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# Fixed points of nearly weak uniformly $L$-Lipschitzian mappings in real Banach spaces

ADESANMI ALAO MOGBADEMU

ABSTRACT. Let $K$ be a nonempty convex subset of a real Banach space $X$. Let $T$ be a nearly weak uniformly $L$-Lipschitzian mapping. A modified Mann-type iteration scheme is proved to converge strongly to the unique fixed point of $T$. Our result is a significant improvement and generalization of several known results in this area of research. We give a specific example to support our result. Furthermore, an interesting equivalence of $T$-stability result between the convergence of modified Mann-type and modified Mann iterations is included.

## 1. INTRODUCTION

Let $X$ be an arbitrary real normed space with the dual $\mathrm{X}^*$. We denote by $J$ the normalized duality mapping from $X$ into $2^{\mathrm{X}^*}$ by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing between elements of $X$ and $X^*$. We first recall and define some concepts as follows.

**Definition 1.1.** Let $K$ be a nonempty subset of a real normed linear space $X$. Let $T : K \to K$. $T$ is called asymptotically nonexpansive if for each $x, y \in K$

$$\|T^n x - T^n y\| \leq k_n \|x - y\|^2, \forall n \geq 1,$$

where $(k_n) \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. $T$ is called asymptotically pseudocontractive with the sequence$(k_n) \subset [1, \infty)$ if and only if $\lim_{n\to\infty} k_n = 1$, and for all $n \in N$ and all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n \|x - y\|^2, \forall n \geq 1.$$

**Remark 1.1.** An asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse may not be true in general (see [2], [3]).

**Definition 1.2.** Let $K$ be a nonempty subset of a real normed linear space $X$. Let $T : K \to K$. A mapping $T$ is called uniformly $L-$ Lipschitzian if, for any $x, y \in K$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \forall n \geq 1.$$

Let $\{\sigma_n\}_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $lim_{n\to\infty}\sigma_n = 0$.

A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{\sigma_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + \sigma_n), \forall\, x, y \in K. \tag{1.1}$$

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A nearly Lipschitzian mapping $T$ with sequence $\{\sigma_n\}$ is said to be nearly uniformly $L$-Lipschitzian if $k_n = L$, for all $n \in \mathbb{N}$.

Observe that the class of nearly uniformly $L$-Lipschitzian mapping is more general than the class of uniformly $L$-Lipschitzian mappings.

The class of nearly uniformly $L$-Lipschitzian have been studied extensively by many authors: for results in this regard, see e.g. Sahu [16], Kim et al. [8] and Mogbademu [10], [11]; for uniformly $L$-Lipschitzian mappings, see e.g. Chang [2], Chang et al. [3], Goebel [6], Ofoedu [12] and Rafiq [13]; see also Berinde [1] and the references therein.

Now, we discuss the following new concept.

**Definition 1.3.** Let $K$ be a subset of a real normed linear space $X$ and $\{a_n\}_{n\geq 1}$ be a sequence in $[0, \infty)$ such that $\lim\limits_{n\to\infty} a_n = 0$. A mapping $T : K \to K$ is called nearly weak uniformly Lipschitzian with respect to the sequence $\{a_n\}$ if for each $n \in N$, there exists a constant $L \geq 1$ such that

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n), \forall\, x \in K, \;\; y \in F(T). \tag{1.2}$$

It is easy to see that if $T$ has a bounded range, then it is nearly weak uniformly Lipschitzian. In fact, since $R(T^n) \subset R(T)$, then $\sup_{x\in K} \|T^n x\| \leq \sup_{x\in K} \|T^{n-1}x\| \leq \cdots \leq \sup_{x\in K} \|Tx\| \leq x$, thus $\|T^n x - T^n y\| \leq \|Tx - Ty\| \leq (\|x - y\|) \leq L(\|x - y\| + a_n)$, where $x \in K,\;\; y \in F(T)$. On the contrary, it may not be true in general. Therefore it is of interest to study the class of mappings in fixed point theory and its applications.

