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Sequences interpolating some geometric inequalities

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ABSTRACT. Using the geometric dynamic of an iterative process (Theorem 2.1), we obtain refinements to some famous geometric inequalities in a triangle by constructing interpolating sequences.

1. INTRODUCTION

For the inequality $a \leq b$, where a, b are real numbers, an increasing sequence $(u_n)_{n\geq 0}$ with the property $a = u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq b$ and $u_n \to b$, is called *increasing interpolating sequence*. Similarly, a decreasing sequence $(v_n)_{n\geq 0}$ satisfying $a \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0 = b$ and $v_n \to a$, is a *decreasing interpolating sequence* for the inequality $a \leq b$.

In this paper we will construct interpolating sequences for some geometric inequalities in a triangle due to Euler, Mitrinović, Weitzenböck, Gordon, Curry, Finsler-Hadwiger, Pólya-Szegö, and Chen.

2. AN USEFUL RESULT ABOUT ITERATIONS IN THE CIRCUMCIRCLE

Let ABC be a triangle with the angles A, B, C measured in radians, with the length-sides a, b, c, the circumradius R, the inradius r, the semiperimeter s, and the area K. For the fixed nonnegative real numbers x, y, z with $x + y + z = 1$, define recursively the sequences $(A_n)_{n\geq 0}, (B_n)_{n\geq 0}, (C_n)_{n\geq 0}$ by $A_{n+1} = xA_n + yB_n + zC_n, B_{n+1} = zA_n + xB_n + yC_n, A_{n+1} = yA_n + zB_n + xC_n, A_0 = A, B_0 = B, C_0 = C, n = 0, 1, \ldots$. Note that $A_n, B_n, C_n > 0$ and $A_n + B_n + C_n = \pi, n = 0, 1, \ldots$ Therefore, we can consider the triangle $A_nB_nC_n$ with the angles A_n, B_n, C_n and having the same circumcircle as triangle ABC, $n = 1, 2 \ldots$. Denote this triangle by $\mathcal{T}_n$. Let $a_n, b_n, c_n, R, r_n, s_n, K_n$ be the length-sides, the circumradius, the inradius, the semiperimeter, and the area of the triangle $\mathcal{T}_n$, respectively.

Let us mention that such recursive systems describing some dynamic geometries are considered by S. Abbot [1], G. Z. Chang and P. J. Davis [6], R. J. Clarke [9], J. Ding, L. R. Hitt and X-M. Zhang [11], L. R. Hitt and X-M. Zhang [14], and D. Ismailescu and J. Jacobs [15].

The first result is contained in the following theorem.

Theorem 2.1. *With the above notations, if at most one of x, y, z is equal to 0, then the sequences $(A_n)_{n\geq 0}, (B_n)_{n\geq 0}, (C_n)_{n\geq 0}$ are convergent and*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}. \tag{2.1}$$

Proof. It is easy to see that the following matrix relations hold

$$\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = U^n \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \tag{2.2}$$

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where U is the circulant matrix given by

$$U = \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}. \tag{2.3}$$

A simple induction argument shows that

$$U^n = \begin{pmatrix} x_n & y_n & z_n \\ z_n & x_n & y_n \\ y_n & z_n & x_n \end{pmatrix}, \tag{2.4}$$

where the sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, (z_n)_{n\geq 1}$ verify the recursive relations $x_{n+1} = xx_n + yy_n + zz_n, y_{n+1} = zx_n + xy_n + yz_n, z_{n+1} = yx_n + zy_n + xz_n, x_1 = x, y_1 = y, z_1 = z, n = 1, 2 \ldots$. Summing down these relations we obtain $x_{n+1} + y_{n+1} + z_{n+1} = x_n + y_n + z_n$, hence the sequence $(x_n + y_n + z_n)_{n\geq 1}$ is constant and equal to 1.

