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Abel extensions of some classical Tauberian theorems

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ABSTRACT. The well-known classical Tauberian theorems given for A_λ (the discrete Abel mean) by Armitage and Maddox in [Armitage, H. D and Maddox, J. I., *Discrete Abel means*, Analysis, **10** (1990), 177–186] is generalized. Similarly the "one-sided" Tauberian theorems of Landau and Schmidt for the Abel method are extended by replacing $\lim As$ with Abel-lim $A\sigma_n^i(s)$. Slowly oscillating of $\{s_n\}$ is a Tauberian condition of the Hardy-Littlewood Tauberian theorem for Borel summability which is also given by replacing $\lim_t (Bs)_t = \ell$, where t is a continuous parameter, with $\lim_n (Bs)_n = \ell$, and further replacing it by Abel-lim$(B\sigma_k^i(s))_n = \ell$, where B is the Borel matrix method.

1. INTRODUCTION

Let $u = \{u_n\}$ be a sequence in $\mathbb{R}$ (or $\mathbb{C}$).

Definition 1.1. ([2], [3], [5]) A series $\sum\limits_{k=0}^{\infty} u_k$ of real (or complex) numbers is called Abel summable to ℓ if the series $\sum\limits_{k=0}^{\infty} s_k x^k$ is convergent for $0 \le x < 1$ and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} s_k x^k = \ell, \quad where \quad s_n = \sum_{k=0}^{n} u_k.$$

In this case, we write $Abel - \lim s_n = \ell$.

Definition 1.2. ([1]) A series $\sum\limits_{k=0}^{\infty} u_k$ of real (or complex) numbers is called A_λ (the discrete Abel mean) convergent to ℓ if the series $\sum\limits_{k=0}^{\infty} s_k x_n^k$ is convergent for all n and

$$\lim_{x_n\to 1^-}(1-x_n)\sum_{k=0}^{\infty} s_k x_n^k = \ell,$$

where $\lambda = \{\lambda_n\}$ is a given sequence such that $1 \le \lambda_0 < \lambda_1 < ... < \lambda_n \to \infty$ and the sequence $\{x_n\}$ is defined by $x_n = 1 - \dfrac{1}{\lambda_n}$. Clearly $0 \le x_0 < x_1 < ... < x_n \to 1$. In this case, we write $A_\lambda - \lim s_n = \ell$.

From definition 1.2, we say that the $\{s_n\}$ is in the domain of the method A_λ if the sequence $(A_\lambda s)_n := (1-x_n)\sum\limits_{k=0}^{\infty} s_k x_n^k$ is convergent for all n.
For $i \in \mathbb{N}$ and $n \in \mathbb{N}^*$, define

$$\sigma_n^i(s) = \begin{cases} \frac{1}{n+1}\sum\limits_{k=0}^{n}\sigma_k^{i-1}(s) & if \quad i \ge 1 \\ s_n & if \quad i = 0. \end{cases}$$

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Abel's well- known Limit Theorem says that the Abel summability method is regular if $\lim s_n = \ell$ implies $Abel - \lim s_n = \ell$. As we know the converse is false in general, e.g Abel-$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2}$ (Abel) but $\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k \neq \frac{1}{2}$. Also, it is obvious that $Abel - \lim s_n = \ell$ implies $A_\lambda - \lim s_n = \ell$. Hence A_λ also defines a regular method. A_λ summability method is regular; that is, if, $A_\lambda - \lim s_n = \ell$ then $A_\lambda - \lim \sigma_n^1(s) = \ell$.

