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On an open problem regarding the spectral radius of the derivatives of a function and of its iterates

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ABSTRACT. The main aim of this note is to investigate empirically the relationship between the spectral radius of the derivative of a function $f : \mathbb{R}^m \to \mathbb{R}^m$ and the spectral radius of the derivatives of its iterates, which is done by means of some numerical experiments for mappings of two and more variables. In this way we give a partial answer to an open problem raised in [Rus, I. A., *Remark on a La Salle conjecture on global asymptotic stability*, Fixed Point Theory, **17** (2016), No. 1, 159–172] and [Rus, I. A., *A conjecture on global asymptotic stability*, communicated at the Workshop "Iterative Approximation of Fixed Points", SYNASC2017, Timişoara, 21-24 September 2017] and also illustrate graphically the importance and difficulty of this problem in the general context. An open problem regarding the domains of convergence is also proposed.

1. PRELIMINARIES

Let X be a nonempty set and $f : X \to X$ be an operator. Denote by

$$F_f = \{x \in X : f(x) = x\}$$

the set of fixed points of f. By definition, see for example [24], f is a Picard operator if
(i) $F_f = \{x^*\}$;
(ii) $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.
Property (ii) above expresses the fact that x^* is *globally asymptotically stable*.

Note also that, if f is a Picard operator, then

$$F_f = F_{f^n} = \{x^*\}, \forall\, n \in \mathbb{N}^*.$$

In this context, the following two conjectures are very natural, see [25].

Conjecture 1. (La Salle, [14]) *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be such that:*
(i) *there exists $x^* \in \mathbb{R}^m$ with $f(x^*) = x^*$;*
(ii) $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$*;*
(iii) *the spectral radius of the differential of f at x, $\rho(df(x))$, is < 1 for all $x \in \mathbb{R}^m$.*
Then, x^* is globally asymptotically stable.

On the other hand, Belitskiĭ and Lyubich in [5] formulated the following conjecture:

Conjecture 2. (Belitskiĭ and Lyubich, [5]) *Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, $\Omega \subset \mathbb{K}^m$ be an open subset, $\Omega_1 \subset \mathbb{K}^m$ be a compact, convex subset with $\Omega_1 \subset \Omega$. Let $f : \Omega \to \mathbb{K}^m$ be a function. We suppose that:*
(i) $f \in C^1(\Omega, \mathbb{K}^m)$*;*
(ii) $f(\Omega_1) \subset \Omega_1$*;*
(iii) $\rho(df(x)) < 1, \forall\, x \in \Omega_1$.
Then, $f\big|_{\Omega_1} : \Omega_1 \to \Omega_1$ is a Picard operator.

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Remark 1.1. ([25]) Let (X,d) be a metric space and $f:X\to X$ be an operator. The following statements are equivalent:

(i) f is a Picard operator;

(ii) for all $k\in\mathbb{N}^*$, f^k is a Picard operator;

(iii) there exists $k\in\mathbb{N}^*$ such that f^k is a Picard operator.

Based on the previous equivalences, very recently, Professor I. A. Rus in [25] proposed the following

Conjecture 3. (Rus, [25]) *Let X be a real Banach space, $\Omega\subset X$ be an open, convex subset and $f:\Omega\to\Omega$ be an operator. We suppose that:*

(i) $f\in C^1(\Omega,X)$;

(ii) $df^k(x):X\to X$ *is a Picard operator, for all* $x\in\Omega$ *and all* $k\in\mathbb{N}^*$;

(iii) $F_f\neq\emptyset$.

Then f is a Picard operator.

Starting from the above facts, the following three open problems related to Conjectures 1-3 were formulated in [28].

Problem 1.1. *There exist counterexamples to La Salle and Belitskiĭ-Lyubich conjectures. Which of them are counterexamples to Rus' conjecture, too ?*

Problem 1.2. *Under which conditions we have that*

$$\varrho(df(x))<1,\ \forall x\in\Omega\ \Rightarrow \varrho(df^k(x))<1,\forall x\in\Omega, \tag{1.1}$$

is valid for all $k\in\mathbb{N}^*$?