On the other hand, the characteristic polynomial of the matrix U is

$$f_U(t) = (t - x - y - z)(t^2 + (y + z - 2x)t + x^2 + y^2 + z^2 - xy - yz - zx). \tag{2.5}$$

The hypothesis $x + y + z = 1$ implies that the roots of the polynomial f_U are $t_1 = 1, t_2 = \alpha, t_3 = \bar{\alpha}$, where $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $|\alpha| < 1$. It follows that we have

$$U = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \bar{\alpha} \end{pmatrix} P^{-1}, \tag{2.6}$$

for some nonsingular matrix P. Therefore, we obtain

$$U^n = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^n & 0 \\ 0 & 0 & \bar{\alpha}^n \end{pmatrix} P^{-1}, \tag{2.7}$$

and we get

$$x_n = a + b\alpha^n + c\bar{\alpha}^n, y_n = a' + b'\alpha^n + c'\bar{\alpha}^n, z_n = a'' + b''\alpha^n + c''\bar{\alpha}^n, n = 1, 2 \ldots$$

for some fixed real numbers $a, b, c, a', b', c', a'', b'', c''$ determined by the initial conditions in the definition of the sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, (z_n)_{n\geq 1}$. Because $\lim_{n\to\infty} \alpha^n = \lim_{n\to\infty} \bar{\alpha}^n = 0$, from the above formulas it follows that the sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, $(z_n)_{n\geq 1}$ are convergent and

$$\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} y_n = a', \lim_{n\to\infty} z_n = a''.$$

From $x_n + y_n + z_n = 1, n = 1, 2 \ldots$, we obtain $a + a' + a'' = 1$. On the other hand, the relation (2.7) shows that the eingenvalues of the matrix U^n are $1, \alpha^n, \bar{\alpha}^n$, that is the characteristic polynomial f_{U^n} of the matrix U^n is

$$f_{U^n}(t) = (t - x_n - y_n - z_n)(t^2 + (y_n + x_n - 2x_n)t + x_n^2 + y_n^2 + z_n^2 - x_n y_n - y_n z_n - z_n x_n)$$

and from the Vieta's relations we have $x_n^2 + y_n^2 + z_n^2 - x_n y_n - y_n z_n - z_n x_n = |\alpha|^{2n}$. When $n \to \infty$, we obtain the relation $a^2 + (a')^2 + (a'')^2 - aa' - a'a'' - a''a = 0$, i.e. $(a - a')^2 + (a' - a'')^2 + (a'' - a)^2 = 0$. Therefore $a = a' = a'' = \frac{1}{3}$, and the desired result follows from the relation (2).

□

We will illustrate the above general iterative process by considering the following special geometric situation also studied by S. Abbot [1] and D. Şt. Marinescu, M. Monea, M. Opincariu and M. Stroe [18]. Recall that, if P is a point in the plane of the triangle ABC, the circumcevian triangle of P with respect to ABC is the triangle defined by the intersections of the Cevians AP, BP, CP with the circumcircle of ABC. We consider $A_1B_1C_1$ to be the circumcevian triangle of the incenter I of ABC, i.e. the circumcircle mid-arc triangle of ABC. In this case we have

$$A_1 = \frac{1}{2}(B+C), B_1 = \frac{1}{2}(C+A), C_1 = \frac{1}{2}(A+B),$$

that is in the general iterative process we have $x = 0, y = \frac{1}{2}, y = \frac{1}{2}$. On the other hand, because $A + B + C = \pi$, we have

$$A_1 = \frac{1}{2}(\pi - A), B_1 = \frac{1}{2}(\pi - B), C_1 = \frac{1}{2}(\pi - C)$$

Define recursively the sequence of triangles $\mathcal{T}_n$ as follows : $\mathcal{T}_{n+1}$ is the circumcircle mid-arc triangle with respect to the incenter of $\mathcal{T}_n$, and $\mathcal{T}_0$ is the triangle ABC. The angles of triangles $\mathcal{T}_n$ are given by the recurrence relations $A_{n+1} = \frac{1}{2}(\pi - A_n), B_{n+1} = \frac{1}{2}(\pi - B_n), C_{n+1} = \frac{1}{2}(\pi - C_n)$, where $A_0 = A, B_0 = B, C_0 = C$. Solving these recurrences we get

$$A_n = \left(-\frac{1}{2}\right)^n A + \frac{\pi}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right),$$
$$B_n = \left(-\frac{1}{2}\right)^n B + \frac{\pi}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right),$$
$$C_n = \left(-\frac{1}{2}\right)^n C + \frac{\pi}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right),$$

and the conclusion in Theorem 2.1 is obviously verified.