By [7], the series $\Sigma_{n=0}^{\infty} u_n$ is Borel summable to ℓ provided that

$$\lim_{t\to\infty} e^{-t}\sum_{k=0}^{\infty}\frac{s_k t^k}{k!} = \ell.$$

Consider as in [4] the summability matrix $B = (b_{nk})$ is given by

$$b_{nk} = \frac{e^{-n}n^k}{k!}$$

By [11], it is known that $\{s_n\}$ is slowly oscillating if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that

$$|s_m - s_n| < \varepsilon \; if \; n \geq N(\varepsilon) \; and \; n \leq m \leq (1+\delta)n,$$

and $\{s_n\}$ of real numbers is slowly decreasing if

$$\liminf(s_m - s_n) \geq 0 \;\; whenever \;\; n \to \infty, \;\; m > n \;\; with \;\; \frac{m}{n} \to 1.$$

Thus, in particular, $\{s_n\}$ is slowly oscillating when $n\Delta s_n$ is bounded and $\{s_n\}$ is slowly decreasing when $n\Delta s_n$ is bounded below.

Also, we say that $\{s_n\}$ is strongly slowly oscillating if

$$(s_m - s_n) \to 0 \;\; whenever \;\; n \to \infty, \;\; m > n \;\; with \;\; \frac{m}{n} = O(1);$$

and $\{s_n\}$ is strongly slowly decreasing if $\{s_n\}$ is real and

$$\liminf(s_m - s_n) \geq 0 \;\; whenever \;\; n \to \infty, \;\; m > n \;\; with \;\; \frac{m}{n} = O(1).$$

Define $t_n = \sum_{k=1}^{n} k u_k = \sum_{k=1}^{n} k\Delta s_k$ and $\Delta s_k = s_k - s_{k-1}, s_{-1} = 0$. We will prove that

$$t_n = \sum_{k=1}^{n} k u_k = (n+1)s_n - \sum_{k=0}^{n} s_k. \tag{1.1}$$

We prove this by using mathematical induction. We show that our claims true for $n = 1$:

$$t_1 = 1u_1 = 2s_1 - (s_0 + s_1) = 2s_1 - s_1 - s_0 = s_1 - s_0 = u_1.$$

$$For \quad n = 2, \quad t_2 = u_1 + 2u_2 = 3s_2 - (s_0 + s_1 + s_2) = 2s_2 - s_0 - s_1 = 2u_2 + u_1.$$

Assume that it is true for $n = m$;

$$t_m = \sum_{k=1}^{m} k u_k = (m+1)s_m - \sum_{k=0}^{m} s_k \tag{1.2}$$

and we prove that it is true for $n = m + 1$: we add both sides $(m+1)u_{m+1}$ of the equality (1.2)

$$t_m + (m+1)u_{m+1} = \sum_{k=1}^{m} k u_k + (m+1)u_{m+1} = (m+1)s_m + (m+1)u_{m+1} - \sum_{k=0}^{m} s_k$$

$$t_{m+1}=\sum_{k=1}^{m+1}ku_k=(m+1)s_{m+1}+s_{m+1}-s_{m+1}-\sum_{k=0}^{m}s_k$$

$$t_{m+1}=\sum_{k=1}^{m+1}ku_k=(m+2)s_{m+1}-\sum_{k=0}^{m+1}s_k.$$

Thus proof is done. We obtain from (1.1)

$$z_n:=\tfrac{t_n}{n+1}=s_n-\sigma_n^1(s)=\tfrac{1}{n+1}\sum_{k=1}^{n}k\Delta s_k=n\Delta\sigma_n^1(s)$$

and

$$\sigma_n^i(z)=\sigma_n^i(s)-\sigma_n^{i+1}(s)=n\Delta\sigma_n^{i+1}(s).$$

Here, $\{z_n\}$ is known as the Kronecker identity. The classical control modulo of the oscillatory behaviour of a sequence $\{s_n\}$ is denoted by $w_n^0(s)=n\Delta s_n$. The general control modulo of the oscillatory behaviour of nonnegative integer order $m\geq 1$ of a sequence $\{s_n\}$ is defined inductively in [3] by $w_n^m(s)=w_n^{m-1}(s)-\sigma_n^1(w_n^{m-1}(s))$. General control modulo is developed by Çanak in [2].

Throughout this paper, the symbols $s_n=o(1)$ and $s_n=O(1)$ mean that $s_n\to 0$ as $n\to\infty$ and that $\{s_n\}$ is bounded for large enough n, respectively.

