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On solutions of functional equations with polynomial translations

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ABSTRACT. In this paper, we study polynomial functional equations of the form $af(p(x))+bf(q(x)) = g(x)$, where $p(x)$, $q(x)$ are given polynomials and $g(x)$ is a given function. Theorems 2.1 and 2.2 contain sufficient conditions under which the functional equation has a solution of the special form. In Section 3 we present an algorithm of constructing polynomial solutions of the functional equations. Other non-polynomial solutions depend on solutions of the homogeneous equation $af(p(x)) + bf(q(x))$ = 0. That case is analyzed in Section 4. Finally, we present a simple method of constructing examples with desirable properties.

1. INTRODUCTION

Theory of functional equations is a large and important domain of mathematics [1, 4, 5, 7, 9, 8]. Formally, a functional equation is a relation between concrete variables where some variables are functions or functions with their derivatives. Some properties of solutions of a given differential equation may be determined without finding their exact form [9, 8]. We study the functional equations without derivatives of solution. One of the general forms of a functional equation is the following:

$$F(x_1, x_2, ..., x_n, f(g_1(x_1, x_2, ..., x_n)), ..., f(g_m(x_1, x_2, ..., x_n))) = 0, \tag{1.1}$$

A solution of this equation is a function f, which satisfies the functional equation. There are many books which analyze concrete equations, but propose only outlines of solving methods [3, 5, 6].

We analyze the functional equations of the form $af(p(x)) + bf(q(x))$ = $g(x)$ in the following two cases:

1. $p(x)$, $q(x)$, $g(x)$ are given polynomials;
2. $p(x)$, $q(x)$ are given polynomials and $g(x)$ is a given function.

The domain of the definition of solutions coincides with the common domain of definition of the given functions $p(x)$, $q(x)$, $g(x)$. We say that $p(x)$ and $q(x)$ are the translation polynomials (functions). If the function $g(x)$ is a polynomial, then we say that the functional equation is polynomial.

Our goal is to present a general approach to solving functional equations with polynomial translations. This approach is important from the didactic point of view. The process of composing problems has the aim of forming the capability to analyze notions and their properties, of consolidating knowledge, of creating premises for their application, of developing school students' mathematical creativity etc. The notions of equation and function are fundamental in the course of elementary mathematics and encapsulate a rich potential for solving problems of an inter- and trans-disciplinary character. The general

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approach allows the structuring of the algorithm of composing concrete equations, whose "spicy" character arises from the way of selecting the coefficients a, b, the coefficients of polynomials $p(x)$, $q(x)$ and the degrees of polynomial $g(x)$. Some problems of this type are sometimes declared as "trick problems". As a rule, functional equations are considered on the field of reals $\mathbb{R}$. However, the main results are true for the field of complex number and, more general, for topological commutative fields.

2. Functional equations with some conditions of symmetry

Fix a topological commutative field $(G, +, \cdot)$.

Theorem 2.1. *The equation*

$$mf(ax+b) + nf(-ax+c) = g(x), \tag{2.2}$$

where $x, m, n, a, b, c \in G$, $m^2 \neq n^2$, $a \neq 0$ *and* $g : G \to G$ *is a function, has a canonical solution* $\varphi : G \to G$, *where* $\varphi(x) = (m^2 - n^2)^{-1}(mg(a^{-1}(x-b)) - ng(a^{-1}(-x+b)))$. *The function* φ *is continuous if and only if the function* g *is continuous. The function* φ *is a polynomial if and only if the function* g *is a polynomial.*

Proof. Let $u = ax + b$. In this case, the equation 2.2 is equivalent with the functional equation

$$mf(u) + nf(-u+b+c) = g(a^{-1}(u-b)). \tag{2.3}$$

If $v = -u + c$, then we obtain the equation

$$mf(-v+b+c) + nf(v) = g(a^{-1}(-v+c)). \tag{2.4}$$

Hence, we obtain the following system of equations

$$\begin{cases} mf(x) + nf(-x+b+c) = g(a^{-1}(x-b)) \\ nf(x) + mf(-x+b+c) = g(a^{-1}(-x+c)) \end{cases} \tag{2.5}$$