Problem 1.3. *Under which conditions we have that*

$$\varrho(df(x))<1\ \forall x\in\Omega\ \Rightarrow\ f\ is\ nonexpansive?$$

We first illustrate the complexity of Problem 1.2 by means of the next examples.

Example 1.1. Let $f\in C^1(\mathbb{R},\mathbb{R})$ such that

$$|f'(x)|<1,\forall x\in\mathbb{R}. \tag{1.2}$$

Then implication (1.1) holds. Indeed, in this case

$$\rho(df(x))=|f'(x)|$$

and

$$\begin{aligned}\rho(df^k(x))&=\rho\left(df(f^{k-1}(x)\circ df(f^{k-2}(x)\circ\cdots\circ df(f(x))\right)=\\&=|f'(f^{k-1}(x))|\cdot|f'(f^{k-2}(x))|\cdot\ldots|f'(x)|<1,\forall x\in\mathbb{R},\end{aligned} \tag{1.3}$$

in view of inequality (1.2).

Example 1.2. (Triangular functions) Let $f:\mathbb{R}^m\to\mathbb{R}^m$ be a triangular function, i.e.,

$$f(x_1,...,x_m)=(f_1(x_1),f_2(x_1,x_2),...,f_m(x_1,...,x_m)),(x_1,...,x_m)\in\mathbb{R}^m,$$

where $f_i:\mathbb{R}^i\to\mathbb{R}^i$ are given first order differentiable functions.

In [13] the authors proved that for triangular functions the LaSalle Conjecture is a theorem. For this class of functions, the implication (1.1) in Problem 1.2 holds, too.

Indeed, in this case we have

$$\rho\left(df(x)\right)=\max\left(|f_1'(x_1)|,\left|\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|,\ldots,\left|\frac{\partial f_m(x_1,\ldots,x_m)}{\partial x_m}\right|\right),$$

$$\rho\left(df^2(x)\right)=\max\left(|f_1'(f_1(x_1))|\cdot|f_1'(x_1)|,\left|\frac{\partial f_2'(f_1(x_1),f_2(x_1,x_2))}{\partial x_2}\right|\cdot\left|\frac{\partial f_2(x_1,x_2)}{\partial x_2}\right|,\ldots\right)$$

Example 1.3. ([12], [25]) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(x_1, x_2, x_3) = \left(\frac{x_1}{2} + x_3(x_1 + x_2x_3)^2, \frac{x_2}{2} - (x_1 + x_2x_3)^2, \frac{x_3}{2} \right),$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Then, f is a counterexample to both La Salle conjecture (Conjecture 2) and Problem 1.2 (see [12], [25]). Indeed, although

$$\rho(df(x_1, x_2, x_3)) < 1, \forall (x_1, x_2, x_3) \in \mathbb{R}^3,$$

and $f(0) = 0$, we have

$$\rho(df^2(2, 0, 2)) > 1.$$

Starting from this background, the main aim of this note is to present an alternative approach to Problem 1.2, by means of some numerical experiments. The information we obtain from these experiments is quite satisfactory and allows us to get a clearer idea on how difficult should be an analytical approach to Problem 1.2 in general.

2. Numerical experiments

We have restricted these experiments to finite dimensional spaces and to mappings in two and more variables. However, for the sake of graphic representation, only numerical experiments for mappings in two variables are reported here. The amplitude of the experiments were seriously limited by the difficulties of computing high order iterates and their derivatives. Indeed, the computation of such iterates involve high complexity, both symbolic and numeric treatment (the attempt to get symbolic iteration, even for a simple mapping, leads to extremely long formulas). However, such numerical experiments provide significant information on the problem and on the theoretical approach that can be done.

We verified for the case of a significant number of mappings whether the implication (1.1):

$$\varrho(df(x)) < 1, \ \forall x \in \Omega \ \Rightarrow \varrho(df^k(x)) < 1, \ \forall x \in \Omega, \ k \in \mathbb{N}^*$$

is true or not.

This has been done in the following way. We considered the sets

$$C_k = \{x \in \Omega | \ \varrho(df^k(x)) < 1\}, k \in \mathbb{N}^*,$$

and have depicted them.