Theorem 1.1. ([1]) *Let $\{\lambda_n\}$ be a strictly increasing sequence of real numbers which tends to infinity such that*

$$\lim_n\frac{\lambda_{n+1}}{\lambda_n}=1.$$

If the $A_\lambda-\lim s_n=\ell$ and $\{s_n\}$ is slowly decreasing, then $\lim s_n=\ell$.

Lemma 1.1. ([1]) *If $\{s_n\}$ is slowly decreasing, then $\left\{\dfrac{t_n}{n}\right\}$ is bounded below.*

Now, we will prove that the hypothesis $A_\lambda-\lim s_n=\ell$ and slowly decreasing of $\{s_n\}$ can be replaced by $A_\lambda-\lim\sigma_n^i(s)=\ell$ and slowly decreasing of $\{z_n\}$. So, we generalize some classical types of Tauberian theorems for given A_λ. Moreover, we extend the "one-sided" Tauberian theorems of Landau and Schmidt's Tauberian theorems for the Abel method by replacing $\lim As$ with Abel-$\lim A\sigma_n^i(s)$.
Before proving our statements, we recall more results that we will need in the sequel.

Theorem 1.2. ([9], [11])
(**1**) *If Abel-*$\lim s_n=\ell$ *and* $n\Delta s_n\geq -c$ *for a positive number c then* $\lim s_n=\ell$.
(**2**) *Let a sequence $\{s_n\}$ of real numbers be slowly decreasing. Then*
Abel $-\lim s_n=\ell$ *implies* $\lim s_n=\ell$.
(**3**) *If $Borel-\lim s_n=\ell$ and $\Delta s_n=o(1)$ then $\lim\sigma_n^1(s)=\ell$.*

2. Main results

Lemma 2.2. *If the $\{s_n\}$ is in the domain of method A_λ for which $\lambda_n=n^\beta$, for some $\beta\geq 1$ and $n\Delta s_n\geq -c$ for some positive c, then the transformed sequence $n\Delta(A_\lambda\sigma_k^i(s))_n$ is also of one-sided, that is $n\Delta(A_\lambda\sigma_k^i(s))_n\geq c_1$ for some positive c_1.*

Proof. By the proof of Theorem 2.5 in [6], if $n\Delta s_n\geq -c$ for a positive number c then $n\Delta\sigma_n^i(s)\geq -c$. In [1] Armitage and Maddox showed that if we let $v_k(x)=\dfrac{x^k}{k}$ then

$$(1-x)\sum_{k=0}^{\infty}\sigma_k^i(s)x^k=\sum_{k=1}^{\infty}\Delta\sigma_k^i(s)x^k=\sum_{k=1}^{\infty}y_k(v_k(x)-v_{k+1}(x)),\quad 0<x<1,$$

where $y_k = \sum_{j=1}^{k} j\Delta\sigma_j^i(s)$. It follows that from $n\Delta\sigma_n^i(s) \geq -c$, $y_k = \sum_{j=1}^{k} j\Delta\sigma_j^i(s) \geq -kc$. Hence we see that since $\{\sigma_k^i(s)\}$ verifies the one-sided Tauberian condition, $\{y_k\}$ is bounded below by $-kM$ for some positive number M. If the Abel transform of $\{\sigma_n^i(s)\}$ is denoted by $A(x) = (A\sigma_k^i(s))_x$ then, for such a positive constant M, we have