Hence, $(m^2 - n^2)f(x) = mg(a^{-1}(x-b)) - ng(a^{-1}(-x+c))$ and the function $\varphi(x) = (m^2 - n^2)^{-1}(mg(a^{-1}(x-b)) - ng(a^{-1}(-x+c)))$ is a solution of the equation 2.2. We say that the function φ is the canonical solution of the equation 2.2. Obviously, the function φ has the desired properties in the dependence of the function g. The proof is complete. $\square$

As a rule, the canonical solution φ of the equation 2.2 is not unique.

Example 2.1. Consider the equation $mf(ax+b) + nf(-ax-b) = g(x)$, where $m^2 \neq n^2$ and $mn \neq 0$, with the canonical solution $\varphi(x) = (m^2 - n^2)^{-1}(mg(a^{-1}(x-b)) - ng(-a^{-1}(-x+b)))$. Let $h : G \longrightarrow G$ be a function with the property $h(x) = -m^{-1}nh(-x)$. Then $f = \varphi + h$ is a solution of the equation. Any solution of the equation has that form. Thus the given equation has infinitely many solutions. The condition $m^2 \neq n^2$ is essential.

Example 2.2. Let G be an algebraic number field. In the field G the following equation $f(x) + f(-x) = 4x^2 + 4x + 1$ has no solutions. Indeed, let $h(x)$ be some solution of the equation $f(x) + f(-x) = 4x^2 + 4x + 1$. In this case, we have $h(x) + h(-x) = 4x^2 + 4x + 1$ and $h(x) + h(-x) = 4x^2 - 4x + 1$. Hence $h(x) + h(-x) = 4x^2 + 1$. Therefore, for any x we have $4x = -4x$, a contradiction.

Example 2.3. In any algebraic number field G, the following equation $f(x) + f(-x) = 4x^4 + 2x^2 - 8$ has infinitely many polynomial solutions of the form $h(x) = 2x^4 + x^2 - 4 + a_1x + a_2x^3 + \ldots + a_nx^{2n-1}$, where n is a natural number and a_1, a_2, ..., a_n are arbitrary elements from G. Indeed, if $h(x) = 2x^4 + x^2 - 4 + a_1x + a_2x^3 + \ldots + a_nx^{2n-1}$, then $h(-x) = 2x^4 + x^2 - 4 - a_1x - a_2x^3 - \ldots - a_nx^{2n-1}$ and $h(x) + h(-x) = 4x^4 + 2x^2 - 8$.

Remark 2.1. Let $m \neq n$, $m^2 \neq n^2$, $a \neq 0$ and the function g is a polynomial. Then the function $\varphi(x) = (m^2 - n^2)^{-1}(mg(a^{-1}(x - b)) - ng(a^{-1}(-x + b)))$ is the unique polynomial solution of the equation 2.2.

Theorem 2.2. *Let $(G, +, \cdot)$ be a topological commutative field. The equation*

$$mf(ax) + nf\left(-\frac{1}{ax}\right) = g(x), \tag{2.6}$$

where x, m, n, $a \in G$, $m^2 \neq n^2$, $ax \neq 0$, $G' = G \setminus \{0\}$ and $g : G' \to G$ is a function, has a canonical solution $\psi(x)$, where $\psi(x) = (m^2 - n^2)^{-1}(mg(a^{-1}x) - ng(-a^{-1}x^{-1}))$. The function ψ is continuous on G' if and only if the function g is continuous on G'.

