho(t)-\rho(x)\right)^2;x\right)=0.$$

Thus, the theorem is proved. □

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