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On the computation of the antiderivatives on $\mathbb{R}$ of a class of continuous periodic functions

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ABSTRACT. In this paper we are concerned with the computation of the antiderivatives on $\mathbb{R}$ of a special class of continuous periodic functions. Finally, some applications of the main result are presented.

1. INTRODUCTION

Let $a \in \mathbb{R}, a > 1$ be given and let $f_a : \mathbb{R} \to \mathbb{R}$ be defined for any $x \in \mathbb{R}$ by

$$f_a(x) = \frac{1}{a + \cos x}. \tag{1.1}$$

Clearly f_a is continuous on $\mathbb{R}$ and consequently it possesses antiderivatives on $\mathbb{R}$.

We note that in the particular case of f_3, the following problem has been proposed (as Problem 16) at the National Entrance Exam to Romanian Technical Universities in July 1988, and has been published in *Gazeta Matematică* no. 11-12 / 1988, page 452:

Problem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \frac{1}{3 + \cos x}, x \in \mathbb{R}.$$

- *Find the antiderivatives of f on $[0, \pi)$;*
- *Find the antiderivatives of f on $[0, 2\pi]$ and compute $\int_0^{2\pi} f(x)dx$.*

A similar problem to Problem 1.1, for the function

$$f(x) = \frac{1}{3 + \sin x + \cos x}, x \in \mathbb{R},$$

has been proposed in 1983 to the Romanian National Olympiad by I. Bârză, see [13].

Note also that Problem 1.1 was the source of many elementary and non elementary developments that were performed by the second author, see [2], [3], [4], [5], [6].

Coming back to the general form (1.1), let us denote by T the following indefinite integral

$$T = \int \frac{dx}{a + \cos x}. \tag{1.2}$$

If $x \in [0, \pi)$, the change of variables $t = \tan \frac{x}{2}$ leads by routine computations to

$$T = \frac{2}{\sqrt{a^2 - 1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan \frac{x}{2}\right) + C, \tag{1.3}$$

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and so, we conclude that the function $G_a : [0, \pi) \to \mathbb{R}$, given by

$$G_a(x) = \frac{2}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) + C, \tag{1.4}$$

is an antiderivative of f_a on $[0, \pi)$, for any constant $C \in \mathbb{R}$.

If we want to compute an antiderivative of f_a on the interval $x \in [0, 2\pi]$, then the problem is more complicated because the function $t = \tan\frac{x}{2}$ is not defined at $x = \pi$. For the particular case of f_3, this problem is solved, for example, in [5] and [6], in Chapter 16.

By using (1.4), we see that an antiderivative of f_a on $[0, 2\pi]$ will have the form

$$F_a(x) = \begin{cases} \frac{2}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) + C_1, & x \in [0, \pi]; \\ C \qquad , & x = \pi; \\ \frac{2}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) + C_2, & x \in [\pi, 2\pi], \end{cases} \tag{1.5}$$

with appropriate values for the constants C, C_2 and C_2.

But in order to be an antiderivative, F_a has to be differentiable, hence continuous on $[0, 2\pi]$. By imposing the continuity of F_a at $x = \pi$, we deduce that the constants C, C_2 and C_2 are related by the following relations

$$-\frac{\pi}{2} + C_1 = C = \frac{\pi}{2} + C_2 \tag{1.6}$$

and therefore, by using (1.5), we obtain the expression for the antiderivative F_a:

$$F_a(x) = \begin{cases} \frac{2}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) + C + \frac{\pi}{2}, & x \in [0, \pi); \\ C, \quad x = \pi; \\ \frac{2}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) + C - \frac{\pi}{2}, & x \in (\pi, 2\pi], \end{cases} \tag{1.7}$$

Remark 1.1. *It is easy to check that F_a given by* (1.7) *is indeed an antiderivative of f_a on $[0, 2\pi]$, i.e.,*

a) F_a is differentiable on $[0, 2\pi]$ and b) $F'_a(x) = f_a(x)$, for all $x \in [0, 2\pi]$.

