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Existence of countably many symmetric positive solutions for system of even order time scale boundary value problems in Banach spaces

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ABSTRACT. This paper establishes the existence and uniqueness of the solutions to the system of even order differential equations on time scales,

$$(-1)^n u_1^{(\Delta\nabla)^n}(t) = \omega_1(t) f_1\big(u_1(t), u_2(t)\big),\ t \in [0, T]_{\mathbb{T}},\ n \in \mathbb{Z}^+,$$
$$(-1)^n u_2^{(\Delta\nabla)^m}(t) = \omega_2(t) f_2\big(u_1(t), u_2(t)\big),\ t \in [0, T]_{\mathbb{T}},\ m \in \mathbb{Z}^+,$$

satisfying two-point Sturm-Liouville integral boundary conditions

$$\alpha_{i+1} u_1^{(\Delta\nabla)^i}(0) - \beta_{i+1} u_1^{(\Delta\nabla)^i\Delta}(0) = \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1,$$
$$\alpha_{i+1} u_1^{(\Delta\nabla)^i}(T) + \beta_{i+1} u_1^{(\Delta\nabla)^i\Delta}(T) = \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1,$$
$$\gamma_{j+1} u_2^{(\Delta\nabla)^j}(0) - \delta_{j+1} u_2^{(\Delta\nabla)^j\Delta}(0) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le m-1,$$
$$\gamma_{j+1} u_2^{(\Delta\nabla)^j}(T) + \delta_{j+1} u_2^{(\Delta\nabla)^j\Delta}(T) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le m-1,$$

by utilizing Schauder fixed point theorem. We also establish the existence of countably many symmetric positive solutions for the above problem by applying Hölder's inequality and Krasnoselskii's fixed point theorem.

1. INTRODUCTION

Recently, researchers are shown much interest on the existence of positive solutions to boundary value problems for dynamic equations on time scales [3, 5, 6, 7, 11, 12, 15]. This has been mainly due to unification of the theory of differential and difference equations in time scale dynamics. The theory is widely applied to various situations, like, in the study of insect population models, neural networks, heat transfer, and epidemic models [2, 7]. For details on time scale calculus we refer to the books by Bohner and Peterson [7, 8], Lakshmikantham et al.[23] and the papers [1, 4, 20].

The boundary value problems with integral boundary conditions occur in the study of nonlocal phenomena in different areas of applied mathematics, physics and engineering, in particular, in heat conduction, chemical engineering, underground waterflow, thermo-elasticity, plasma physics [3, 11, 12, 21, 22, 25, 32, 35]. Recently, much attention is paid to establish the existence of positive solutions to boundary value problems with integral boundary conditions on time scales [10, 13, 14, 19, 24, 26, 31, 33] and for the existence of symmetric positive solutions for higher order boundary value problems with different types of boundary conditions [9, 17, 18, 28, 29].

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In [26], Oguz and Topal studied following system of boundary value problems on time scales,

$$(-1)^n u_k^{(\Delta\nabla)^n}(t) = f_k\big(t, u_1(t), u_2(t)\big),\ t \in [a,b]_{\mathbb{T}},\ k = 1,2,$$
$$\alpha u_k^{(\Delta\nabla)^i}(a) - \beta u_k^{(\Delta\nabla)^i\Delta}(a) = 0,\ 0 \le i \le n-1,\ k = 1,2,$$
$$\alpha u_k^{(\Delta\nabla)^i}(b) + \beta u_k^{(\Delta\nabla)^i\Delta}(b) = 0,\ 0 \le i \le n-1,\ k = 1,2,$$

under the conditions that $f_k\,(k=1,2)$ are non-increasing with respect to u_1, u_2 and established a necessary condition for the existence and uniqueness of symmetric positive solutions by the method of monotone iterative technique.

Motivated by the work mentioned above, we consider the system of even order differential equations on time scales,

$$\begin{cases} (-1)^n u_1^{(\Delta\nabla)^n}(t) = \omega_1(t) f_1\big(u_1(t), u_2(t)\big),\ t \in [0,T]_{\mathbb{T}}, \\ (-1)^m u_2^{(\Delta\nabla)^m}(t) = \omega_2(t) f_2\big(u_1(t), u_2(t)\big),\ t \in [0,T]_{\mathbb{T}}, \end{cases} \tag{1.1}$$