$$
\begin{aligned}
n\Delta A(x_n) &= n\Big(A(x_n) - A(x_{n-1})\Big) = n\Big(\sum_{k=1}^{\infty} y_k \int_{x_{n-1}}^{x_n} y^{k-1}(1-y)dy\Big) \\
&\geq -n\Big(M\sum_{k=1}^{\infty} k \int_{x_{n-1}}^{x_n} y^{k-1}(1-y)dy\Big) = -n\Big(M \int_{x_{n-1}}^{x_n} \sum_{k=1}^{\infty} k y^{k-1}(1-y)dy\Big) \\
&= -n\Big(M \int_{x_{n-1}}^{x_n} (1-y)^{-1}dy\Big) \geq n\Big(-M\log\tfrac{n^\beta}{(n-1)^\beta}\Big) \geq -M\log\Big(\tfrac{n^\beta+1}{(n-1)^\beta}\Big)^n \\
&= -M\log\Big(\tfrac{n^\beta}{(n-1)^\beta} + \tfrac{1}{(n-1)^\beta}\Big)^n = -M\log\Big(C^\beta + D^\beta\Big)^n \\
&= -M\log C^{n\beta}\Big[1 + (\tfrac{D}{C})^\beta\Big]^n = -M\Big[\log(\tfrac{n}{n-1})^{n\beta} + \log(1+\tfrac{1}{n^\beta})^n\Big] \\
&= -M\left[\beta\log\Big(1+\tfrac{1}{n-1}\Big)^n + \log\Big[\Big[1+\Big(\tfrac{1}{n}\Big)^\beta\Big]^{n^\beta}\Big]^{n^{1-\beta}}\right] \\
&= -M\left[\beta\log\Big[\Big(1+\tfrac{1}{n-1}\Big)^{n-1}\Big(1+\tfrac{1}{n-1}\Big)\Big] + \log\Big[\Big[1+\Big(\tfrac{1}{n}\Big)^\beta\Big]^{n^\beta}\Big]^{n^{1-\beta}}\right].
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\liminf_n n\Delta A_\lambda(x_n) \geq \\
&\liminf_n -M\left[\beta\log\Big[(1+\tfrac{1}{n-1})^{n-1}(1+\tfrac{1}{n-1})\Big] + \log\Big[\Big[1+(\tfrac{1}{n})^\beta\Big]^{n^\beta}\Big]^{n^{1-\beta}}\right] \\
&= -M\limsup_n \left[\beta\log\Big[(1+\tfrac{1}{n-1})^{n-1}(1+\tfrac{1}{n-1})\Big] + \log\Big[\Big[1+(\tfrac{1}{n})^\beta\Big]^{n^\beta}\Big]^{n^{1-\beta}}\right] \\
&= -M(\beta + \log e^0) = -M\beta.
\end{aligned}
$$

Consequently, we see that the sequence $(A_\lambda\sigma_k^i(s))_n$ obeys the one-sided Tauberian condition. □

Lemma 2.3. *If the $\{s_n\}$ is in the domain of the method A_λ and is of slowly decreasing then the transformed sequence $(A_\lambda\sigma_k^i(s))_n$ is also of slowly decreasing.*

Proof. Proof is similar to one of Lemma 2.2. □

Lemma 2.4. *If $\{s_n\}$ is slowly decreasing then $\{\sigma_n^i(s)\}$ for all $i \geq 1$ is slowly decreasing.*

Proof. We claim that $\{\sigma_n^i(s)\}$ for all $i \geq 1$ is slowly decreasing. We will prove this by using mathematical induction. We show that our claims true for $i = 1$. Let $\{s_n\}$ be slowly decreasing. By Lemma 1.1, $z_n = \dfrac{t_n}{n} = s_n - \sigma_n^1(s)$ is bounded below. Hence, $z_n = n\Delta\sigma_n^1(s)$ is bounded below. Consequently, $\{\sigma_n^1(s)\}$ is slowly decreasing. Assume that it is true for

$i=t-1$, and we will prove that it is true for $i=t$. By assumption, since $\{\sigma_n^{t-1}(s)\}$ is slowly decreasing and applying Lemma 1.1, we obtain $\sigma_n^{t-1}(z)) = n\Delta\sigma_n^t(s)$ is bounded below. Hence, there exits a positive constant M such that $n\Delta\sigma_n^t(s) \geq -M$ for all n. For n large enough, $n > N_1, \sigma_m^t(s) - \sigma_n^t(s) = \sum_{k=n+1}^{m} \Delta\sigma_k^t(s) \geq -\sum_{k=n+1}^{m} \frac{M}{k} = -M(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{m}) \geq -M(\frac{1}{n+1}+\frac{1}{n+1}+...+\frac{1}{n+1}) \geq -M(1-\frac{m}{n}) \geq -M(1-1) = 0, \ by \ \frac{m}{n} \to 1.$ □