In all figures that are presented in this paper, the black area represents the region in which the implication is true. We will use the term *domain of convergence* for the set C_k and *condition of convergence* for $\varrho(df^k(x)) < 1$.

Example 2.4. Consider the function $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f_1((x_1, x_2)^T) = \begin{pmatrix} 0.3\sin(x_1) + x_1x_2 \\ x_1^3 - 0.5x_2 \end{pmatrix}, (x_1, x_2) \in \mathbb{R}^2.$$

We note that $p = (0, 0)^T$ is a fixed point$of f_1$. Let r_k denote the spectral radius of $df^k(p)$. The sets C_k and values of $r_k = \varrho(df_1^k(p))$ are depicted in Figure 1, for $k = 1, ..., 6$.

Remark 2.2. Based on the results we obtained by numerical tests performed for function f_1 in Example 2.4, it appears that the sequence of sets $\{C_k\}$, starting with $k = 2$, is ascending, every set C_k *almost* covers the previous set C_{k-1}. Note that the scalar sequence $\{r_k\}$ is strictly decreasing: $r_1 = 0.5, \ r_2 = 0.25, r_3 = 0.125, r_4 = 0.0625, r_5 = 0.03125, r_6 = 0.015625$ and so on, since, by (1.3),

$$\varrho(df_1^k(p)) = |f_1'(p)|^k.$$

In Figure 1 the values r_k are written under the corresponding surfaces.

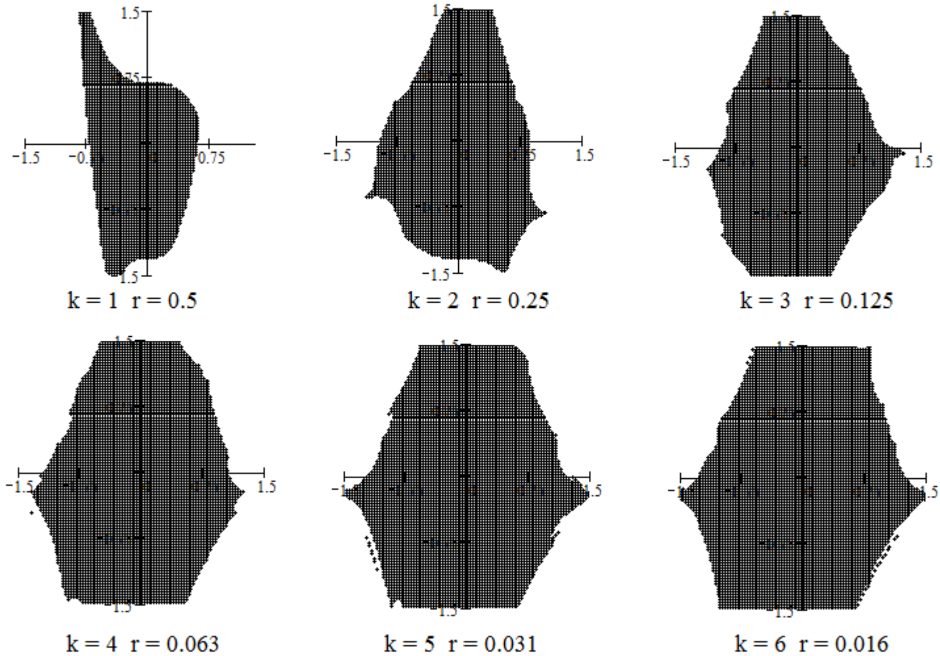

FIGURE 1. The sets $C_k (k = 1, 2, 3, 4, 5, 6)$, corresponding to f_1

Remark 2.3. We used the term "almost covers" because we can only make a "visual" comparison between two successive sets (it can be verified this conjecture rigorously). It seems also that the shape of sets C_k (the black regions) stabilizes, i.e., they do not change anymore.

It follows that, by using a stronger computer, like Blue-gin from the Laboratories at West University of Timişoara, the sequence $\{C_k\}$ could be obtained for more values of k and we could even compare two successive sets C_k and C_{k+1}, to decide about the inclusion mentioned above.