**Example 1.1.** Let $X = R,\;\; K = [0, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0, 1], \\ 0, & x > 1. \end{cases}$$

Then $T\rho = \rho$ if and only if $\rho = 0$. In fact, for a real sequence $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \to 0$ as $n \to \infty$, we easily compute to have

$$||T^n x - T^n \rho|| \leq \frac{1}{2}(||x - \rho|| + \frac{1}{2^n}), \quad \forall\, x \in K, \quad \rho = 0.$$

Hence, $T$ satisfies the nearly Lipschitzian condition.

**Remark 1.2.** It is obvious that every nearly Lipschitz map with a fixed point satisfies inequality (1.2). Clearly, the class of nearly weak uniformly $L$-Lipschitzian mappings is a generalization of the class of nearly uniformly $L$-Lipschitzian mappings which in turn is a generalization of the class of uniformly $L$-Lipschitzian mappings (see [5].

It is the interest of this paper to discuss the following modified Mann-type iteration scheme associated with nearly weak uniformly $L$-Lipschitzian mappings to have a strong convergence in the real Banach spaces setting.

Let $x_1 \in K$ be a nonempty convex subset of a real normed linear space $X$ and $T : K \to K$ be a map. For a sequence $\{v_n = f(x_n)\}$ in $K$ where $f : K \to K$ is a mapping, define $\{x_n\}_{n=1}^{\infty}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n v_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ . We observe that the iteration process (1.3) is well defined and is a generalization of the modified Mann and modified Ishikawa iterations used by several authors (see [2]-[16]). This is true in the sense that, when $v_n = f(x_n) = x_n$, then (1.3) reduces to the modified Mann iteration define by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \tag{1.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$. If in (1.3), $v_n = f(x_n) = (1-\beta_n)x_n + \beta_n T^n x_n$ where $\beta_n$ is a sequence in $(0,1)$, then it will reduce to the modified Ishikawa iteration method (see [14]).

## 2. Preliminaries

In the sequel, we shall need the following lemmas.

**Lemma 2.1.** [4]. *Let $X$ be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x+y\|^2 \le \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

**Lemma 2.2.** [4, 10] *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=1}^{\infty} b_n = +\infty \quad and \quad \lim_{n\to\infty} b_n = 0.$$

*Suppose that $\{a_n\}_{n=1}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying*

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\Phi(a_{n+1}), \quad \forall n \ge \mathbb{N}_0$$

*where $\lim\limits_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

## 3. Main results

**Theorem 3.1.** *Let $K$ be a nonempty convex subset of a real Banach space $X$. Let $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ as defined in equation (1.2). Let $\{\epsilon_n\} \in (0,1)$ and $\{k_n\} \subset [1,\infty)$ be sequences with $lim_{n\to\infty}\epsilon_n = 0$ and $lim_{n\to\infty}k_n = 1$. For a sequence $\{v_n\}$ in $K$, define a sequence $\{x_n\}$ in $K$ satisfying $\lim_{n\to\infty} ||v_n - x_n|| = 0$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n v_n, \; n \ge 1,$$

*where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that (i)$\sum_{n\ge 1} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n = 0$. There exists $\tau_0 > 0$ such that $\alpha_n \le \tau_0 \; \forall n \ge n_0$, for some $n_0 \in N$. Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that*

$$< T^n x - T^n \rho, j(x-\rho) > \le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

*for all $x \in K, \rho \in F(T)$. Then $\{x_n\}_{n\ge 1}$ converges strongly to the unique fixed point of $T$.*