Now, if the requirement is to find an antiderivative of f on $[0, 4\pi]$, the problem will be more difficult, because in that case $t = \tan\frac{x}{2}$ is not defined at $x = \pi$ and $x = 3\pi$ and hence its antiderivative F_a will have 6 branches, and so on.

Hence, it is now quite clear that, by using the technique presented above for the interval $[0, 2\pi]$, it is not possible to find an antiderivative of f_a on $\mathbb{R}$.

Starting from this difficulty, the main aim of the next section is to present a method that allows us to compute in a simple manner the antiderivatives of f_a on any interval $I \subset \mathbb{R}$. The starting point of this question is Problem 523, page 69 from [9].

We also present some applications of the obtained formula and indicate further developments around this topic.

2. The main result and some applications

The key tool in obtaining the antiderivative of f_a on the entire real axis, is to avoid having in its expression the function $\tan\frac{x}{2}$. To this end, we shall use the well known identity

$$\arctan u - \arctan v = \arctan\frac{u-v}{1+uv}, \tag{2.8}$$

valid for any $u, v \in \mathbb{R}$ such that $uv \neq -1$. On any interval $I \subset \mathbb{R}$ that does not contain a point of the form $(2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, by using (1.3) we have

$$\arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) = \arctan\left(\tan\frac{x}{2}\right) + \arctan\left(\sqrt{\frac{a-1}{a+1}} \cdot \tan\frac{x}{2}\right) - \arctan\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2} - \left[\arctan\left(\tan\frac{x}{2}\right) - \arctan\left(\sqrt{\frac{a-1}{a+1}}\tan\frac{x}{2}\right)\right].$$

By denoting $u = \tan\frac{x}{2}$ and $v = \sqrt{\frac{a-1}{a+1}}\tan\frac{x}{2}$, we have

$$u - v = \left(1 - \sqrt{\frac{a-1}{a+1}}\right)\tan\frac{x}{2}, 1 + uv = 1 + \sqrt{\frac{a-1}{a+1}}\tan^2\frac{x}{2},$$

and hence, by using the formula $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$, we can remove $\tan\frac{x}{2}$ in the following way

$$\frac{u-v}{1+uv} = \tan\frac{x}{2}\left(1 - \sqrt{\frac{a-1}{a+1}}\right) \cdot \frac{\sqrt{a+1}(1+\cos x)}{\sqrt{a+1}(1+\cos x) + \sqrt{a-1}(1-\cos x)}$$

$$= \frac{\sin x}{1+\cos x} \cdot \left(1 - \sqrt{\frac{a-1}{a+1}}\right) \frac{\sqrt{a+1}(1+\cos x)}{\sqrt{a+1} + \sqrt{a-1} + (\sqrt{a+1} - \sqrt{a-1})\cos x}$$

$$= \sin x \cdot \frac{\sqrt{a+1} - \sqrt{a-1}}{(\sqrt{a+1} - \sqrt{a-1})\cos x + \sqrt{a+1} + \sqrt{a-1}} = \frac{\sin x}{\cos x + \frac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}}$$

$$= \frac{\sin x}{\cos x + \frac{\left(\sqrt{a+1}+\sqrt{a-1}\right)^2}{2}} = \frac{\sin x}{a + \sqrt{a^2-1} + \cos x}.$$

Therefore, for all values of x in an interval $I \subset \mathbb{R}$ that does not contain points of the form $(2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, the following identity is valid

$$\arctan\left(\tan\frac{x}{2}\right) - \arctan\left(\frac{a-1}{\sqrt{a^2-1}}\tan\frac{x}{2}\right) = \arctan\left(\frac{\sin x}{a + \sqrt{a^2-1} + \cos x}\right). \tag{2.9}$$

Remark 2.2. We stress on the fact that the identity (2.9) has been obtained only for an interval $I \subset \mathbb{R}$ that does not contain points of the form $(2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, because in all calculations that lead to (2.9), the function $\tan\frac{x}{2}$ was still involved.