Example 2.5. We now consider the function $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f_2(x_1, x_2) = \begin{pmatrix} 0.2x_1 + x_2^2 \\ x_1x_2 - \cos(x_2) \end{pmatrix}, (x_1, x_2) \in \mathbb{R}^2.$$

Note that $p = (0, 0)^T$ that we used in the numerical tests is no more a fixed point of f_2.

In Figure 2 are depicted the sets C_k for $k = 1, ..., 6$. The sequence of sets C_k is now descending starting with $k = 3$: every set C_{k-1} almost covers the next one, C_k. Note also that, as a consequence of the fact that $p = (0, 0)^T$ is not a fixed point of f_2, the scalar sequence $r_k = \varrho(df_2^k(p))$ behaves differently, i.e., starting with $k = 2$, it is increasing: $r_2 = 0.04,\ r_3 = 0.224, ..., r_6 = 3.263$.

It is then not surprising that in this case the convergence condition is lost at $p = (0, 0)^T$, which is not a fixed point of f_2.

Note also that the descending characteristics of the successive convergence domains is, in some extent, relative. For example, it is obvious that there exists convergence points in C_5 which are not found in the convergence domain of C_4.

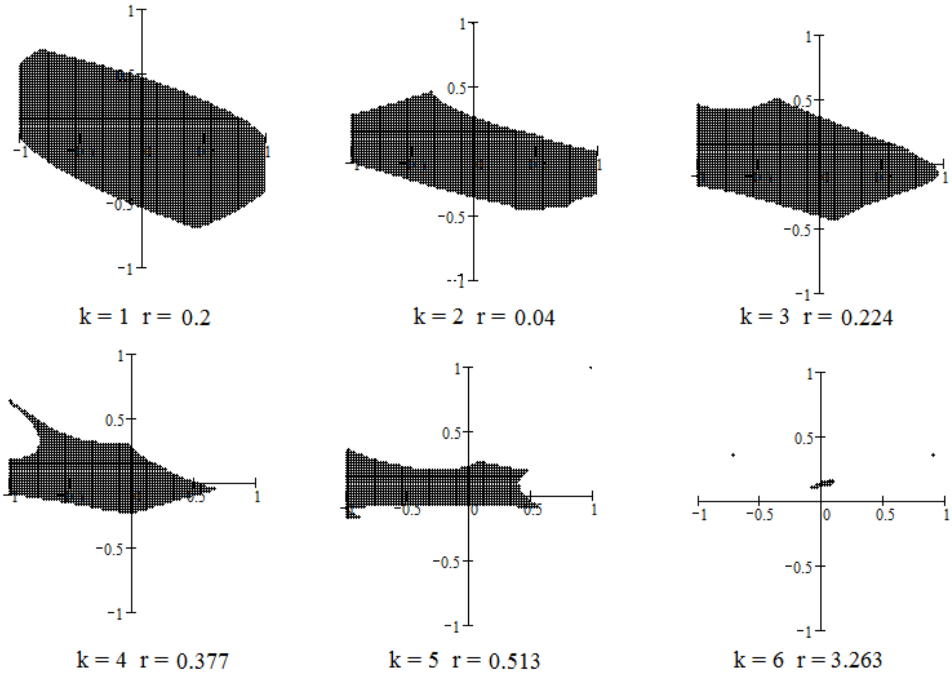

FIGURE 2. The sets $C_k, k = 1, 2, 3, 4, 5, 6$, corresponding to f_2

Remark 2.4. For $k = 6$ the convergence domain is a very small set of points. We should underline that in our computer program we used the option "Continue on error" (which is common in mathematical software) when the spectral radius is computed. It is rather possible that a lot of good points to be lost. In particular, at the fixed point p of f_2, the computer program finds an error when calculating $\varrho(df_2(p))$, for $k = 6$.