Proof. Let $u = ax$ and $v = -\frac{1}{ax}$. In this case, the equation 2.6 is equivalent with the functional equations

$$mf(u) + nf\left(-\frac{1}{u}\right) = g\left(\frac{u}{a}\right), \tag{2.7}$$

and

$$mf\left(-\frac{1}{v}\right) + nf(v) = g\left(-\frac{1}{av}\right). \tag{2.8}$$

Hence, we obtain the following system of equations

$$\begin{cases} mf(x) + nf\left(-\frac{1}{x}\right) = g\left(\frac{x}{a}\right) \\ nf(x) + mf\left(-\frac{1}{x}\right) = g\left(-\frac{1}{ax}\right) \end{cases} \tag{2.9}$$

which has the form

$$\begin{cases} m^2f(x) + mnf\left(-\frac{1}{x}\right) = mg\left(\frac{x}{a}\right) \\ -n^2f(x) - mnf\left(-\frac{1}{x}\right) = -ng\left(-\frac{1}{ax}\right) \end{cases} \tag{2.10}$$

Hence $(m^2-n^2)f(x) = mg\left(\frac{x}{a}\right) - ng\left(-\frac{1}{ax}\right)$ and the function $\psi(x) = (m^2-n^2)^{-1}(mg(a^{-1}x) - ng(-a^{-1}x^{-1}))$ is a solution of the equation 2.6. We say that the function ψ is the canonical solution of the equation 2.6. The proof is complete. □

Remark 2.2. As in the case of Theorem 2.1, one may want to establish the essentiality of the condition $m^2 \neq n^2$ and non uniqueness of the solutions of the equation 2.6. Indeed, if ψ is the canonical solution of the equation 2.6 and $h(x) = -f\left(-\frac{1}{x}\right)$, then $f = \psi + h$ is a solution of the equation 2.6. Any solution of the equation 2.6 has this form.

If $m + n \neq 0$ and $g(x) = g(x^{-1})$ for each $x \in G'$, then $\phi(x) = (m + n)^{-1}g(x)$ is the second canonical solution of the equation 2.6.

Remark 2.3. The equation 2.6 has polynomial solutions only in special cases. Assume that $k \geq 1$, $g(x) = a_kx^k + ... + a_1x + c + b_kx^{-k} + ... + b_1x^{-1}$, $a_nb_n \neq 0$ and $mb_i = na_i$ for each $i \leq k$. Let $a_0 = (m + n)^{-1}c$. Then $\pi(x) = m^{-1}(a_kx^k + ... + a_1x) + a_0$ is a polynomial solution of the equation 2.6.

3. General method of solving polynomial equations

In the present section we examine the functional equation

$$af(p(x)) + bf(q(x)) = g(x), \tag{3.11}$$

where $p(x)$, $q(x)$, $g(x)$ are given polynomials.

Assume that

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \\ q(x) &= b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0, \\ g(x) &= c_k x^k + c_{k-1}x^{k-1} + \ldots + c_1 x + c_0, \end{aligned} \tag{3.12}$$

where $ab \neq 0$ and $a_n b_m c_k \neq 0$.

The equation 3.11 determines the homogeneous equation

$$af(p(x)) + bf(q(x)) = 0. \tag{3.13}$$

Let S be the set of all solutions of the equation 3.11 and S_0 be set of all solutions of the equation 3.13. Always $S_0 \neq \emptyset$: the set S_0 contains the function $f = 0$ as a solution of 3.13.

The following elementary fact establishes the relation between S and S_0.

Proposition 3.1. $S = \{f_0 + h : h \in S_0\}$, where $f_0 \in S$.

Corollary 3.1. Either $S = \emptyset$, or $|S| = |S_0|$.

By virtue of the proposition 3.1, it is important to find some solution f_0 of the equation 3.11. Assume that f_0 is a polynomial solution of the equation 3.11 and $f_0(x) = e_l x^l + e_{l-1}x^{l-1} + \ldots + e_1 x + e_0$. The polynomials $p(x)$, $q(x)$, $g(x)$, $f_0(x)$ have the degrees n, m, k, l respectively.