3. AN INTERPOLATING SEQUENCE FOR EULER'S INEQUALITY $R \geq 2r$

The Euler's inequality is a central result in triangle geometry (see T. Andreescu, O. Mushkarov and L. Stoyanov [3], D. Andrica [4], D. S. Mitrinovic, J. Pecaric and V. Volonec [20], and G. Popescu, I. V. Maftei, J. L. Diaz-Barrero and M. Dincă [22]). It is a direct consequence of Blundon's inequality and it has numerous and various refinements (see for instance D. Andrica [4], D. Andrica and D. Şt. Marinescu [5], and D. S. Mitrinovic, J. Pecaric and V. Volonec [20]). In this section we use the result in Theorem 2.1 to construct an increasing interpolating sequence for the Euler's inequality.

Theorem 3.2. With the above notations the sequence of inradii $(r_n)_{n\geq 0}$ is increasing and we have

$$\lim_{n\to\infty} r_n = \frac{R}{2}. \tag{3.8}$$

Proof. Using the known formula $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ we have to prove the following the following inequality

$$\sin\frac{A_{n+1}}{2}\sin\frac{B_{n+1}}{2}\sin\frac{C_{n+1}}{2} \geq \sin\frac{A_n}{2}\sin\frac{B_n}{2}\sin\frac{C_n}{2}. \tag{3.9}$$

Denote $\frac{A_n}{2} = u$, $\frac{B_n}{2} = v$, $\frac{C_n}{2} = t$ and the inequality (3.8) is equivalent to

$$\sin(xu + yv + zt)\sin(zu + xv + yt)\sin(yu + zv + xt) \geq \sin u \sin v \sin t. \tag{3.10}$$

To prove the inequality (3.10), let us consider the function $f : (0, \pi) \to \mathbb{R}$, defined by $f(s) = \ln \sin s$, which is concave on the interval $(0, \pi)$. Applying the Jensen's inequality we get the inequalities

$$f(xu + yv + zt) \geq xf(u) + yf(v) + zf(t),$$

$$f(zu + xv + yt) \geq zf(u) + xf(v) + yf(t),$$
$$f(yu + zv + xt) \geq yf(u) + zf(v) + xf(t).$$

Summing these inequalities and using the relation $x + y + z = 1$, the inequality (3.10) follows.

From relation

$$\frac{r_n}{R} = 4 \sin\frac{A_n}{2} \sin\frac{B_n}{2} \sin\frac{C_n}{2}$$

and from (1) we obtain the limit (3.8).

□

Corollary 3.1. With the above notations, the sequence of inradii $(r_n)_{n\geq 0}$ is an increasing interpolating sequence for the Euler's inequality, i.e. we have the inequalities

$$r = r_0 \leq r_1 \leq \ldots \leq r_n \leq \ldots \leq \frac{R}{2}. \tag{3.11}$$

4. An interpolating sequence for Mitrinović's inequality $s \leq \frac{3\sqrt{3}}{2}R$

The inequality $s \leq \frac{3\sqrt{3}}{2}R$ is known in the literature as Mitrinović's inequality. It is a simple consequence of Blundon's inequality but also there are different direct proofs. It has as a counterpart the inequality $3\sqrt{3}r \leq s$. Combining these two inequalities, we obtain a refinement to Euler's $R \geq 2r$:

$$3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}}{2}R.$$

In what follows we use the result in Theorem 2.1 to construct an increasing interpolating sequence for the Mitrinović's inequality and a decreasing interpolating sequence for its counterpart.