*Proof.* Let $\rho \in F(T)$, it is easy to see that $\rho$ is unique. For, if $\rho'$ is another fixed point of $T$, then $T^n\rho = \rho$ and $T^n\rho' = \rho'$, $\forall n \ge 1$. That is, $\|\rho-\rho'\|^2 \le k_n\|\rho-\rho'\|^2 - \Phi(\|\rho-\rho'\|) + \epsilon_n, \; \forall n \ge 1$. Taking limits of bothsides as $n \to \infty$, we get

$$\|\rho-\rho'\|^2 \le \|\rho-\rho'\|^2 - \Phi(\|\rho-\rho'\|) < \|\rho-\rho'\|^2,$$

a contradiction. Hence, $\rho$ is unique.
Since $T$ is nearly weak uniformly Lipschitzian and $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ is a strictly increasing continuous function such that

$$\langle T^n x - T^n\rho, j(x-\rho)\rangle \le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n, \tag{3.5}$$

for $x \in K, \; \rho \in F(T)$, implying that

$$\Phi(\|x-\rho\|) \le k_n\|x-\rho\|^2 + L(\|x-\rho\| + a_n)\|x-\rho\| + \epsilon_n.$$

Taking limit of both sides as $n \to \infty$, we get

$$\Phi(\|x-\rho\|) \le (1+L)\|x-\rho\|^2.$$

If $\|x-\rho\| = 0 \ \forall n \in N$, then we are done. So, we assume $x_1 \neq Tx_1$ for some $x_1 \in K$ such that

$$\epsilon_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi)$$

and denote that $a_0 = \epsilon_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2$, $R(\Phi)$ is the range of $\Phi$. Indeed, if $\Phi(a) \to +\infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0,\infty]\} = a_1 < +\infty$ with $a_1 < a_0$, then for $\rho \in K$, there exists a sequence $\{u_n\}$ in $K$ such that $u_n \to \rho$ as $n \to \infty$ with $u_n \neq \rho$. Clearly, $Tu_n \to T\rho$ as $n \to \infty$ thus $\{u_n - Tu_n\}$ is a bounded sequence. Therefore, there exists a natural number $n_0$ such that

$$\epsilon_n + (k_n + L)\|u_n - \rho\|^2 + L\|u_n - \rho\|^2 < \frac{a_1}{2}$$

for $n \geq n_0$, then we redefine $x_1 = u_{n_0}$ and

$$\epsilon_n + (k_n + L)\|x_1 - \rho\|^2 + L\|x_1 - \rho\|^2 \in R(\Phi).$$

Step 1. We first prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Set $R = \Phi^{-1}(a_0)$, then from above (3.5), we obtain that $\|x_1 - \rho\| \leq R$.

Denote

$$B_1 = \{x \in K : \|x - \rho\| \leq R\}, \quad B_2 = \{x \in K : \|x - \rho\| \leq 2R\}. \tag{3.6}$$

Now, we want to prove that $x_n \in B_1$. If $n = 1$, then $x_1 \in B_1$. Now, assume that it holds for some $n$, that is, $x_n \in B_1$. Suppose that, it is not the case, then $\|x_{n+1} - \rho\| > R > \frac{R}{2}$. Since $\{a_n\} \in [0,\infty]$ with $a_n \to 0$ as $n \to \infty$, set $M = \sup\{a_n : n \in N\}$. Define $\tau_0 \in R^+$ by

$$\tau_0 = min\left\{1, \frac{R}{L(2R+M)}, \frac{R}{(L(2R+M)+R)}, \frac{\Phi(\frac{R}{2})}{32R^2}, \frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}, \frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}, \frac{\Phi(\frac{R}{2})}{8}\right\}. \tag{3.7}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} k_n = 1$. Without loss of generality, let $0 \leq \alpha_n, k_n - 1$, $\epsilon_n \leq \tau_0$ for any $n \geq 1$. We get

