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On Wijsman asymptotically lacunary $\mathcal{I}$-statistical equivalence of weight g of sequence of sets

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ABSTRACT. This paper presents the following definition which is a natural combination of the definitions of asymptotically equivalence, $\mathcal{I}$-convergence, statistical limit, lacunary sequence, and Wijsman convergence of weight g; where $g : \mathbb{N} \to [0, \infty)$ is a function satisfying $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$ for sequence of sets. Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman $\mathcal{I}$-asymptotically lacunary statistical equivalent of multiple L of weight g if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

(denoted by $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$). We mainly investigate their relationship and also make some observations about these classes.

1. INTRODUCTION

Before continuing with this paper we present some definitions and preliminaries.

The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et al. in a metric space [7]. Later it was further studied by ([2], [5], [6], [12], [13], [14], [15], [16], [17], [21]) and many others. $\mathcal{I}$-convergence is a generalization form of statistical convergence, which was introduced by Fast (see [3]) and that is based on the notion of an ideal of the subset of positive integers $\mathbb{N}$. The following definitions and notions will be needed.

Definition 1.1. ([7]) A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$, each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper, $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 1.2. ([7]) A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$, each $B \supseteq A$ we have $B \in \mathcal{F}$.

Proposition 1.1. ([7]) *If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ is a filter of $\mathbb{N}$ it is called the filter associated with the ideal.*

Definition 1.3. ([7]) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

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If we take $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$, then the corresponding convergence coincides with the usual convergence.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 1.4. ([4]) A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$. In this case we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$.

The upper density of weight g was defined in [1] by the formula

$$\overline{d}_g(A) = \lim_{n\to\infty} \sup \frac{A(1,n)}{g(n)}$$

for $A \subseteq \mathbb{N}$ where as before $A(1,n)$ denotes the cardinality of the set $A \cap [1,n]$. Then, the family

$$\mathcal{I}_g = \left\{A \subseteq \mathbb{N} : \overline{d}_g(A) = 0\right\}$$

forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$ as $n \to \infty$. So we additionally assume that $\frac{n}{g(n)} \nrightarrow 0$ as $n \to \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G. As a natural consequence we can introduce the following definition.

Definition 1.5. ([1]) A sequence $\{x_n\}$ of real numbers is said to d_g-statistically convergent to x if for any given $\varepsilon > 0$, $\overline{d}_g(A_\varepsilon) = 0$, where A_ε is the set defined in Definition 1.3.

In [15], Savaş combined the approaches of [6] and [1] and introduced new and further general summability methods, namely, $\mathcal{I}$-statistical convergence of weight g and $\mathcal{I}$-lacunary statistical convergence of weight g for sequence of sets.

Definition 1.6. ([15]) Let (X,d) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For non-empty closed subsets A, $A_k \subset X$, we say that the sequences $\{A_k\}$ is Wijsman $\mathcal{I}$-statistical convergent of weight g to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{g(n)} \left|\{k \leq n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(S(\mathcal{I}_w)^g\right)$.

Definition 1.7. ([15]) Let (X,d) be a metric space, θ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For non-empty closed subsets A, $A_k \subset X$, we say that the sequences $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistical convergent of weight g to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{g(n)} \left|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(S_\theta(\mathcal{I}_w)^g\right)$.

In 1993, Marouf [8] peresented definitions for asymptotically equivalent sequences and asymptotic regular matrices.

Definition 1.8. ([8]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

In 2003, Patterson [10] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Let (X, ρ) a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if

$$\sup_k d(x, A_k) < \infty$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Definition 1.9. ([22]) Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The concept of Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray & Rhoades [9] in 2012. Similar to the concept, the concept of Wijsman lacunary statistical convergence of sequences of sets presented by Ulusu & Nuray [18] in 2012.

In [6] asymptotically $\mathcal{I}$-equivalence, asymptotically $\mathcal{I}$-statistical equivalence and asymptotically $\mathcal{I}$-lacunary statistical equivalence which are a natural combinations of the definitions of asymptotical equivalence, $\mathcal{I}$-convergence and lacunary sequence for sequence of sets were studied.