Theorem 4.3. 1) With the above notations, the sequence of semiperimeters $(s_n)_{n\geq 0}$ is increasing and we have

$$\lim_{n\to\infty} s_n = \frac{3\sqrt{3}}{2}R. \tag{4.12}$$

2) The sequence $(\frac{s_n}{r_n})_{n\geq 0}$ is decreasing and

$$\lim_{n\to\infty} \frac{s_n}{r_n} = 3\sqrt{3}. \tag{4.13}$$

Proof. 1) We have $s_n = R(\sin A_n + \sin B_n + \sin C_n)$ and the function $g : (0, \pi) \to \mathbb{R}$, $g(u) = \sin u$ is concave on the interval $(0, \pi)$. From the Jensen's inequality we obtain $\sin A_{n+1} = \sin(xA_n + yB_n + zC_n) \geq x \sin A_n + y \sin B_n + z \sin C_n$. Similarly, we get other two inequalities $\sin B_{n+1} \geq z \sin A_n + x \sin B_n + y \sin C_n$ and $\sin C_{n+1} \geq y \sin A_n + z \sin B_n + x \sin C_n$. Summing up these inequalities it follows $s_n \leq s_{n+1}$. The relation $\lim_{n\to\infty} s_n = \frac{3\sqrt{3}}{2}R$ follows from Theorem 2.1.

2) From the relation $\cot\frac{A_n}{2} = \frac{s_n - a_n}{r_n}$ and the other two, we obtain

$$\cot\frac{A_n}{2} + \cot\frac{B_n}{2} + \cot\frac{C_n}{2} = \frac{3s_n - 2s_n}{r_n} = \frac{s_n}{r_n}.$$

Because the function $h(u) = \cot u$ is convex on the interval $(0, \frac{\pi}{2})$, with a similar argument as in the proof of part 1), it follows

$$\cot\frac{A_{n+1}}{2} + \cot\frac{B_{n+1}}{2} + \cot\frac{C_{n+1}}{2} \leq \cot\frac{A_n}{2} + \cot\frac{B_n}{2} + \cot\frac{C_n}{2},$$

that is $\frac{s_{n+1}}{r_{n+1}} \leq \frac{s_n}{r_n}$. The limit $\lim_{n\to\infty} \frac{s_n}{r_n} = 3\sqrt{3}$ follows from Theorem 2.1. □

Corollary 4.2. 1) With the above notations, the sequence of semiperimeters $(s_n)_{n\geq 0}$ is an increasing interpolating sequence for the Mitrinović's inequality, i.e. we have

$$s = s_0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq \frac{3\sqrt{3}}{2}R. \tag{4.14}$$

2) The sequence $(\frac{s_n}{r_n})_{n\geq 0}$ is a decreasing interpolating sequence for the counterpart of Mitrinović's inequality, i.e. we have

$$3\sqrt{3} \leq \ldots \leq \frac{s_n}{r_n} \leq \ldots \leq \frac{s_1}{r_1} \leq \frac{s_0}{r_0} = \frac{s}{r}. \tag{4.15}$$

5. Interpolating Weitzenböck's inequality

In a triangle ABC, the Weitzenböck's inequality [25] is

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}K, \tag{5.16}$$

where a, b, c are the length of the sides of the triangle and K denotes the area of ABC. To construct an interpolating sequence for (5.16) we use the special case $x = 0, y = \frac{1}{2}, y = \frac{1}{2}$ in the general iterative process described in Section 2. Considering the sequence $(u_n)_{n\geq 0}$, where

$$u_n = \frac{a_n^2 + b_n^2 + c_n^2}{4K_n}, n = 0, 1, \ldots$$

we obtain the following result :

Theorem 5.4. With the above notations, the sequence $(u_n)_{n\geq 0}$ is decreasing and we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{a_n^2 + b_n^2 + c_n^2}{4K_n} = \sqrt{3}. \tag{5.17}$$

Proof. Clearly, we have

$$u_n = \frac{4R^2(\sin^2 A_n + \sin^2 B_n + \sin^2 C_n)}{8R^2 \sin A_n \sin B_n \sin C_n} = \frac{1}{2}\left(\frac{\sin A_n}{\sin B_n \sin C_n} + \frac{\sin B_n}{\sin C_n \sin A_n} + \frac{\sin C_n}{\sin A_n \sin B_n}\right) =$$

$$\frac{1}{2}(\cot B_n + \cot C_n + \cot C_n + \cot A_n + \cot A_n + \cot B_n) = \cot A_n + \cot B_n + \cot C_n.$$

It is easy to show that the inequality $2\cot\frac{x+y}{2} \leq \cot x + \cot y$ holds for every $x, y \in (0, \pi)$ with $x + y < \pi$. Applying this property and using the recursive relations in the process, it follows

$$u_{n+1} = \cot A_{n+1} + \cot B_{n+1} + \cot C_{n+1} \leq \cot A_n + \cot B_n + \cot C_n = u_n,$$

that is the sequence is decreasing. Because $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$, we obtain immediately the limit (5.17).