satisfying the Sturm-Liouville integral boundary conditions

$$\begin{cases} \alpha_{i+1} u_1^{(\Delta\nabla)^i}(0) - \beta_{i+1} u_1^{(\Delta\nabla)^i\Delta}(0) = \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1, \\ \alpha_{i+1} u_1^{(\Delta\nabla)^i}(T) + \beta_{i+1} u_1^{(\Delta\nabla)^i\Delta}(T) = \int_0^T a_{i+1}(s) u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1, \\ \gamma_{j+1} u_2^{(\Delta\nabla)^j}(0) - \delta_{j+1} u_2^{(\Delta\nabla)^j\Delta}(0) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le m-1, \\ \gamma_{j+1} u_2^{(\Delta\nabla)^j}(T) + \delta_{j+1} u_2^{(\Delta\nabla)^j\Delta}(T) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s)\nabla s,\ 0 \le j \le m-1, \end{cases} \tag{1.2}$$

where $n, m \in \mathbb{Z}^+$(positive integers), $\mathbb{T}$ is a symmetric time scale, $T \in \mathbb{T}$, $f_k \in C\big(\mathbb{R}\times\mathbb{R}, \mathbb{R}\big)$, $\omega_k(t) \in L^p_\nabla[0,1]_{\mathbb{T}}(k=1,2)$ for some $p \ge 1$ and establish the existence and uniqueness of the solutions for the above system by applying Schauder fixed point theorem and the existence of countably many symmetric positive solutions by allowing $\omega_k(t)(k=1,2)$ to have countably many singularities in $(0, \frac{T}{2})_{\mathbb{T}}$ using the Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach Space.

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which are useful to study the behavior of solution of the boundary value problem (1.1)-(1.2). In Section 3, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2), estimate bounds for the Green's function, and some lemmas which are needed in establishing our main results are provided. In Section 4, we obtain existence and uniqueness of a solution for (1.1)-(1.2), due to Schauder fixed point theorem. In Section 5, we establish a criteria for the existence of countably many symmetric positive solutions for the boundary value problem (1.1)-(1.2) by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space. Finally, we provide an example of a family of functions $\omega(t)$ that satisfy required conditions.

2. Preliminaries

In this section, we provide some definitions and lemmas which are useful for our later discussions; for details, see [3, 5, 6, 7, 16, 30, 34].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$,

$$\sigma(t) = \inf\{r \in \mathbb{T} : r > t\}, \quad \rho(t) = \sup\{r \in \mathbb{T} : r < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise let $\mathrm{T}^k = \mathbb{T}$.

Definition 2.2. An interval time scale $\mathbb{T} = [a, b]_{\mathbb{T}}$ is said to be symmetric if for any given $t \in \mathbb{T}$, we have $b + a - t \in \mathbb{T}$ and a function $u : \mathbb{T} \to \mathbb{R}$ is said to be symmetric on $\mathbb{T}$ if for any given $t \in \mathbb{T},\ u(t) = u(b + a - t)$.

By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Similarly other intervals can be defined.

Definition 2.3. A function $u : \mathbb{T} \to \mathbb{R}$ is said to be concave if for any $t_1, t_2 \in \mathbb{T}$ and $c \in [0, 1],\ u(ct_1 + (1 - c)t_2) \geq cu(t_1) + (1 - c)u(t_2)$.

Definition 2.4. Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that P holds on $A \backslash E_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $E_2 \subset A$ with $\mu_\nabla(E_2) = 0$ such that P holds on $A \backslash E_2$, then P is said to hold ∇–a.e. on A.

Definition 2.5. Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M,\ \nabla - a.e.\ on\ E \text{ if } p = +\infty.$$

Lemma 2.1. *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big)f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

For convenience, we introduce the following notation throughout the paper: For $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$,

$$\xi_i := \int_0^T a_i(r)\nabla r,\ \zeta_i := \frac{\alpha_i}{\alpha_i - \xi_i},\ \xi'_i := \int_0^T b_i(r)\nabla r,\ \zeta'_i := \frac{\gamma_i}{\gamma_i - \xi'_i},$$

$$g_i := \int_0^T G_i(r, r)\nabla r,\ g'_i := \int_0^T \mathcal{G}_i(r, r)\nabla r,\ g_i(\tau) := \int_\tau^{T-\tau} G_i(r, r)\nabla r,$$

$$g^*_i(\tau) := \int_\tau^{T-\tau} \mathcal{G}_i(r, r)\nabla r,$$

where

$$G_i(t,s) := \frac{1}{\alpha_i d_i}\begin{cases} (\beta_i+\alpha_i t)(\beta_i+\alpha_i(T-s)), \ t \le s, \\ (\beta_i+\alpha_i s)(\beta_i+\alpha_i(T-t)), \ s \le t, \end{cases} \text{in which } d_i = \alpha_i T + 2\beta_i,$$

$$\mathcal{G}_i(t,s) := \frac{1}{\gamma_i d'_i}\begin{cases} (\delta_i+\gamma_i t)(\delta_i+\gamma_i(T-s)), \ t \le s, \\ (\delta_i+\gamma_i s)(\delta_i+\gamma_i(T-t)), \ s \le t, \end{cases} \text{in which } d'_i = \gamma_i T + 2\delta_i.$$