$$\begin{aligned}
\|x_{n+1} - \rho\| &\leq (1 - \alpha_n)\|x_n - \rho\| + \alpha_n\|T^n v_n - \rho\| \\
&\leq R + \tau_0 L(2R + M) \\
&\leq 2R, \\
\|x_{n+1} - x_n\| &\leq \alpha_n\|T^n v_n - x_n\| \\
&\leq \alpha_n(\|T^n v_n - \rho\| + \|x_n - \rho\|) \\
&\leq \tau_0(L(2R + M) + R), \\
\|v_n - x_{n+1}\| &\leq \|v_n - x_n\| + \alpha_n\|T^n v_n - x_n\| \\
&\leq \|v_n - x_n\| + \alpha_n(\|T^n v_n - \rho\| + \|x_n - \rho\|) \\
&\leq R + \tau_0(L(2R + M) + R).
\end{aligned} \tag{3.8}$$

Using Lemma 2.1 and the above estimates, we compute

$$
\begin{aligned}
\|x_{n+1}-\rho\|^2 \le\ & (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n v_n - x_n, j(x_{n+1}-\rho) > \\
=\ & (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1}-\rho) > \\
& + 2\alpha_n < x_{n+1} - x_n, j(x_{n+1}-\rho) > \\
& + 2\alpha_n < T^n v_n - T^n x_{n+1}, j(x_{n+1}-\rho) > \\
\le\ & \|x_n-\rho\|^2 + 2\alpha_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|) + \epsilon_n) \\
& - 2\alpha_n\|x_{n+1}-\rho\|^2 + 2\alpha_n L(\|v_n - x_{n+1}\| + a_n)\|x_{n+1}-\rho\| \\
& + 2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
\le\ & \|x_n-\rho\|^2 - 2\alpha_n\Phi(\frac{R}{2}) + 2\alpha_n\frac{\Phi(\frac{R}{2})}{32R^2}4R^2 + 2\alpha_n\frac{\Phi(\frac{R}{2})}{8} \\
& + 2\alpha_n L\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}2R[2(L(2R+M)+R)+M] \\
& + 2\alpha_n\frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}2R[L(2R+M)+R] \\
\le\ & \|x_n-\rho\|^2 - \alpha_n\Phi(\frac{R}{2}) \\
\le\ & R^2.
\end{aligned}
\tag{3.9}
$$

which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

Step 2. We want to prove that $\|x_n-\rho\| \to 0$ as $n \to \infty$. By step 1, we obtain that $\{\|x_n-\rho\|\}$ is a bounded sequence. Let $M_1 = \sup\{\|x_n-\rho\|\} + sup\{\|v_n-\rho\|\}$. Observe that

$$
\begin{aligned}
\|x_{n+1}-\rho\| \le\ & (1-\alpha_n)\|x_n-\rho\| + \alpha_n\|T^n v_n - \rho\| \\
\le\ & R + \tau_0 L(2R+M) \\
\le\ & 2R, \\
\|x_{n+1}-x_n\| \le\ & \alpha_n\|T^n v_n - x_n\| \\
\le\ & \alpha_n(\|T^n v_n-\rho\| + \|x_n-\rho\|) \\
\le\ & \alpha_n(L(\|v_n-\rho\| + a_n) + \|x_n-\rho\|) \\
\le\ & \alpha_n(L(M_1+a_n)+M_1), \\
\|v_n - x_{n+1}\| \le\ & \|v_n-x_n\| + \alpha_n\|T^n v_n - x_n\| \\
\le\ & \|v_n-x_n\| + \alpha_n(\|T^n v_n-\rho\| + \|x_n-\rho\|) \\
\le\ & \|v_n-x_n\| + \alpha_n(L(M_1+a_n)+M_1).
\end{aligned}
\tag{3.10}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty} k_n = 1$ and $\{x_n\}_{n=1}^{\infty}$ is bounded. From (3.9), we observed that