□

Corollary 5.3. With the above notations, the sequence $(u_n)_{n\geq 0}$ is a decreasing interpolating sequence for the Weitzenböck's inequality, i.e. we have

$$\sqrt{3} \leq \ldots \leq u_n \leq \ldots \leq u_1 \leq u_0 = \frac{a^2 + b^2 + c^2}{4K}. \tag{5.18}$$

6. Interpolating Gordon's inequality

In a triangle ABC the following inequality holds

$$ab + bc + ca \geq 4\sqrt{3}K, \tag{6.19}$$

where a, b, c are the length of the sides of the triangle and K denotes the area of ABC, and it is known as Gordon's inequality [12]. Denote by a_n, b_n, c_n, K_n the length of the sides and the area of the triangle $\mathcal{T}_n, n = 0, 1, 2, \ldots$, where $a_0 = a, b_0 = b, c_0 = c$. Consider the sequence $(t_n)_{n\geq 0}$, where

$$t_n = \frac{a_n b_n + b_n c_n + c_n a_n}{K_n}, n = 0, 1, \ldots$$

Theorem 6.5. With the above notations, the sequence $(t_n)_{n\geq 0}$ is decreasing and we have

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{a_n b_n + b_n c_n + c_n a_n}{K_n} = 4\sqrt{3}. \tag{6.20}$$

Proof. Using the area formula for triangle $\mathcal{T}_n$, we have

$$\frac{a_n b_n + b_n c_n + c_n a_n}{K_n} = 2\left(\frac{1}{\sin A_n} + \frac{1}{\sin B_n} + \frac{1}{\sin C_n}\right).$$

Consequently, the property is equivalent to

$$\frac{1}{\sin A_n} + \frac{1}{\sin B_n} + \frac{1}{\sin C_n} \geq \frac{1}{\sin A_{n+1}} + \frac{1}{\sin B_{n+1}} + \frac{1}{\sin C_{n+1}}.$$

The function $f : (0, \pi) \to \mathbb{R}$, defined by $f(s) = \frac{1}{\sin s}$, is convex on the interval $(0, \pi)$. With a similar argument as in Theorem 3.2, the conclusion follows.

Because $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$, we obtain the limit (6.20) from the first relation. □

Corollary 6.4. With the above notations, the sequence $(t_n)_{n\geq 0}$ is an decreasing interpolating sequence for the Weitzenböck's inequality, i.e. we have

$$4\sqrt{3} \leq \ldots \leq t_n \leq \ldots \leq t_1 \leq t_0 = \frac{ab + bc + ca}{K}. \tag{6.21}$$

7. Interpolating Curry's inequality

Curry's inequality [10] is :

$$4\sqrt{3}K \leq \frac{9abc}{a + b + c}, \tag{7.22}$$

and it is an improvement to inequality (6.19).

Clearly, putting together these inequalities, we obtain the following interpolating inequalities to Weitzenböck inequality :

$$4\sqrt{3}K \leq \frac{9abc}{a + b + c} \leq ab + bc + ca \leq a^2 + b^2 + c^2. \tag{7.23}$$

With the above notations, consider the sequence $(v_n)_{n\geq 0}$, where

$$v_n = \frac{9a_n b_n c_n}{4K_n(a_n + b_n + c_n)}, n = 0, 1, \ldots$$

Theorem 7.6. The sequence $(v_n)_{n\geq 0}$ is decreasing and we have

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{9a_nb_nc_n}{4K_n(a_n+b_n+c_n)} = \sqrt{3}. \tag{7.24}$$

Proof. Because the relation

$$v_n = \frac{9R}{a_n+b_n+c_n} = \frac{9}{2(\sin A_n + \sin B_n + \sin C_n)},$$

and $\sin A_{n+1} = \sin(xA_n + yB_n + zC_n) \geq x\sin A_n + y\sin B_n + z\sin C_n$, $\sin B_{n+1} \geq z\sin A_n + x\sin B_n + y\sin C_n$, and $\sin C_{n+1} \geq y\sin A_n + z\sin B_n + x\sin C_n$, we get immediately the inequality $v_{n+1} \leq v_n$. The limit follows from Theorem 2.1. □