We make the following assumptions: $J := [0,T]_{\mathbb{T}}$ and for $1 \le i \le 2$:

(H1) there exists a sequence $\{t_k\}_{k=1}^{\infty}$ $(k \in \mathbb{N})$, $t_1 < \frac{T}{2}$, $\lim_{k\to\infty} t_k = t^* \ge 0$ and $\lim_{t\to t_k} \omega_i(t) = +\infty$ for $k = 1,2,3,\cdots$,

(H2) $\omega_i \in L^p_\nabla(J)$ for some $1 \le p \le +\infty$ and there exists $\epsilon > 0$ such that $\omega_i(t) \ge \epsilon$ for all $[t^*, 1-t^*]_{\mathbb{T}}$,

(H3) $\alpha_i, \beta_i, \gamma_j, \delta_j \ge 0$ such that $d_i := \alpha_i T + 2\beta_i > 0$, $d'_j := \gamma_j T + 2\delta_j > 0$ for each $1 \le i \le n$ and $1 \le j \le m$,

(H4) $a_j, b_j \in L^1_\nabla(J)$ for all $1 \le i \le n, 1 \le j \le m$ are nonnegative and $\alpha_i > \xi_i$, $\gamma_j > \xi'_j$ for all $1 \le i \le n$, $1 \le j \le m$ on J.

3. Green's function and bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function.

Lemma 3.2. *Let* $(H3), (H4)$ *hold. Then for any* $g_1(t) \in C(J)$, *the boundary value problem,*

$$-u_1^{\Delta\nabla}(t) = g_1(t), \ t \in J, \tag{3.3}$$

$$\alpha_i u_1(0) - \beta_i u_1^{\Delta}(0) = \int_0^T a_i(s)u_1(s)\nabla s, \ 1 \le i \le n, \tag{3.4}$$

$$\alpha_i u_1(T) + \beta_i u_1^{\Delta}(T) = \int_0^T a_i(s)u_1(s)\nabla s, \ 1 \le i \le n, \tag{3.5}$$

has a unique solution

$$u_1(t) = \int_0^T H_i(t,s)g_1(s)\nabla s, \ for\ 1 \le i \le n, \tag{3.6}$$

where

$$H_i(t,s) = G_i(t,s) + \frac{1}{\alpha_i - \xi_i}\int_0^T G_i(r,s)a_i(r)\nabla r, \tag{3.7}$$

for $1 \le i \le n$.

Proof. Suppose u_1 is a solution of (3.3), then, we have

$$\begin{aligned} u_1(t) &= -\int_0^t\int_0^s g_1(r)\nabla r\Delta s + At + B \\ &= -\int_0^t (t-s)g_1(s)\nabla s + At + B \end{aligned}$$

where $A = \lim_{t\to 0^+} u^{\Delta}(t)$ and $B = u(0)$. Using the boundary conditions (3.4), (3.5), we can determined A and B as

$$A = \frac{1}{d_i}\int_0^T [\alpha_i(T-s) - \beta_i]g_1(s)\nabla s$$

$$B = \frac{1}{d_i}\Big[\int_0^T \frac{\beta_i}{\alpha_i}[(\alpha_i(T-s)+\beta_i)g_1(s)\nabla s + \int_0^T \frac{1}{\alpha_i}[\alpha_i T + 2\beta_i]a_i(s)u_1(s)\nabla s\Big]$$

Thus, we have

$$u_1(t) = \frac{1}{\alpha_i d_i}\Big[\int_0^t (\beta_i + \alpha_i s)(\beta_i + \alpha_i(T-t))g_1(s)\nabla s + \int_t^T (\beta_i + \alpha_i t)(\beta_i + \alpha_i(T-s))g_1(s)\nabla s\Big] + \frac{1}{\alpha_i}\int_0^T a_i(s)u_1(s)\nabla s$$

from which, we obtain

$$u_1(t) = \int_0^T G_i(t,s)g_1(s)\nabla s + \frac{1}{\alpha_i}\int_0^T a_i(s)u_1(s)\nabla s. \tag{3.8}$$