For non-empty closed subsets A_k and B_k of X, define $d\left(x; A_k, B_k\right)$ as follows:

$$d\left(x; A_k, B_k\right) = \begin{cases} \dfrac{d\left(x, A_k\right)}{d\left(x, B_k\right)} & , \quad \text{if, } x \notin A_k \cup B_k \\ \\ L & , \quad \text{if, } x \in A_k \cup B_k \end{cases}$$

Definition 1.10. ([6]) Let (X, ρ) be a metric space and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically $\mathcal{I}$-equivalent of multiple L (Wijsman sense) if every $\varepsilon > 0$, for each $x \in X$,

$$\{k \in \mathbb{N} : |d\left(x; A_k, B_k\right) - L| \geq \varepsilon\} \in \mathcal{I}.$$

$\left(\text{denoted by } A_k \overset{\mathcal{I}_W^L}{\sim} B_k\right)$ and simply asymptotically $\mathcal{I}$-equivalent if $L = 1$.

As an example, consider the following sequences of circles in the (x, y)-plane.

$$A_k = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 2ky = 0\} & , \quad \text{if, } k \text{ is a square integer} \\ \\ \{1, 1\} & , \quad \text{otherwise.} \end{cases}$$

and

$$B_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\right\} & , \quad \text{if, } k \text{ is a square integer} \\ \\ \{1, 1\} & , \quad \text{otherwise.} \end{cases}$$

If we take $\mathcal{I} = \mathcal{I}_d$ we have $\{k \in \mathbb{N} : |d(x; A_k, B_k) - 1| \geq \varepsilon\} \in \mathcal{I}_d$. Thus, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically $\mathcal{I}$-equivalent (Wijsman sense). i.e. $A_k \overset{\mathcal{I}_W^1}{\sim} B_k$.

In [19] asymptotically lacunary statistical equivalent set sequences were studied. In addition, they also presented asymptotically equivalent (Wijsman sense) analogs of theorems in [11] due to Patterson and Savaş.

In this paper we combine the approaches of [6] and [1] and introduce new concepts, namely, asymptotically $\mathcal{I}$-statistical equivalence, asymptotically $\mathcal{I}$-lacunary statistical equivalence, strongly $\mathcal{I}$-asymptotically lacunary equivalence, strongly $\mathcal{I}$-Cesáro asymptotically equivalence of multiple L of weight g for sequence of sets. We mainly investigate their relationship and also make some observations about these classes and most importantly the study leaves a lot of interesting open problems.

2. Main results

Definition 2.11. Let (X, ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For non-empty closed subsets $A_k, B_k \subset X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman asymptotically $\mathcal{I}$-statistical equivalent of multiple L of weight g provided that for every $\varepsilon > 0$ and $\delta > 0$ the set

$$\left\{n \in \mathbb{N} : \frac{1}{g(n)} \left|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \overset{S^L(\mathcal{I}_W)^g}{\sim} B_k$.

Definition 2.12. Let (X, ρ) be a metric space, θ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of multiple L of weight g provided that for every $\varepsilon > 0$ and $\delta > 0$ the set

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$.

Definition 2.13. Let (X, ρ) be a metric space, θ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman strongly asymptotically $\mathcal{I}$-lacunary equivalent of multiple L of weight g provided that for every $\varepsilon > 0$ the set

$$\left\{r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geq \varepsilon\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \overset{N_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$.

Definition 2.14. Let (X, ρ) be a metric space, θ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman strongly Cesáro $\mathcal{I}$-asymptotically equivalent of multiple L of weight g provided that for every $\varepsilon > 0$ the set

$$\left\{ n \in \mathbb{N} : \frac{1}{g(n)} \sum_{k=1}^{n} |d(x; A_k, B_k) - L| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \overset{\sigma_1^L(\mathcal{I}_W)^g}{\sim} B_k$.