However, by taking advantage of the right hand side of (2.9) one can prove by direct computation of the derivatives that the following result holds.

Theorem 2.1. *The function $F_a : \mathbb{R} \to \mathbb{R}$, given by*

$$F_a(x) = \frac{1}{\sqrt{a^2-1}}\left(x - 2\arctan\frac{\sin x}{a + \sqrt{a^2-1} + \cos x}\right) + C, x \in \mathbb{R}, \tag{2.10}$$

is an antiderivative on $\mathbb{R}$ of the function f_a given by (1.1).

We end this section by giving some examples on how one can apply formula (2.10) for solving difficult related problems.

Example 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{3 + \cos x}, x \in \mathbb{R}.$$

Compute $\int_0^{2\pi} f(x)dx$.

Solution. We have $a = 3$ and by (2.10), we deduce that

$$F(x) = \frac{1}{2\sqrt{2}} \left(x - 2\arctan\frac{\sin x}{3 + 2\sqrt{2} + \cos x} \right) + C, x \in \mathbb{R},$$

is an antiderivative of f on $\mathbb{R}$. Therefore

$$\int_0^{2\pi} f(x)dx = F(2\pi) - F(0) = \frac{\pi}{\sqrt{2}},$$

which is the result obtained in [5] and [6] to Problem 1.1 but by using formula (1.7).

Example 2.2 (problem 450, page 41, [11])**.** Compute the definite integral

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 3}. \tag{2.11}$$

Solution. The integral can be written in the form

$$I = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{dx}{\frac{3}{2} + \cos x} \tag{2.12}$$

and applying (2.10) with $a = \frac{3}{2}$, one obtains

$$I = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{9}{4} - 1}} \left(x - 2\arctan\frac{\sin x}{\frac{3}{2} + \sqrt{\frac{9}{4} - 1} + \cos x} \right) \Bigg|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{5}} \left(\frac{\pi}{2} - 2\arctan\frac{1}{\frac{3}{2} + \frac{\sqrt{5}}{2}} \right) = \frac{1}{\sqrt{5}} \left(\frac{\pi}{2} - 2\arctan\frac{2}{3 + \sqrt{5}} \right) =$$

$$= \frac{2}{\sqrt{5}} \left(\arctan 1 - \arctan\frac{2}{3 + \sqrt{5}} \right) = \frac{2}{\sqrt{5}} \cdot \arctan\frac{1 - \frac{2}{3+\sqrt{5}}}{1 + \frac{2}{3+\sqrt{5}}} =$$

$$= \frac{2}{\sqrt{5}} \arctan\frac{1 + \sqrt{5}}{\sqrt{5}(1 + \sqrt{5})} = \frac{2}{\sqrt{5}} \arctan\frac{1}{\sqrt{5}}.$$

Example 2.3 (problem 451, page 41, [11])**.** Compute the definite integral

$$I = \int_0^{4\pi} \frac{dx}{5 + 4\cos x}. \tag{2.13}$$

Solution. We have

$$I = \frac{1}{4} \int_0^{4\pi} \frac{dx}{\frac{5}{4} + \cos x} = \frac{1}{4} \cdot \frac{1}{\sqrt{\frac{25}{16} - 1}} \left(x - 2\arctan\frac{\sin x}{\frac{5}{4} + \sqrt{\frac{25}{16} - 1} + \cos x} \right) \Bigg|_0^{4\pi} =$$