Corollary 7.5. With the above notations, the sequence $(v_n)_{n\geq 0}$ is a decreasing interpolating sequence for the Curry's inequality, i.e. we have

$$\sqrt{3} \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0 = \frac{9abc}{4K(a+b+c)}. \tag{7.25}$$

8. Interpolating Finsler-Hawiger inequality

It is well-known that in every triangle ABC the following inequality holds

$$a^2+b^2+c^2 \geq 4\sqrt{3}K + (a-b)^2 + (b-c)^2 + (c-a)^2 \tag{8.26}$$

where a, b, c are the length of the sides of the triangle and K denotes the area of ABC. Clearly, it is a direct improvement to the Weitzenböck's inequality (5.16). The inequality (8.26) is known as the Finsler-Hawiger inequality [24], [13], and it is equivalent to

$$ab+bc+ca \geq 4\sqrt{3}K + \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2].$$

Therefore, it is also a strong improvement to the Gordon's inequality (6.19). The inequality (8.26) was intensively investigated by many authors ; C. Alsina and R. Nelsen [2], A. Cipu [9], C. Lupu and C. Pohoaţă [16], C. Lupu, C. Mateescu, V. Matei and M. Opincariu [17], and D. Şt. Marinescu, M. Monea, M. Opincariu and M. Stroe [19].

In what follows we use the result in Theorem 2.1 to construct a decreasing interpolating sequence for the Finsler-Hawiger inequality. With the above notations, consider the sequence $(w_n)_{n\geq 0}$, where

$$w_n = \frac{a_n^2+b_n^2+c_n^2-(a_n-b_n)^2-(b_n-c_n)^2-(c_n-a_n)^2}{4K_n}, n = 0, 1, \ldots$$

Theorem 8.7. The sequence $(w_n)_{n\geq 0}$ is decreasing and we have

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{a_n^2+b_n^2+c_n^2-(a_n-b_n)^2-(b_n-c_n)^2-(c_n-a_n)^2}{4K_n} = \sqrt{3}. \tag{8.27}$$

Proof. Firstly, we will prove the following relation

$$\frac{a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2}{4K} = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}. \tag{8.28}$$

Indeed, starting with the left hand side, we have

$$\frac{2ab+2bc+2ca-a^2-b^2-c^2}{4K} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} - \frac{\sin A}{2\sin B\sin C} - \frac{\sin B}{2\sin C\sin A}$$

$$-\frac{\sin C}{2\sin A\sin B} = \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} - \frac{1}{2}(\cot B + \cot C + \cot C + \cot A + \cot A + \cot B) =$$

$$\frac{1}{\sin A} - \cot A + \frac{1}{\sin B} - \cot B + \frac{1}{\sin C} - \cot C = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2},$$

since

$$\frac{1}{\sin A} - \cot A = \frac{1-\cos A}{\sin A} = \frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \tan\frac{A}{2}.$$

Now, to prove that the sequence $(w_n)_{n\geq 0}$ is decreasing we are using relation (8.28) to obtain

$$w_{n+1} = \tan\frac{A_{n+1}}{2} + \tan\frac{B_{n+1}}{2} + \tan\frac{C_{n+1}}{2}.$$

The function $f : (0, \frac{\pi}{2}) \to \mathbb{R}$, defined by $f(s) = \tan s$, is convex on the interval $(0, \frac{\pi}{2})$. With a similar argument as in Theorem 2, we obtain the inequality $w_{n+1} \leq w_n$ and the conclusion follows.

Because $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$ we obtain the limit (8.27) from the first relation.

□

Corollary 8.6. With the above notations, the sequence $(w_n)_{n\geq 0}$ is a decreasing interpolating sequence for the Finsler-Hadwiger inequality, i.e. we have

$$\sqrt{3} \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = \frac{a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2}{4K}. \tag{8.29}$$

9. Interpolating Pólya-Szegö inequality

Recall that in every triangle ABC the following inequality is due by Pólya and Szegö [21] :

$$K \leq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}. \tag{9.30}$$

Using our method we can construct an interpolating sequence for (21).