After certain computations we can determined,

$$\int_0^T a_i(s)u(s)\nabla s = \frac{\alpha_i}{\alpha_i - \xi_i}\int_0^T\Big[\int_0^T G_i(s,r)a_i(s)\nabla s\Big]g_1(r)\nabla r. \tag{3.9}$$

Hence

$$\begin{aligned} u(t) &= \int_0^T G_i(t,s)g_1(s)\nabla s + \frac{1}{\alpha_i - \xi_i}\int_0^T\Big[\int_0^T G_i(s,r)a_i(s)\nabla s\Big]g_1(r)\nabla r \\ &= \int_0^T\Big[G_i(t,s) + \frac{1}{\alpha_i - \xi_i}\int_0^T G_i(s,r)a_i(s)\nabla s\Big]g_1(r)\nabla r \\ &= \int_0^T H_i(t,s)g_1(s)\nabla s, \end{aligned}$$

where $H_i(t,s)$ is defined in (3.7). □

Lemma 3.3. *Assume that* $(H3), (H4)$ *hold and for* $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$ *define* $\eta_i(\tau) = \frac{\alpha_i\tau+\beta_i}{\alpha_i T+\beta_i}$. *Then* $G_i(t,s)$ *satisfies the following properties for* $1 \le i \le n$,

(i) $0 < G_i(t,s) \le G_i(s,s)$ *for all* $t, s \in J$,
(ii) $\eta_i(\tau)G_i(s,s) \le G_i(t,s)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J$,
(iii) $G_i(1-t, 1-s) = G_i(t,s)$ *for all* $t, s \in J$,
(iv) *For each* $s \in J$, *the functions* $G_i(.,s)$ *are concave in the first argument on* J.

Lemma 3.4. *Assume that* $(H3), (H4)$ *hold and for* $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. *Then* $H_i(t,s)$ *have the following properties for* $1 \le i \le n$,

(i) $0 < H_i(t,s) \le \zeta_i G_i(s,s)$ *for all* $t, s \in J$,
(ii) $\zeta_i\eta_i(\tau)G_i(s,s) \le H_i(t,s)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J$,
(iii) $H_i(1-t, 1-s) = H_i(t,s)$ *for all* $t, s \in J$,
(iv) *For each* $s \in J$, *the functions* $H_i(.,s)$ *are concave in the first argument on* J.

Lemma 3.5. *Assume that* $(H3), (H4)$ *hold and* $H_i(t,s)$ *is given in* (3.7) *for* $1 \le i \le n$,. *Let* $K_1(t,s) = H_1(t,s)$ *and define recursively*

$$K_i(t,s) = \int_0^T K_{i-1}(t,r)H_i(r,s)\nabla r, \quad for \quad 2 \le i \le n. \tag{3.10}$$

Then $K_n(t,s)$ *is the Green's function for the homogeneous boundary value problem*

$$(-1)^n u_1^{(\Delta\nabla)^n}(t) = 0,\ t \in J,$$

$$\alpha_{i+1}u_1^{(\Delta\nabla)^i}(0) - \beta_{i+1}u_1^{(\Delta\nabla)^i\Delta}(0) = \int_0^T a_{i+1}(s)u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1,$$

$$\alpha_{i+1}u_1^{(\Delta\nabla)^i}(T) + \beta_{i+1}u_1^{(\Delta\nabla)^i\Delta}(T) = \int_0^T a_{i+1}(s)u_1^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \le i \le n-1.$$

Lemma 3.6. *Assume that $(H3), (H4)$ hold and for $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. Define*

$$g_n^* = \prod_{i=1}^{n} g_i,\ \zeta_n^* = \prod_{i=1}^{n} \zeta_i,\ L_n(\tau) = \prod_{i=1}^{n} \zeta_i \eta_i(\eta),\ g_n(\tau) = \prod_{i=1}^{n-1} g_i(\tau),$$

then the Green's function $K_n(t, s)$ satisfies the following inequalities:

(i) $0 < K_n(t, s) \leq \zeta_n^* g_n^* G_n(s, s)$, *for all $t, s \in J$ and*
(ii) $K_n(t, s) \geq L_n(\tau) g_n(\tau) G_n(s, s)$, *for all $t \in [\tau, T - \tau]_{\mathbb{T}}$ and $s \in J$,*