Example 2.6. Consider the function $f_3 : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f_3(x) = \begin{pmatrix} x_1 - \sin(x_2) \\ x_1^2 + 0.5x_2 \end{pmatrix}.$$

In Figure 3 are depicted the sets $C_k,\ k = 1, ..., 6$. In this case $p = (0, 0)^T$ is a fixed point of f_3 but the spectral radius of $df_3^k(p),\ k = 1, 2, ...$ is constant, i.e., $\varrho(df_3(p)) = 1$. However, the convergence domains seem to be descending starting with $k = 5$.

Apart of the three examples presented above, we performed a significant number of numerical tests with different functions in two and more variables, rather casually chosen. In most of those cases, the sequence $\{r_k\}$ was strictly decreasing.

The properties revealed for the function f_1 held *exactly in the same manner for all these examples* reported in this note. When the sequence $\{\varrho_k\}$ is monotone increasing, the behaviour of the convergence domains are, in general, unpredictable.

We therefore can state at the end of this note the following open problem.

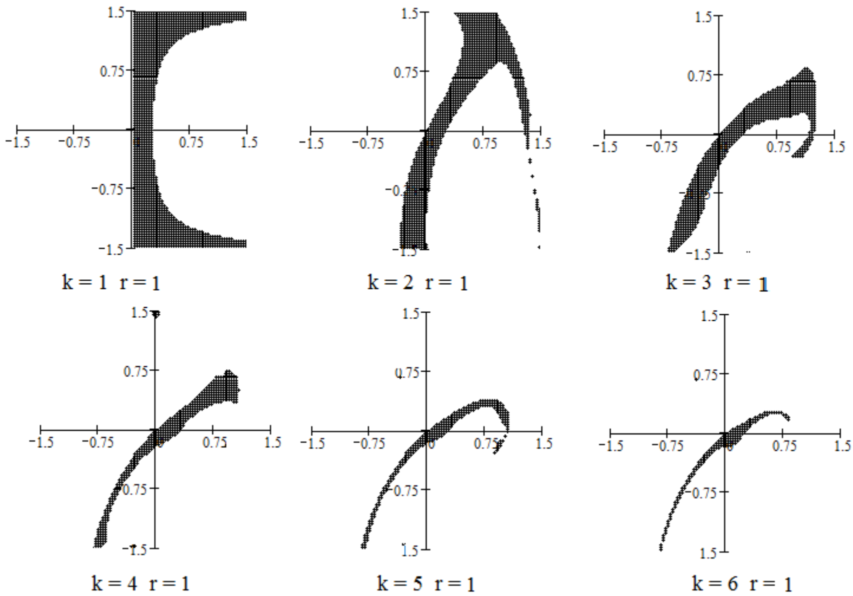

FIGURE 3. The sets $C_k, k = 1, 2, 3, 4, 5, 6$, corresponding to f_3

Problem 2.4. *Suppose that Ω contains a fixed point p and that the sequence $\{\varrho(df^k(p))\}$ is strictly decreasing. Then the convergence domains is ascending starting with some k_0, i.e., $C_{k-1} \subset C_k,\ k \geq k_0$.*

3. Conclusions

More numerical tests and developments in the study of the convergence sets for various iterative methods were performed by the second author and his collaborators and were published in some recent papers [18]-[23].

References

[1] Abts, D. and Reinermann, J., *A fixed point theorem for holomorphic mappings in locally convex spaces*, Nonlinear Anal., **3** (1979), No. 3, 353–359

[2] Aksoy, A. G. and Martelli, M., *Global convergence for discrete dynamical systems and forward neutral networks*, Turk J. Math., **25** (2001), 345–354

[3] Alarcón, B., Castro, S. B. S. D., Labourian, I. S., *A local but not global attractor for a $\mathbb{Z}_n$-symmetric map*, J. Singularities, **6** (2012), 1–14

[4] Appell, J., De Pascale, E. and Vignoli, A., *Nonlinear spectral theory*, De Gruyter Series in Nonlinear Analysis and Applications, 10. Walter de Gruyter & Co., Berlin, 2004

[5] Belitskiĭ, G. R. and Lyubich, Yu. I., *Matrix norms and their applications*. Translated from the Russian by A. Iacob. Operator Theory: Advances and Applications, 36. Birkhäuser Verlag, Basel, 1988

[6] Berinde, V., *Iterative Approximation of Fixed Points*, Lectures Notes in Math., 1912 (2007), Springer

[7] Berinde, V. and Păcurar, M., *Stability of k-step fixed point iterative methods for some Prešić type contractive mappings*, J. Inequal. Appl. 2014, 2014:149, 12 pp.