Theorem 9.8. With the above notations, the sequence $(\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}})_{n\geq 0}$ is increasing and we have

$$\lim_{n\to\infty} \frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}} = \frac{\sqrt{3}}{4}. \tag{9.31}$$

Proof. Using the area formula for triangle $A_nB_nC_n$, we have

$$\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}} = \frac{(a_nb_nc_n)^{\frac{1}{3}}}{4R_n} = \frac{\sqrt[3]{\sin A_n \sin B_n \sin C_n}}{2},$$

and the property is equivalent to

$$\sin A_n \sin B_n \sin C_n \leq \sin A_{n+1} \sin B_{n+1} \sin C_{n+1}.$$

From the argument in the proof of Theorem 3.2 for the concave function $f : (0, \pi) \to \mathbb{R}$ defined by $f(s) = \ln \sin s$, the conclusion follows. The limit (16) follows from the first relation and the limits $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$. □

Corollary 9.7. With the above notations, the sequence $(\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}})_{n\geq 0}$ is an increasing interpolating sequence for the Pólya-Szegö inequality, i.e. we have

$$\frac{K}{(abc)^{\frac{2}{3}}} \leq \frac{K_1}{(a_1b_1c_1)^{\frac{2}{3}}} \leq \ldots \leq \frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}} \leq \ldots \frac{\sqrt{3}}{4}. \tag{9.32}$$

10. Interpolating Chen's inequality

In the recent paper of Y-D. Wu, V. Lokesha and H. M. Srivastava [26] it is presented a refinement of inequality (5.16) in the form

$$K \leq \frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}(\frac{2r}{R})^{\frac{1}{3}}, \tag{10.33}$$

mentioning that this inequality is due by S.-L. Chen [7]. Using our method, we can obtain an interpolating sequence to this result.

Theorem 10.9. With the above notations, the sequence $(\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}}(\frac{2r_n}{R_n})^{\frac{1}{3}})_{n\geq 0}$ is increasing and we have

$$\lim_{n\to\infty} \frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}}(\frac{2r_n}{R_n})^{\frac{1}{3}}) = \frac{\sqrt{3}}{4}. \tag{10.34}$$

Proof. The property does not result from the monotony of the two sequences because they are of the opposite monotony. Observe that

$$\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}}(\frac{2r_n}{R_n})^{\frac{1}{3}} = \frac{\sqrt[3]{\sin A_n \sin B_n \sin C_n}}{2} \cdot \frac{1}{\sqrt[3]{8 \sin \frac{A_n}{2} \sin \frac{B_n}{2} \sin \frac{C_n}{2}}} =$$

$$= \frac{\sqrt[3]{\cos \frac{A_n}{2} \cos \frac{B_n}{2} \cos \frac{C_n}{2}}}{2}.$$

The function $f : (0, \pi/2) \to \mathbb{R}$ defined by $f(s) = \ln \cos s$ is concave, and we obtain as in the proof of Theorem 2, the inequality

$$\ln \cos \frac{A_{n+1}}{2} + \cos \frac{B_{n+1}}{2} + \cos \frac{C_{n+1}}{2} \geq \ln \cos \frac{A_n}{2} + \cos \frac{B_n}{2} + \cos \frac{C_n}{2}.$$

The limit (10.34) can be obtained by using the relations $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$. □

Corollary 10.8. The sequence $(\frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}}(\frac{2r_n}{R_n})^{\frac{1}{3}})_{n\geq 0}$ is an increasing interpolating sequence for the Chen's inequality, i.e. we have

$$\frac{K}{(abc)^{\frac{2}{3}}}(\frac{2r}{R})^{\frac{1}{3}} \leq \frac{K_1}{(a_1b_1c_1)^{\frac{2}{3}}}(\frac{2r_1}{R_1})^{\frac{1}{3}} \leq \ldots \leq \frac{K_n}{(a_nb_nc_n)^{\frac{2}{3}}}(\frac{2r_n}{R_n})^{\frac{1}{3}} \leq \ldots \leq \frac{\sqrt{3}}{4}. \tag{10.35}$$

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