Proof. It is clear that Green's function $H_n(t, s) \geq 0$, for all $t, s \in J$. Now we prove the inequality by induction on n and denote the statement by $p(n)$.
From (3.7) we have $K_1(t, s) = H_1(t, s) \leq \zeta_1 G_1(s, s)$ and

$$\begin{aligned} K_2(t, s) &= \int_0^T K_1(t, r) H_2(r, s) \nabla r \\ &\leq \int_0^T \zeta_1 G_1(r, r) \zeta_2 G_2(s, s) \nabla r \\ &\leq \prod_{i=1}^{2} \zeta_i \prod_{i=1}^{1} g_i G_2(s, s) \end{aligned}$$

Now for $t \in [\tau, 1 - \tau]_{\mathbb{T}}$, we have $K_1(t, s) = H_1(t, s) \geq \zeta_1 \eta_1(\tau) G_1(s, s)$, and

$$\begin{aligned} K_2(t, s) &= \int_0^T K_1(t, r) H_2(r, s) \nabla r \\ &\geq \zeta_1 \eta_1(\tau) \int_\tau^{T-\tau} G_1(r, r) H_2(r, s) \nabla r \\ &\geq \zeta_1 \eta_1(\tau) \int_\tau^{T-\tau} G_1(r, r) \zeta_2 \eta_2(\tau) G_2(s, s) \nabla r \\ &\geq \prod_{i=1}^{2} \zeta_i \eta_i(\tau) \prod_{i=1}^{1} g_i(\tau) G_2(s, s). \end{aligned}$$

Hence, $p(1), p(2)$ are true. Suppose $p(k)$ is true, then from (3.11), we have

$$\begin{aligned} K_{k+1}(t, s) &= \int_0^T K_k(t, r) H_{k+1}(r, s) \nabla r \\ &\leq \prod_{i=1}^{k} \zeta_i \prod_{i=1}^{k-1} g_i \int_0^T G_k(r, r) H_{k+1}(r, s) \nabla r \\ &\leq \prod_{i=1}^{k} \zeta_i \prod_{i=1}^{k-1} g_i \int_0^T G_k(r, r) \zeta_{k+1} G_{k+1}(s, s) \nabla r \\ &\leq \prod_{i=1}^{k+1} \zeta_i \prod_{i=1}^{k} g_i G_{k+1}(s, s) \end{aligned}$$

and for $t \in [\tau, 1-\tau]_{\mathbb{T}}$,

$$\begin{aligned} K_{k+1}(t,s) &= \int_0^T K_k(t,r)H_{k+1}(r,s)\nabla r \\ &\geq \prod_{i=1}^{k} \zeta_i\eta_i(\tau) \prod_{i=1}^{k-1} g_i(\tau) \int_0^T G_k(r,r)G_{k+1}(r,s)\nabla r \\ &\geq \prod_{i=1}^{k} \zeta_i\eta_i(\tau) \prod_{i=1}^{k-1} g_i(\tau) \int_\tau^{T-\tau} G_k(r,r)G_{k+1}(r,s)\nabla r \\ &\geq \prod_{i=1}^{k+1} \zeta_i\eta_i(\tau) \prod_{i=1}^{k} g_i(\tau) G_{k+1}(s,s). \end{aligned}$$

So, $p(k+1)$ holds. This completes the proof. □

Lemma 3.7. *The Green's function $K_i(t,s)$ for $1 \leq i \leq n$, satisfies the following conditions*

$$K_i(t,s) = K_i(1-t, 1-s)\, \forall\, t, s \in J \tag{3.11}$$

and for each $s \in J$, $K_i(\cdot, s)$ $(1 \leq i \leq n)$ is concave in the first argument on J.

Proof. The proof is by induction. For $i = 1$, the equation (3.11) is clear and assume that the equation (3.11) is true for fixed $i \geq 2$. Then from (3.10) and using transformation $r_1 = 1-r$, we have

$$\begin{aligned} K_{i+1}(t,s) &= \int_0^T K_i(t,r)H_{j+1}(r,s)\nabla r \\ &= \int_0^T K_i(1-t, 1-r)H_{i+1}(1-r, 1-s)\nabla r \\ &= \int_0^T K_i(1-t, r_1)H_{i+1}(r_1, 1-s)\nabla r_1 \\ &= K_{i+1}(1-t, 1-s). \end{aligned}$$

Now, to prove concavity of $K_n(.,s)$, let $c \in [0,1]$ and $t, r, s \in J$ with $t \leq r$ and using Lemma 3.3. For $n = 1$,

$$\begin{aligned} K_1(ct + (1-c)r, s) &= H_1(ct + (1-c)r, s) \\ &\geq cH_1(t,s) + (1-c)H_1(r,s) \\ &\geq cK_1(t,s) + (1-c)K_1(r,s). \end{aligned}$$