Theorem 2.1. *Let g_1, $g_2 \in G$ be such that there exist $M > 0$ and $j_0 \in \mathbb{N}$ such that $\frac{g_1(n)}{g_2(n)} \leq M$ for all $n \geq j_0$. Then, $A_k \overset{S^L(\mathcal{I}_W)^{g_1}}{\sim} B_k \subseteq A_k \overset{S^L(\mathcal{I}_W)^{g_2}}{\sim} B_k$.*

Proof. For any $\varepsilon > 0$,

$$\begin{aligned} \frac{|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}|}{g_2(n)} &= \frac{g_1(n)}{g_2(n)} \frac{|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}|}{g_1(n)} \\ &\leq M \frac{|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}|}{g_1(n)} \end{aligned}$$

for all $n \geq j_0$. Hence, for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}|}{g_2(n)} \geq \delta \right\}$$

$$\subset \left\{ n \in \mathbb{N} : \frac{|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}|}{g_1(n)} \geq \frac{\delta}{M} \right\} \cup \{1, 2, ..., j_0\}$$

If $A_k \overset{S^L(\mathcal{I}_W)^{g_1}}{\sim} B_k$, then the set on the right hand side belongs to the ideal $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $A_k \overset{S^L(\mathcal{I}_W)^{g_1}}{\sim} B_k \subseteq A_k \overset{S^L(\mathcal{I}_W)^{g_2}}{\sim} B_k$. □

Theorem 2.2. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A_k, B_k be non-empty closed subsets of X. Then, $A_k \overset{N_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$ implies $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$.*

Proof. Let $\theta = \{k_r\}$ be a lacunary sequence, and $A_k \overset{N_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$. Then, we can write

$$\begin{aligned} \sum_{k \in I_r} |d(x; A_k, B_k) - L| &\geq \sum_{k \in I_r, |d(x; A_k, B_k) - L| \geq \varepsilon} |d(x; A_k, B_k) - L| \\ &\geq \varepsilon . |\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}| \end{aligned}$$

and so

$$\frac{1}{\varepsilon g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geq \frac{1}{g(h_r)} |\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}| .$$

Then, for any $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} |\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}| \geq \delta \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geq \varepsilon\delta \right\} \in \mathcal{I}.$$

Hence, we have $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$. □

Remark 2.1. In [6] it was further proved that

(i) $\{A_k\}, \{B_k\} \in L_\infty$ and $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k$ implies $A_k \overset{N_\theta^L(\mathcal{I}_W)}{\sim} B_k$,

(ii) $A_k \overset{S_\theta^L(\mathcal{I}_W)}{\sim} B_k \cap L_\infty = A_k \overset{N_\theta^L(\mathcal{I}_W)}{\sim} B_k \cap L_\infty$.

It is not clear whether these results hold for any $g \in G$ and we leave it as an open problem.

We will now investigate the relationship betweenWijsman asymptotically $\mathcal{I}$-statistical equivalent of multiple L and Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of weight g for sequence of sets.

Theorem 2.3. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A_k, B_k be non-empty closed subsets of X. Then, $A_k \overset{S^L(\mathcal{I}_W)^g}{\sim} B_k$ implies $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$ if*

$$\liminf_r \frac{g(h_r)}{g(k_r)} > 1.$$

Proof. Since $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$, so we can find a $H > 1$ such that for sufficiently large r we have $\frac{g(h_r)}{g(k_r)} \geq H$.

Since $A_k \overset{S^L(\mathcal{I}_W)^g}{\sim} B_k$, for every $\varepsilon > 0$ and sufficiently large r we have

$$\begin{aligned}\frac{1}{g(k_r)} \left|\{k \leq k_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| &\geq \frac{1}{g(k_r)} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \\ &\geq H \frac{1}{g(k_r)} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right|.\end{aligned}$$

Then, for any $\delta > 0$, we get

$$\left\{r \in \mathbb{N} : \tfrac{1}{g(h_r)} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{g(k_r)} \left|\{k \leq k_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \geq H\delta\right\} \in \mathcal{I}.$$

This shows that $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$. □

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$, $\{\cup n : k_{r-1} < n \leq k_r, r \in C\} \in \mathcal{F}(\mathcal{I})$.