[8] Berinde, V., Păcurar, M. and Rus, I. A., *From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators*, Carpathian J. Math., **30** (2014), No. 3, 283–292

[9] Chamberland, M., *Dynamics of maps with nilpotent Jacobians*, J. Difference Equ. Appl., **12** (2006), No. 1, 49–56

[10] Cheban, D., *Belitskiĭ-Lyubich conjecture for C-analytic dynamical systems*, Discrete Contin. Dyn. Syst., Ser. B, **20** (2015), No. 3, 945–959

[11] Coll, B., Gasull, A. and Prohens, R., *On a criterium of global attraction for discrete dynamical systems*, Commun. Pure Appl. Anal., **5** (2006), No. 3, 537–550

[12] Cima, A., van den Essen, A., Gasull, A., Hubbers, E. and Manosas, F., *A polynomial counterexample to the Markus-Yamabe conjecture*, Adv. Math.,**131** (1997), No. 2, 453–457

[13] Cima, A., Gasull, A. and Manosas, F., *The discrete Markus-Yamabe problem*, Nonlinear Anal., **35** (1999), No. 3, Ser. A: Theory Methods, 343–354

[14] La Salle, J. P., *The stability of dynamical systems*. With an appendix: "Limiting equations and stability of nonautonomous ordinary differential equations" by Z. Artstein. Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1976

[15] Măruşter, Şt., *Experiments on the regions of asymptotic stability*, An. Univ. Timişoara Ser. Ştiinţ. Mat., **26** (1988), No. 3, 53–66

[16] Măruşter, Şt., *The stability of gradient-like methods*, Appl. Math. Comput., **177** (2001), 103–115

[17] Măruşter, L. and Măruşter, Şt., *On the error estimation and T-stability of the Mann iteration*, J. Comput. Appl. Math., **276** (2015), 110–116

[18] Măruşter, Şt., *A note on the convergence of Mann iteration*, Creat. Math. Inform., **26** (2017), No. 1, 85–88

[19] Măruşter, Şt., *Local convergence and radius of convergence for modified Newton method*, An. Univ. Vest Timiş. Ser. Mat.-Inform, **55** (2017), No. 2, 157–169

[20] Măruşter, Şt., *Estimating the local radius of convergence for Picard iteration*, Algorithms (Basel), **10** (2017), No. 1, Paper No. 10, 11 pp.

[21] Măruşter, Şt. and Măruşter, L., *Local convergence of generalized Mann iteration*, Numer. Algorithms, **76** (2017), No. 4, 905–916

[22] Argyros, I. K. and Măruşter, Şt., *Predetermining the number of periodic steps in multi-step Newton-like methods for solving equations and systems of equations*, Appl. Math. Comput., **329** (2018), 420–431

[23] Măruşter, Şt. and Măruşter, L., *Sharp estimation of local convergence radius for the Picard iteration*, J. Fixed Point Theory Appl., **20** (2018), No. 1, Art. 28, 13 pp.

[24] Rus, I. A., *Picard operators and applications*, Sci. Math. Jpn., **58** (2003), No. 1, 191–219

[25] Rus, I. A., *Remark on a La Salle conjecture on global asymptotic stability*, Fixed Point Theory, **17** (2016), No. 1, 159–172

[26] Rus, I. A., *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai Math., **61** (2016), No. 3, 343–358

[27] Rus, I. A., *Some properties of the solutions of those equations for which the Krasnoselskii iteration converges*, Carpathian J. Math., **28** (2012), No. 2, 329–336

[28] Rus, I. A., *A conjecture on global asymptotic stability*, Workshop "Iterative Approximation of Fixed Points", SYNASC 2017, Timişoara, 21-24 Sept. 2017

[29] Wang, C., *A note on the error estimation of the Mann iteration*, J. Comput. Appl. Math., **285** (2015), 226–229

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