Lemma 2.5. *If $\{s_n\}$ is slowly oscillating then $\{\sigma_n^i(u)\}$ for all $i \geq 1$ is slowly oscillating.*

Proof. Proof is similar to one of Lemma 2.4. □

Lemma 2.6. *If $\{s_k\}$ is slowly oscillating then $\{(B\sigma_k^i(s))_n\}$ is slowly oscillating.*

Proof. By Lemma 2.5, slow oscillating of $\{s_k\}$ implies both $\{\sigma_k^i(s)\}$ for all $i \geq 1$ is slowly oscillating and $\Delta\sigma_k^i(s) = o(1)$. It follows that $\left|\sum_{k=n+1}^{m} \Delta\sigma_k^i(s)\right| \leq \frac{\varepsilon}{2}$ for n large enough. Thus we have

$$\begin{aligned}
\left|(B\sigma_k^i(s))_m - (B\sigma_k^i(s))_n\right| &= \left|(B\sigma_k^i(s))_{n+r} - (B\sigma_k^i(s))_n\right| \\
&= \Big|\sum_{p=0}^{\infty}\sum_{j=0}^{p} b_{r,j}b_{n,p-j}\sigma_p^i(s) - \sum_{k=0}^{\infty} b_{n,k}\sigma_k^i(s)\Big| \\
&= \Big|\sum_{j=0}^{\infty} b_{r,j}\sum_{p=j}^{\infty} b_{n,p-j}\sigma_p^i(s) - \sum_{k=0}^{\infty} b_{n,k}\sigma_k^i(s)\Big| \\
&= \Big|\sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\sigma_{k+j}^i(s) - \sum_{k=0}^{\infty} b_{n,k}\sigma_k^i(s)\Big| \\
&= \Big|\sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}(\sigma_{k+j}^i(s) - \sigma_k^i(s))\Big| \\
&= \Big|\textstyle\sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\sum_{p=1}^{j} \Delta\sigma_{k+p}^i(s)\Big| \\
&\leq \sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\sum_{p=1}^{j_0} |\Delta\sigma_{k+p}^i(s)| + \sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\Big|\sum_{p=j_0+1}^{j} \Delta\sigma_{k+p}^i(s)\Big| \\
&\leq \sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\sum_{p=1}^{j_0} \frac{1}{k+p} + \sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\frac{\varepsilon}{2} \\
&\leq \sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k}\sum_{p=1}^{j} \frac{1}{k+1} + \frac{\varepsilon}{2}\sum_{j=0}^{\infty} b_{r,j}\sum_{k=0}^{\infty} b_{n,k} \\
&\leq \sum_{j=0}^{\infty} jb_{r,j}\sum_{k=0}^{\infty} b_{n,k}\frac{1}{k+1} + \frac{\varepsilon}{2} = \sum_{k=0}^{\infty} b_{n,k}\frac{1}{k+1} + \frac{\varepsilon}{2} \\
&= \sum_{k=0}^{\infty} b_{n,k}\frac{n}{n(k+1)} + \frac{\varepsilon}{2} = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}\frac{n}{k+1} + \frac{\varepsilon}{2} \\
&= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k+1} + \frac{\varepsilon}{2} = \frac{1}{n} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for $n \geq n_0$ large enough. □

Theorem 2.3 extends the Theorem 1.1 which is given in [1].

Theorem 2.3. *Let* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$. *If* $A_\lambda - \lim \sigma_n^i(s) = \ell$ *and* $\{z_n\}$ *is slowly decreasing then* $\lim s_n = \ell$.