The following two facts serve as the principle of accordance of the degrees of the polynomials $p(x)$, $q(x)$, $g(x)$, $f_0(x)$.

Proposition 3.2. $k \leq \max\{nl; ml\}$.

Proof. Indeed, $f_0(p(x))$ is a polynomial of the degree nl and $f_0(q(x))$ is a polynomial of the degree ml. The sum $af_0(p(x)) + bf_0(q(x))$ is a polynomial of the degree $\leq \max\{nl; ml\}$. Hence $k \leq \max\{nl; ml\}$. The proof is complete. □

Proposition 3.3. If $m < n$, then $k = nl$.

Proof. The proof is similar with the proof of Proposition 3.2. □

The principle of accordance of the degrees permits to propose a general method of find polynomial solutions of the polynomial equation 3.11.

Assume that $m \leq n$. Firstly, we mention the following facts:

F1. If $m < n$ and $k \neq nl$ for any $l \in \mathbb{N} = \{0, 1, 2, ...\}$, then the equation 3.11 has no polynomial solutions.

F2. If $m = n$, $aa_n + bb_n \neq 0$ and $k \neq nl$, then the equation 3.11 has no polynomial solutions of degree l. Moreover, if $m = n$, $aa_n + bb_n \neq 0$ and $k \neq nl$ for any $l \in \mathbb{N}$, then the equation 3.11 has no polynomial solutions.

Now we propose the method of construction of polynomial solutions:

Step 1. Fix a natural number l where $k \leq nl$. Excluding the cases F1 and F2, set one of the variants: $m < n$ and $k = nl$, or $m = n$, $aa_n + bb_n \neq 0$ and $k = nl$, or $m = n$, $aa_n + bb_n = 0$ and $k < nl$.

Step 2. Fix a polynomial $f_0(x) = e_l x^l + e_{l-1}x^{l-1} + \ldots + e_1 x + e_0$, $e_l \neq 0$.
Step 3. Compute the polynomial $f_0(p(x)) = a_n e_l x^{nl} + d_{nl-1}x^{nl-1} + \ldots + d_1 x + d_0$.
Step 4. Compute the polynomial $f_0(q(x)) = b_m e_l x^{ml} + r_{ml-1}x^{ml-1} + \ldots + r_1 x + r_0$.
Step 5. Compute the polynomial $af_0(p(x)) + bf_0(q(x))$.
Step 6. Analyze the final results of computing.

Since we suppose that f_0 is a solution of the equation 3.11, we have $g(x) = af_0(p(x)) + bf_0(q(x))$ and we obtain nl relations between coefficients of $g(x)$ and $af_0(p(x))+bf_0(q(x))$. These relations between coefficients form a system of equations for the coefficients of the solution and permit to establish if for the equation 3.11 there exists or not some polynomial solutions.

In particular, polynomial solutions exist if and only if the obtained system of equations is compatible. It is convenient to calculate the coefficients of the solution f_0 in the descending order: $a_l, a_{l-1}, ..., a_1, a_0$.

In the case $n = m = 1$ the following fact is true.

Proposition 3.4. If $n = m = 1$ and $aa_1^l + bb_1^l \neq 0$ for any $l \in \mathbb{N}$, then the equation 3.11 has a unique polynomial solution.

Now we illustrate the algorithm with the following examples.

Example 3.4. Consider the equation $3f(x^2 + x - 1) - f(2x) = g(x)$, where $g(x) = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$.

Any polynomial solution is of the degree 2. Let $f_0(x) = l_2x^2 + l_1x + l_0$ be a solution of the equation. Then $3f(x^2 + x - 1) - f(2x) = 3l_2x^4 + 6l_2x^3 + (3l_1 - 7l_2)x^2 + (l_1 - 6l_2)x + (2l_0 - 3l_1 + 3l_2)$.