Next, we assume that $K_i(ct + (1-c)r, s) \geq cK_i(t,s) + (1-c)K_i(r,s)$ for fixed $i \geq 2$. Then

$$\begin{aligned} K_{i+1}(ct + (1-c)r, s) &= \int_0^T K_i(ct + (1-c)r, s_1)H_{i+1}(s_1, s)\nabla s \\ &\geq \int_0^T \Big[cK_i(t, s_1) + (1-c)K_i(r, s_1)\Big] H_{i+1}(s_1, s)\nabla s \\ &\geq c\int_0^T K_i(t, s_1)H_{i+1}(s_1, s)\nabla s \\ &\qquad + (1-c)\int_0^T K_i(r, s_1)G_{i+1}(s_1, s)\nabla s \\ &\geq cK_{i+1}(t,s) + (1-c)K_{i+1}(r,s). \end{aligned}$$

This completes the proof. □

We can also formulate similar results as Lemmas 3.2–3.7 above follows:

Lemma 3.8. *Let $(H3), (H4)$ hold. Then for any $g_2(t) \in C(J)$, the boundary value problem,*

$$-u_2^{\Delta\nabla}(t) = g_2(t),\ t \in J, \tag{3.12}$$

$$\gamma_j u_2(0) - \delta_j u_2^{\Delta}(0) = \int_0^T b_j(s) u_2(s) \nabla s,\ 1 \le j \le m, \tag{3.13}$$

$$\gamma_j u_2(T) + \delta_j u_2^{\Delta}(T) = \int_0^T b_j(s) u_2(s) \nabla s,\ 1 \le j \le m, \tag{3.14}$$

has a unique solution

$$u_2(t) = \int_0^T \mathcal{H}_j(t,s) g_2(s) \nabla s,\ for\ 1 \le j \le m, \tag{3.15}$$

where

$$\mathcal{H}_j(t,s) = \mathcal{G}_j(t,s) + \frac{1}{\gamma_j - \xi_j'} \int_0^T \mathcal{G}_j(r,s) b_j(r) \nabla r, \tag{3.16}$$

for $1 \le j \le m$,

Lemma 3.9. *Assume that $(H3), (H4)$ hold and for $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$ define $\eta_j^*(\tau) = \frac{\gamma_j \tau + \delta_j}{\gamma_j T + \delta_j}$. Then $\mathcal{G}_j(t,s)$ for $1 \le j \le m$, satisfies the following properties:*

(i) $0 < \mathcal{G}_j(t,s) \le \mathcal{G}_j(s,s)$ *for all* $t, s \in J$,
(ii) $\eta_j^*(\tau)\mathcal{G}_j(s,s) \le \mathcal{G}_j(t,s)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J$,
(iii) $\mathcal{G}_j(1-t, 1-s) = \mathcal{G}_j(t,s)$ *for all* $t, s \in J$.
(iv) *For each* $s \in J$, *the functions* $\mathcal{G}_j(.,s)$ *are concave in the first argument on* J.

Lemma 3.10. *Assume that $(H3), (H4)$ holds and for $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. Then $\mathcal{H}_j(t,s)$ for $1 \le j \le m$, have the following properties:*

(i) $0 < \mathcal{H}_j(t,s) \le \zeta_j' \mathcal{G}_j(s,s)$ *for all* $t, s \in J$,
(ii) $\zeta_j' \eta_j^*(\tau)\mathcal{G}_j(s,s) \le \mathcal{H}_j(t,s)$ *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J$
(iii) *For each* $s \in J$, *the functions* $\mathcal{H}_j(.,s)$ *are concave in the first argument on* J.

Lemma 3.11. *Assume that $(H3), (H4)$ hold and $\mathcal{H}_j(t,s)$ for $1 \le j \le m$, is given in* (3.16). *Let $\mathcal{K}_1(t,s) = \mathcal{H}_1(t,s)$ and recursively define*

$$\mathcal{K}_j(t,s) = \int_0^T \mathcal{K}_{j-1}(t,r) \mathcal{H}_j(r,s) \nabla r, \quad for \quad 2 \le j \le m. \tag{3.17}$$