Proof. Since A_λ method is regular, $A_\lambda - \lim \sigma_n^{i+1}(s) = \ell$. Hence, we have $A_\lambda - \lim \sigma_n^i(z) = 0$. By Lemma 2.4, as $z_n = n\Delta\sigma_n^1(s)$ is slowly decreasing, $(\sigma_n^i(z)) = n\Delta\sigma_n^{i+1}(s)$ for all $i \geq 1$ is slowly decreasing. Since $\{\sigma_n^i(z)\}$ is A_λ summability to 0, $\lim \sigma_n^i(z) = \lim n\Delta\sigma_n^{i+1}(s) = 0$. $\lim n\Delta\sigma_n^{i+1}(s) = 0$ implies $n\Delta\sigma_n^{i+1}(s)$ is bounded below, that is, $n\Delta\sigma_n^{i+1}(s) \geq -c$ for some positive c. It follows that $\{\sigma_n^{i+1}(s)\}$ is slowly decreasing. If $\sigma_n^i(z) = \sigma_n^i(s) - \sigma_n^{i+1}(s)$ is slowly decreasing, then $\{\sigma_n^i(s)\}$ is slowly decreasing. From $A_\lambda - \lim \sigma_n^i(s) = \ell$, we have $\lim \sigma_n^i(s) = \ell$. By the fact that every sequence (C, 1) limitable is Abel limitable, we have $Abel - \lim \sigma_n^{i-1}(s) = \ell$. $Abel - \lim \sigma_n^{i-1}(s) = \ell$ implies $A_\lambda - \lim \sigma_n^{i-1}(s) = \ell$. If we continue in that way, we obtain, $A_\lambda - \lim s_n = \ell$. By Theorem 1.1, $\lim s_n = \ell$. □

Theorem 2.3 generalises Theorem 1.1. For example, if we consider the case $i = 1$ then the sequence $\{s_n\}$ which is the Taylor coefficients of the function f defined by $f(t) = \sin(1-t)^{-1}$ on $0 < t < 1$ is not A_λ convergent however, Cesaro of the sequence $\{s_n\}$ is A_λ convergent.

An immediate consequence of Theorem 2.3 is that the boundedness below of $n\Delta z_n$ is a Tauberian condition for A_λ.

Corollary 2.1. *Let* $\lim_{n\to\infty} \dfrac{\lambda_{n+1}}{\lambda_n} = 1$. *If* $A_\lambda - \lim \sigma_n^i(s) = \ell$ *and* $n\Delta z_n \geq -c$ *for some positive* c *, then* $\lim s_n = \ell$.

Also, by considering $\{s_n\}$ as a complex sequence we deduce the following result.

Corollary 2.2. *Let* $\lim_n \dfrac{\lambda_{n+1}}{\lambda_n} = 1$. *If* $A_\lambda - \lim \sigma_n^i(s) = \ell$ *and slowly oscillating of* $\{z_n\}$ *, then* $\lim s_n = \ell$.

The proof of Lemma 1.1 in [1] can be modified to show that if $\{z_n\}$ is strongly slowly decreasing , then $\lim inf\left(\dfrac{t_n}{n}\right) \geq 0$. In view of this, the proof of Theorem 2.3 can be adapted to yield the following result:

Theorem 2.4. *Let* $\dfrac{\lambda_{n+1}}{\lambda_n} = O(1)$. *If* $A_\lambda - \lim \sigma_n^i(s) = \ell$ *and* $\{z_n\}$ *is the strongly slowly decreasing, then* $\lim s_n = \ell$.

It follows that for a complex sequence $\{\sigma_n^i(s)\}$ the strongly slowly oscillating of $\{z_n\}$ is a Tauberian condition for A_λ when $\dfrac{\lambda_{n+1}}{\lambda_n} = O(1)$.

Theorem 2.4 is a generalization of the Theorem 7 in [1]. The strongly slowly decreasing of $\{z_n\}$ does not imply the strongly slowly decreasing of $\{s_n\}$. As an example, if we take $s_n = \sum_{j=1}^n \frac{1}{j^{\frac{1}{2}}} + \sum_{k=1}^n \frac{1}{k} \sum_{j=1}^n \frac{1}{j^{\frac{1}{2}}}$, we see that $z_n = \sum_{j=1}^n \frac{1}{j^{\frac{1}{2}}}$ is slowly decreasing but clearly, $\{s_n\}$ is not slowly decreasing.