In this case: $3l_2 = c_4$, $6l_2 = c_3$, $3l_1 - 7l_2 = c_2$, $l_1 - 6l_2 = c_1$, $2l_0 - 3l_1 + 3l_2 = c_0$. If $c_3 \neq 2c_4$, the equation has no polynomial solutions. If $c_3 = 2c_4$, the equation has an unique polynomial solution $f_0(x)$, where $l_2 = \frac{1}{3}c_4 = \frac{1}{6}c_3$, $l_1 = \frac{1}{3}(c_2 + 7l_2)$, $l_0 = \frac{1}{2}(c_0 + 3l_0 - 3l_2)$.

Example 3.5. $4f(3x^2 - x) - 9f(2x) = g(x)$, where $g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$.

Let $f_0(x) = l_2x^2 + l_1x + l_0$ be a solution of the equation. Then $4f(3x^2 - x) - 9f(2x^2 + x)$ $= -60l_2x^3 - (6l_1 + 5l_2)x^2 - 13l_1x - 5l_0$.

In this case: $-60l_2 = c_3$, $5l_2 + 6l_1 = -c_2$, $13l_1 = -c_1$, $5l_0 = -c_0$.

The coefficient l_2 is calculated in an unique way $l_2 = -\frac{1}{60}c_3$, but for l_1 we obtain two conditions, that must be satisfied simultaneously: $l_1 = -\frac{1}{6}c_2 + \frac{1}{72}c_3$ and $l_1 = -\frac{1}{13}c_1$. If these conditions are not satisfied simultaneously, then the equation has no polynomial solutions. In other case, we calculate $l_0 = -\frac{1}{5}c_0$ and we write the polynomial solution $f_0(x) = -\frac{1}{60}c_3x^2 - \frac{1}{13}c_1x - \frac{1}{5}c_0$.

Example 3.6. $f(3x^2 + x) - f(2x^2 - x) = g(x)$, where $g(x) = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$. Any polynomial solution is of the degree 2. Let $f_0(x) = l_2x^2 + l_1x + l_0$ be a solution of the equation. Then $f(3x^2 + x) - f(2x^2 - x) = 5l_2x^4 + 10l_2x^3 + l_1x^2 + 2l_1x$.

Hence: $5l_2 = c_4$, $10l_2 = c_3$, $l_1 = c_2$, $2l_1 = c_1$, $0 = c_0$. The equation has polynomial solutions under the conditions: $c_3 = 2c_4$, $c_1 = 2c_2$, $c_0 = 0$.

Remark 3.4. The above method can be applied for finding polynomial solutions of the functional equations of the form

$$af(p(x))^n + bf(q(x))^m = g(x), \tag{3.14}$$

where $p(x)$, $q(x)$, $g(x)$ are the given polynomials and $n, m \in \mathbb{N}$.

4. Existence of periodic solutions of the homogeneous equation

Consider the equation

$$af(p(x)) + bf(kp(x) + c) = \varphi(p(x)) + d, \tag{4.15}$$

where c, d are constant numbers, $|k| = 1$, $a \neq 0$ and $p(x)$, $\varphi(x)$ are non-constant functions.

In this case, we let $t = p(x)$ and obtain the equation $af(t) + bf(kt + c) = \varphi(t) + d$. For $k = -1$ this equation is solved in Section 2. Assume that $k = 1$ and $\varphi(p(x)) + d = 0$.

If $a \neq 0$ and $b = 0$, then $f(x) = 0$ is the unique solution of the equation 4.15.

If $b \neq 0$, $c = 0$ and $a + b \neq 0$, then $f(x) = 0$ is the unique solution of the equation 4.15.

Assume now that $ab \neq 0$, $a + b \neq 0$ and $c \neq 0$.

In this case, the equation 4.15 has the form

$$f(t) = -ba^{-1}f(t + c). \tag{4.16}$$

This fact permits to determine all solutions of the equation 4.15, applying the following algorithm.