Then $\mathcal{K}_m(t,s)$ is the Green's function for the homogeneous boundary value problem

$$(-1)^n u_2^{(\Delta\nabla)^m}(t) = 0,\ t \in J,$$

$$\gamma_{j+1} u_2^{(\Delta\nabla)^j}(0) - \delta_{j+1} u_2^{(\Delta\nabla)^j \Delta}(0) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s) \nabla s,\ 0 \le j \le m-1,$$

$$\gamma_{j+1} u_2^{(\Delta\nabla)^j}(T) + \delta_{j+1} u_2^{(\Delta\nabla)^j \Delta}(T) = \int_0^T b_{j+1}(s) u_2^{(\Delta\nabla)^j}(s) \nabla s,\ 0 \le j \le m-1.$$

Lemma 3.12. *Assume that $(H3), (H4)$ hold and for $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. Define*

$$g_m^* = \prod_{j=1}^{m} g_j',\ \zeta_m^* = \prod_{j=1}^{m} \zeta_j',\ L_m(\tau) = \prod_{j=1}^{m} \zeta_j' \eta_j^*(\tau),\ g_m(\tau) = \prod_{j=1}^{m-1} g_j^*(\tau),$$

then the Green's function $K_n(t,s)$ satisfies the following inequalities:

(i) $0 < \mathcal{K}_m(t,s) \le \zeta_m^* g_m^* \mathcal{G}_m(s,s)$, *for all* $t, s \in J$ *and*

(ii) $\mathcal{K}_m(t,s) \geq L_m(\tau)g_m(\tau)\mathcal{G}_m(s,s)$, *for all* $t \in [\tau, T-\tau]_{\mathbb{T}}$ *and* $s \in J$,

Lemma 3.13. *The Green's function* $\mathcal{K}_j(t,s)$ *for* $1 \leq j \leq m$, *satisfies the following conditions*

$$\mathcal{K}_j(t,s) = \mathcal{K}_j(1-t, 1-s)\,\forall\, t,s \in J \tag{3.18}$$

and for each $s \in J$, $\mathcal{K}_j(\cdot, s)$ $(1 \leq j \leq m)$ *is concave in the first argument on* J.

4. Existence and uniqueness

In this section, we establish the existence and local uniqueness of a solution to the system (1.1)-(1.2). Consider the Banach space $E = C(J)$ with supremum norm $\|\cdot\|$ and the Banach space $X = E \times E$ with the norm $\|(u_1,u_2)\|_X = \|u_1\| + \|u_2\|$.

For $k = 1,2$, we consider three possible cases for $\omega_k \in L^p_\nabla(J) : p > 1$, $p = 1$, $p = \infty$. When $p > 1$ we have the following theorem.

Theorem 4.1. *Assume that the functions* $f_i(u_1,u_2)$ *are continuous with respect to* $(u_1,u_2) \in \mathbb{R} \times \mathbb{R}$ *for* $i = 1,2$. *If* M *satisfies* $\Lambda \leq \frac{M}{\varepsilon}$, *where*

$$\varepsilon = \max\left\{2\|G_n\|_{L^q_\nabla}\|\omega_1\|_{L^p_\nabla}, 2\|\mathcal{G}_m\|_{L^q_\nabla}\|\omega_2\|_{L^p_\nabla}\right\}$$

and $\Lambda > 0$ *satisfies*

$$\Lambda \geq \max_{\|(u_1,u_2)\| \leq M}\left\{|f_1(u_1,u_2)|, |f_2(u_1,u_2)|\right\},$$

then the system (1.1)-(1.2) *has a solution.*

Proof. Let $P = \{(u_1,u_2) \in X : \|(u_1,u_2)\| \leq M\}$. Then P is a cone in X. The cone P is closed, bounded and convex subset of X and hence the Schauder fixed point theorem is applicable. Define $T : P \to X$ by

$$T(u_1,u_2)(t) = \big(T_n(u_1,u_2)(t), T_m(u_1,u_2)(t)\big)$$

where

$$T_n(u_1,u_2)(t) = \int_0^T K_n(t,s)\omega_1(s)f_1(u_1(s),u_2(s))\nabla s$$

and

$$T_m(u_1,u_2)(t) = \int_0^T \mathcal{K}_m(t,s)\omega_2(s)f_2(u_1(s),u_2(s))\nabla s$$

for $t \in J$. Clearly the solution of the system $(1.1)-(1.2)$ is the fixed point of operator T. It can be shown that the $T : P \to X$ is continuous. We claim that $T : P \to P$. If $(u_1,u_2) \in X$, then