Next theorem extends the classical Tauberian theorems of Hardy and Littlewood in [7] and [10] respectively.

Theorem 2.5. *Let* $\lim_{n\to\infty} \dfrac{\lambda_{n+1}}{\lambda_n} = 1$ *and* $\lambda_n = n^\beta$, *for some* $\beta \geq 1$. *If Abel*-$\lim(A_\lambda \sigma_n^i(s))_k = \ell$ *and* $n\Delta s_n \geq -c$, *then* $\lim s_n = \ell$.

Proof. By the proof of Theorem 2.5 of in [6], $n\Delta s_n \geq -c$ for a positive number c implies $n\Delta\sigma_n^i(s) \geq -c$. Hence, by Lemma 2.2, we see that $(A_\lambda\sigma_n^i(s))_k$ obeys the one-sided Tauberian condition. From Abel-$\lim(A_\lambda\sigma_n^i(s))_k = \ell$, we have $\lim A_\lambda\sigma_n^i(s) = \ell$, by above (1) in Theorem 1.2. Now by Theorem 1.1 implies that $\sigma_n^i(s)$ is Abel summable to ℓ. Since $n\Delta\sigma_n^i(s) \geq -c$, $\lim\sigma_n^i(s) = \ell$. By the fact that every sequence (C, 1) limitable is Abel limitable, we have $Abel - \lim\sigma_n^{i-1}(s) = \ell$. Since $Abel - \lim\sigma_n^{i-1}(s) = \ell$ and $n\Delta\sigma_n^{i-1}(s) \geq -c$, we obtain that $\lim\sigma_n^{i-1}(s) = \ell$. If we continue in that way, we obtain, $Abel - \lim s_n = \ell$. By (1) in Theorem 1.2, $\lim s_n = \ell$. □

Remark 2.1. The following result, which is analogous to Theorem 2.5, may be proved for the slow decrease condition by using the Tauberian theorems results provided in [11] and Lemma 2.4. This then extends the classical Tauberian theorem of Schmidt [11].

Theorem 2.6. *Let* $\lim_{n\to\infty}\dfrac{\lambda_{n+1}}{\lambda_n} = 1$. *If Abel-*$\lim(A_\lambda\sigma_n^i(s))_k = \ell$ *and the* $\{s_n\}$ *is slowly decreasing then* $\lim s_n = \ell$.

Final theorem is a Abel extension of Hardy and Littlewood's Tauberian theorem in [8] for Borel summability.

Theorem 2.7. *If Abel-*$\lim(B\sigma_n^i(s))_k = \ell$ *and* $\{s_n\}$ *is slowly oscillating then* $\lim s_n = \ell$.

Proof. By Lemma 2.3, slowly oscillating of $\{s_n\}$ implies both slowly oscillating of $\{\sigma_n^i(s)\}$ and $\Delta\sigma_n^i(s) = o(1)$. By Lemma 2.6, we conclude that $(B\sigma_n^i(s))_k$ is slowly oscillating. This allows us to apply (2) in Theorem 1.2 that $\lim(B\sigma_n^i(s))_k = \ell$. Now (3) in Theorem 1.2 gives $\lim\sigma_n^{i+1}(s) = \ell$. By the fact that every sequence (C, 1) limitable is Abel limitable, we have $Abel - \lim\sigma_n^i(s) = \ell$. Since $Abel - \lim\sigma_n^i(s) = \ell$ and $\{\sigma_n^i(s)\}$ is slowly decreasing, we obtain $\lim\sigma_n^i(s) = \ell$. If we continue in that way, we obtain $Abel - \lim s_n = \ell$. Again by (2) in Theorem 1.2, we get $\lim s_n = \ell$. □

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