Step 1. We set $I_n = [nc, (n + 1)c)$ for any integer $n \in \mathbb{Z}$.

Step 2. On I_0, we fix some function $h_0(x)$.

Step 3. On I_n, we construct the function $h_n(x) = (-ab^{-1})^n h_0(x - nc)$. Since $x - nc \in I_0$ if and only if $x \in I_n$, the function h_n is correctly constructed.

Step 4. We let $h = h|I_n$ for any $n \in \mathbb{Z}$. Then h is a solution of the equation 4.15.

Remark 4.5. Any solution of the equation 4.15 can be constructed by the above algorithm.

Remark 4.6. If $\left|ab^{-1}\right| = 1$, then the solutions 4.15 are periodic functions with the period $2c$.

Remark 4.7. If we set $h_c = -ab^{-1}h_0(0)$ and the function h is continuous on $[0, c]$, then the solution h is continuous on $\mathbb{R}$. All continuous solutions of the equation 4.15 can be obtained in this way.

This algorithm permits to construct "spicy" solutions of functional equations.

Example 4.7. $f(2x^2 + 4x + 3) - f(2x^2 + 4x + 1) = 8(x + 1)^2$

Prove that a continuous solution $s(x)$ such that $s(x) = 2x$ exists for any $x \in [0, 2)$.

The equation has no solutions of the degree ≤ 1.

Assume that $f(x) = ax^2 + bx + c$ is a solution of the given equation. Since $f(2x^2 + 4x + 3) - f(2x^2 + 4x + 1) = 8ax^2 + 16ax + 8a + 2b$, from $8ax^2 + 16ax + 8a + 2b = 8x^2 + 16x + 8$ we obtain $a = 1$, $b = 0$ and c is arbitrary. Hence, the polynomials $f(x) = x^2 + c$ are solutions of the equation.

The homogeneous equation $f(2x^2 + 4x + 3) - f(2x^2 + 4x + 1) = 0$ can be considered of the form $f(t + 2) - f(t) = 0$.

The periodic function with the period 2 forms all solutions of that equation. If $I_n = [2n, 2n + 2)$, $h(x) = -x^2 + 2x$ for $x \in I_0$ and $h(x) = -(x - 2n)^2 + 2(x - 2n)$ for $x \in I_n$, then the function for h is a solution of the given homogeneous equation. Fix the solution $f(x) = x^2$ for the given equation. Then $s(x) = f(x) + h(x) = (4n + 2)x - 4(n^2 - n)$ for any $x \in I_n$ and $n \in \mathbb{N}$ is a solution of the given equation. We have $s(x) = 2x$ for any $x \in I_0$.

Hence, for composing functional equations may be useful the following algorithm:

Step 1. Fix two polynomials $p(x)$ and $q(x)$.

Step 2. Fix two numbers a and b.

Step 3. Fix a polynomial $f_0(x)$ as the solution.

Step 4. Compute the polynomial $g(x) = af_0(p(x)) + bf_0(q(x))$.

We obtain the functional equation $af(p(x))+bf(q(x))=g(x)$ with the polynomial solution $f_0(x)$.

After that we have one of the following cases:

Case 1. If $q(x)=p(x)+c$, then we fix a periodic solution $h(x)$ of the homogeneous functional equation $af(x)+bf(x+c)=0$. The function $f(x)=f_0(x)+h(x)$ is a non-polynomial solution of the equation $af(p(x))+bf(q(x))=g(x)$. We may fix a priori the form of the solution $f(x)$ on the interval $[0,c)$.

Case 2. We may select the polynomials $p(x)$, $q(x)$, $h_0(x)$ and the coefficients a,b for which the functional equation $af(p(x))+bf(q(x))=g(x)$ has infinitely many polynomials solutions.

Case 3. We may select the polynomial $g(x)$ for which the functional equation $af(p(x))+bf(q(x))=g(x)$ has no solutions.

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