$$= \frac{1}{4} \cdot \frac{4}{3} \cdot 4\pi = \frac{4\pi}{3}.$$

Example 2.4 (problem 507, page 45, [11])**.** Compute the limit

$$l = \lim_{n \to \infty} \int_0^n \frac{dx}{1 + n^2 \cos^2 x}. \tag{2.14}$$

Solution. Let I_n be the definite integral under the limit. We have

$$I_n = \int_0^n \frac{dx}{1+n^2\cos^2 x}dx = \int_0^n \frac{1}{1+\frac{n^2}{2}(1+\cos 2x)}dx = 2\int_0^n \frac{1}{n^2+2+n^2\cos 2x}dx$$

$$= \frac{2}{n^2}\int_0^n \frac{1}{\frac{n^2+2}{n^2}+\cos 2x}dx = \frac{1}{n^2}\int_0^{2n} \frac{dt}{\frac{n^2+2}{n^2}+\cos t} =$$

$$= \frac{1}{n^2}\cdot\frac{1}{\sqrt{(\frac{n^2+2}{n^2})^2-1}}\left(t - 2\arctan\frac{\sin t}{\frac{n^2+2}{n^2}+\sqrt{(\frac{n^2+2}{n^2})^2-1}+\cos t}\right)\Bigg|_0^{2n}$$

$$= \frac{1}{\sqrt{4n^2+4}}\left(2n - 2\arctan\frac{\sin 2n}{\frac{n^2+2}{n^2}+\sqrt{(\frac{n^2+2}{n^2})^2-1}+\cos 2n}\right)$$

$$= \frac{1}{\sqrt{1+\frac{1}{n^2}}}\left(1 - \frac{1}{n}\arctan\frac{\sin 2n}{\frac{n^2+2}{n^2}+\sqrt{(\frac{n^2+2}{n^2})^2-1}+\cos 2n}\right).$$

Now,

$$\lim_{n\to\infty}\frac{1}{n}\arctan\frac{\sin 2n}{\frac{n^2+2}{n^2}+\sqrt{(\frac{n^2+2}{n^2})^2-1}+\cos 2n} = 0 \Rightarrow l = \lim_{n\to\infty} I_n = 1.$$

3. More developments

We end this paper by indicating some elementary and non elementary developments that were obtained by the second author in [2]-[6].

Starting from Problem 1.1, in [5] and [6] the following generalization of the fundamental formula of the integral calculus (also called Leibniz-Newton formula, in the Romanian mathematical literature) has been obtained.

Theorem 3.2 (Theorem 2, Chapter 16, [5]). *Let $f : [a,b] \to \mathbb{R}$ be such that*
(i) f is Riemann integrable on $[a,b]$;
(ii) f possesses antiderivatives on $[a,b]$.
Let $c \in (a,b)$ and $F : [a,b]\setminus\{c\} \to \mathbb{R}$ be a differentiable function with the property

$$F'(x) = f(x), \text{ for all } x \in [a,b]\setminus\{c\}.$$

Then F has lateral limits at the point $x = c$ and

$$\int_a^b f(x)dx = F(b) - F(a) + F(c-0) - F(c+0).$$

Example 3.5. Consider the function f given in Problem 1.1. Although the function G_3 given by (1.4) is not an antiderivative of f on the interval $[0, 2\pi]$, however G_3 satisfies all assumptions of Theorem 3.2 and therefore

$$\int_0^{2\pi} f(x)dx = G_3(2\pi) - G_3(0) + G_3(\pi-0) + G_3(\pi+0)$$

$$= 0 - 0 + \frac{\pi}{2\sqrt{2}} - \left(-\frac{\pi}{2\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

Remark 3.3. If F in Theorem 3.2 is actually an antiderivative of f, then we obtain the following extension of Leibniz-Newton formula, stated and proven in [7].

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}$ be such that*
(i) f is Riemann integrable on $[a, b]$;
(ii) f possesses antiderivatives on $[a, b]$.
Then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is an antiderivative of f on $[a, b]$.

In particular, by Theorem 3.3 we obtain the first fundamental theorem of integral calculus which states that, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Theorem 3.2 has been extended further to the case of an infinite but numerable set of points c, in the papers [3] and [4].

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