$$
\lim_{n\to\infty} L\|v_n - x_{n+1}\| = 0. \tag{3.11}
$$

So from (1.3), we have

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &\le (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n v_n - x_n, j(x_{n+1}-\rho) > \qquad (3.12)\\ &= (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1}-\rho) > \\ &\quad + < x_{n+1}-x_n, j(x_{n+1}-\rho) > \\ &\quad + < T^n v_n - T^n x_{n+1}, j(x_{n+1}-\rho) > \\ &\le \ \|x_n-\rho\|^2 + 2\alpha_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|) + \epsilon_n) \\ &\quad - 2\alpha_n\|x_{n+1}-\rho\|^2 + 2\alpha_n L(\|v_n - x_{n+1}\| + a_n)\|x_{n+1}-\rho\| \\ &\quad + 2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\ &\le \|x_n-\rho\|^2 + 2\alpha_n(k_n-1)M_1^2 \\ &\quad - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) + 2\alpha_n\epsilon_n \\ &\quad + 2\alpha_n L(\|v_n-x_n\| + \alpha_n(L(M_1+a_n)+M_1)) + a_n)M_1 \\ &\quad + 2\alpha_n(\alpha_n(L(M_1+a_n)+M_1))M_1 \\ &= \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) + o(\alpha_n), \end{aligned}$$

where

$$\begin{aligned} &2\alpha_n(k_n-1)M_1^2 + 2\alpha_n L(\|v_n-x_n\| + \alpha_n(L(M_1+a_n)+M_1)) + a_n)M_1 \\ &+2\alpha_n(\alpha_n(L(M_1+a_n)+M_1))M_1 + 2\alpha_n\epsilon_n \\ &= o(\alpha_n). \end{aligned}$$

Thus, by Lemma 2.2, we obtain that $\lim_{n\to\infty}\|x_n-\rho\| = 0$. This completes the proof. □

From Theorem 3.1, we have the following corollary.

**Corollary 3.1.** *Let $K$ be a nonempty convex subset of a real Banach space $X$. Let $T : K \to K$ be a nearly weak uniformly $L$-Lipschitzian mapping with sequence $\{a_n\}$ as defined in equation* (1.2). *Let $\{\epsilon_n\} \in (0,1)$ and $\{k_n\} \subset [1,\infty)$ be sequences with $lim_{n\to\infty}\epsilon_n = 0$ and $lim_{n\to\infty}k_n = 1$. For some $x_0 \in K$, define the modified Mann iterative sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ \ n \ge 1,$$

*where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that (i)$\sum_{n\ge1}\alpha_n = \infty$ (ii) $\lim_{n\to\infty}\alpha_n = 0$. Then, there exists $\tau_0 > 0$ such that $\alpha_n \le \tau_0\ \forall n \ge n_0$, for some $n_0 \in N$. Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that*

$$< T^n x - T^n\rho, j(x-\rho) > \le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

*for all $x \in K, \rho \in F(T)$. Then $\{x_n\}_{n\ge0}$ converges strongly to the unique fixed point of $T$.*

*Proof.* By letting $v_n = x_n$ in Theorem 3.1, we get the convergence of modified Mann iteration (1.4). □

**Remark 3.3.** Unlike as in several existing results in the literature (see [2, 3, 6], [10]-[16]), Theorem 3.1 and Corollary 3.1 are applicable for any nearly weak uniformly $L$-Lipschitzian mapping. It is also well known that whenever a theorem is proved using Mann-type iteration (without error term), the method of proof follows easily to the case of Mann-type iteration process with error term.
Now, we give an example to support practical application of our main result.