Theorem 2.4. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A_k, B_k be non-empty closed subsets of X. Then, $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$ implies $A_k \overset{S^L(\mathcal{I}_W)^g}{\sim} B_k$ if $\sup_r \sum_{i=0}^{r-1} \frac{g(h_{i+1})}{g(k_{r-1})} = B < \infty$.*

Proof. Suppose that $A_k \overset{S_\theta^L(\mathcal{I}_W)^g}{\sim} B_k$ and for ε, δ, $\delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N} : \tfrac{1}{g(h_r)} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{g(n)} \left|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in \mathcal{F}(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$K_j = \frac{1}{g(h_j)} \left|\{k \in I_j : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Now

$$\frac{1}{g(n)} \left|\{k \leq n : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| \leq \frac{1}{g(k_{r-1})} \left|\{k \leq k_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right|$$

$$= \frac{1}{g(k_{r-1})} \left|\{k \in I_1 : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| + ... + \frac{1}{g(k_{r-1})} \left|\{k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right|$$

$$= \frac{g(k_1)}{g(k_{r-1})} \frac{1}{g(h_1)} \left|\{k \in I_1 : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right| + \frac{g(k_2 - k_1)}{g(k_{r-1})} \frac{1}{g(h_2)} \left|\{k \in I_2 : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right|$$

$$+ ... + \frac{g(k_r - k_{r-1})}{g(k_{r-1})} \frac{1}{g(h_r)} \left|\{k \in I_2 : |d(x; A_k, B_k) - L| \geq \varepsilon\}\right|$$

$$= \frac{g(k_1)}{g(k_{r-1})} K_1 + \frac{g(k_2 - k_1)}{g(k_{r-1})} K_2 + ... + \frac{g(k_r - k_{r-1})}{g(k_{r-1})} K_r \leq \sup_{j \in J} K_j . \sup_r \sum_{i=0}^{r-1} \frac{g(k_{i+1} - k_i)}{g(k_{r-1})} < B\delta.$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\cup \{n : k_{r-1} < n \leq k_r, r \in C\} \subset T$ where $C \in \mathcal{F}(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Theorem 2.5. *Let (X, ρ) be a metric space and A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ be a lacunary sequence with* $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$ *then,*

$$A_k \overset{\sigma_1^L(\mathcal{I}_W)^g}{\sim} B_k \Rightarrow A_k \overset{N_\theta^L(\mathcal{I}_W)^g}{\sim} B_k.$$

Proof. Let $\liminf_r \frac{g(h_r)}{g(k_r)} > 1$. so we can find a $H > 1$ such that for sufficiently large r we have $\frac{g(h_r)}{g(k_r)} \geq H$. Let $\varepsilon > 0$ be given. Now observe that

$$\begin{aligned} \frac{1}{g(k_r)} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L| &\geq \frac{1}{g(k_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \\ &\geq \frac{g(h_r)}{g(k_r)} \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \\ &\geq H . \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \end{aligned}$$

Thus, we have

$$\frac{1}{g(k_r)} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L| < \varepsilon$$

implies

$$\frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| < \frac{\varepsilon}{H}.$$

So we can conclude that

$$\left\{ r \in \mathbb{N} : \frac{1}{g(k_r)} \sum_{k=1}^{k_r} |d(x; A_k, B_k) - L| < \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{g(h_r)} \sum_{k \in I_r} |d(x; A_k, B_k) - L| < \frac{\varepsilon}{H} \right\}.$$

Finally, since the set defined in the first inclusion is in the filter $\mathcal{F}(\mathcal{I})$, then the set defined in the second inclusion is also in the filter. This proves the theorem. □

Theorem 2.6. *Let* (X, ρ) *be a metric space and* A_k, B_k *be non-empty closed subsets of* X. *If* $\theta = \{k_r\}$ *be a lacunary sequence with* $\sup_r \sum_{i=0}^{r-1} \frac{g(h_{i+1})}{g(k_{r-1})} = B < \infty$, *then,*

$$A_k \overset{N_\theta^L(\mathcal{I}_W)^g}{\sim} B_k \Rightarrow A_k \overset{\sigma_1^L(\mathcal{I}_W)^g}{\sim} B_k.$$

Proof. The proof is similar to the proof of theorem 2.4. □

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