**Example 3.2.** Let $X = R$ be the set of real numbers with the usual norm, $K = [0,\infty)$ and $T : K \to K$ be a mapping define by

$$Tx = \frac{x^2}{1+x^3}, \forall x \in K.$$

It is easy to show that $T$ is nearly weak uniformly $L$-Lipschitzian with sequence $\{a_n\}$ having fixed point $\rho = 0$ and strictly monotonic increasing. Observe that for any $x \in K$, $T^n x \leq T^{n-1}x \leq ... \leq Tx$. Let define a function $\Phi : [0, \infty) \to [0, \infty)$ by $\Phi(t) = \frac{t^2}{1+t^2}$. Then $\Phi$ is a strictly increasing continuous function with $\Phi(0) = 0$. For all $x \in K$ and $\rho \in F(T)$, set $k_n = 1 + \frac{1}{n}$, $\epsilon_n = \frac{1}{1+n}$, $a_n = \frac{1}{2^n}$ and $L \geq 1$, then we get

$$\begin{aligned} < T^n x - T^n 0, j(x-0) > &\leq < Tx - 0, j(x-0) > \\ &\leq |x-0|^2 - \Phi(|x-0|) + 0 \\ &\leq k_n|x-0|^2 - \Phi(|x-0|) + \epsilon_n, \forall n \geq 1 \end{aligned}$$

and

$$\begin{aligned} |T^n x - T^n 0| &\leq |Tx - 0| \\ &\leq L(|x-0| + 0) \\ &\leq L(|x-0| + a_n), \forall n \geq 1. \end{aligned}$$

Clearly, $T$ is nearly weak uniformly $L$-Lipschitzian and applicable to Theorem 3.1 and Corollary 3.1.

**Prototype**. An example of our control sequence $\alpha_n$ in our Theorem 3.1 and Corollary 3.1 is $\alpha_n = \frac{1}{1+n}$.

## 4. The equivalence of stability between modified Mann-type and modified Mann iterations

The following definition is well known (see [7]).

**Definition 4.4.** Let $X$ be a real Banach space. Suppose that $F(T)$, the fixed point set of $T$, is nonempty and that the sequence $\{x_n\}$ converges to a point $\rho \in F(T)$.
(i). Let $\{y_n\} \subset X$, and define $\epsilon_n = \|y_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n f(y_n)\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = \rho$, then the modified Mann-type iteration scheme (1.3) is said to be $T$-stable or stable with respect to $T$.
(ii). Let $\{y_n\} \subset X$, and define $\epsilon_n = \|y_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = \rho$, then the modified Mann iteration scheme (1.4) is said to be $T$-stable or stable with respect to $T$.
According to Definition 4.4, we shall prove that the modified Mann-type and modified Mann iterations are equivalent for a nearly weak uniformly $L$-Lipschitzian map:

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| = 0 \implies \lim_{n\to\infty} u_n = \rho. \tag{4.13}$$

$$\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| = 0 \implies \lim_{n\to\infty} x_n = \rho. \tag{4.14}$$

**Theorem 4.2.** *Let $X$ be a real Banach space . Let $T$ be a nearly weak uniformly $L$-Lipschitzian self mapping with sequence $\{a_n\}$ as defined in equation* (1.2), *$\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, then the following are equivalent:*
*(i). The modified Mann-type iteration is $T$ stable.*
*(ii). The modified Mann iteration is $T$ stable.*

*Proof.* Firstly, we prove that (4.13) $\implies$ (4.14).
Suppose $\lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| = 0$, then

$$\begin{aligned} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| &\leq \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| + \|\alpha_n T^n u_n - \alpha_n T^n v_n\| \\ &\leq \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| + \alpha_n L(\|u_n - v_n\| + a_n) \\ &\to 0. \end{aligned}$$

So according to (4.13), we have $\lim_{n\to\infty} u_n = \rho$.
Conversely, we prove that (4.14) $\implies$ (4.13).
Suppose $\lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| = 0$, then

$$\begin{aligned}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| &\leq \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| + \|\alpha_n T^n v_n - \alpha_n T^n x_n\| \\ &\leq \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| + \alpha_n L(\|v_n - x_n\| + a_n) \\ &\to 0.\end{aligned}$$

□

By (4.14), we obtain $\lim\limits_{n\to\infty} x_n = \rho$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LAGOS
LAGOS, NIGERIA
*E-mail address*: amogbademu@unilag.edu.ng