$$\begin{aligned}
\|T(u_1,u_2)\|_X &= \|T_n(u_1,u_2)\| + \|T_m(u_1,u_2)\| \\
&= \max_{t \in J}\Big|\int_0^T G_n(t,s)\omega_1(s)f_1(u_1,u_2)\nabla s\Big| \\
&\qquad + \max_{t \in J}\Big|\int_0^T \mathcal{G}_m(t,s)\omega_2(s)f_2(u_1,u_2)\nabla s\Big| \\
&\leq \int_0^T \max_{t \in J}|G_n(t,s)||\omega_1(s)||f_1(u_1,u_2)|\nabla s \\
&\qquad + \int_0^T \max_{t \in J}|\mathcal{G}_m(t,s)||\omega_2(s)||f_2(u_1,u_2)|\nabla s \\
&\leq \|G_n\|_{L^q_\nabla}\|\omega_1\|_{L^p_\nabla}\Lambda + \|\mathcal{G}_m\|_{L^q_\nabla}\|\omega_2\|_{L^p_\nabla}\Lambda \\
&\leq \varepsilon\Lambda.
\end{aligned}$$

Thus, we have $\|T(u_1,u_2)\|_X \leq M$, where M satisfies $\Lambda \leq \frac{M}{\varepsilon}$. ☐

The following two Corollaries deal with the cases when $p = \infty$ and $p = 1$, respectively.

Corollary 4.1. *Assume that the functions $f_i(u_1, u_2)$ are continuous with respect to $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$ for $i = 1, 2$. If M satisfies $\Lambda \leq \frac{M}{\varepsilon}$, where*

$$\varepsilon = \max\Big\{2\|G_n\|_{L^1_\nabla}\|\omega_1\|_{L^\infty_\nabla}, 2\|\mathcal{G}_m\|_{L^1_\nabla}\|\omega_2\|_{L^\infty_\nabla}\Big\}$$

and $\Lambda > 0$ satisfies

$$\Lambda \geq \max_{\|(u_1,u_2)\|\leq M}\Big\{|f_1(u_1, u_2)|, |f_2(u_1, u_2)|\Big\},$$

then the system (1.1)-(1.2) *has a solution.*

Corollary 4.2. *Assume that the functions $f_i(u_1, u_2)$ are continuous with respect to $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$ for $i = 1, 2$. If M satisfies $\Lambda \leq \frac{M}{\varepsilon}$, where*

$$\varepsilon = \max\Big\{2\|G_n\|_{L^\infty_\nabla}\|\omega_1\|_{L^1_\nabla}, 2\|\mathcal{G}_m\|_{L^\infty_\nabla}\|\omega_2\|_{L^1_\nabla}\Big\}$$

and $\Lambda > 0$ satisfies

$$\Lambda \geq \max_{\|(u_1,u_2)\|\leq M}\Big\{|f_1(u_1, u_2)|, |f_2(u_1, u_2)|\Big\},$$

then the system (1.1)-(1.2) *has a solution.*

Corollary 4.3. *If the functions $f_i(u_1, u_2)$ are continuous and bounded on $\mathbb{R} \times \mathbb{R}$ for $i = 1, 2$, then the system* (1.1)-(1.2) *has a solution.*

Proof. Choose $Q > \sup\{|f_1(u_1, u_2)|, |f_2(u_1, u_2)|\}$. Pick $M > 0$ large enough so that $Q < \frac{M}{\varepsilon}$, where ε is defined in the Theorem 4.1. Then there is a number $\Lambda > 0$ such that $Q > \Lambda$ where $\Lambda \geq \max_{\|(u_1,u_2)\|\leq M}\Big\{|f_1(u_1, u_2)|, |f_2(u_1, u_2)|\Big\}$. Hence, $\varepsilon < \frac{M}{Q} < \frac{M}{\Lambda}$ and thus the system has a solution by Theorem 4.1. □

5. Existence of countably infinitely many positive solutions

In this section, we establish the existence of countably infinitely many symmetric positive solutions to the system (1.1)-(1.2) by applying Hölder's inequality and Krasnoselskii's fixed point theorem in cones. Assume throughout this section that $\omega_k, (k = 1, 2)$ have countably many singularities in $(0, \frac{T}{2})_{\mathbb{T}}$.

Theorem 5.2. (Krasnoselskii fixed point theorem, [15]). *Let $\mathcal{B}$ be a Banach space and let $P \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1, Ω_2 are open with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$, or
(ii) $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Theorem 5.3. (Hölder's inequality, [5, 27]) *Let $f \in L^p_\nabla(J^*)$ with $p > 1, g \in L^q_\nabla(J^*)$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\nabla(J^*)$ and $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$.*
where

$$\|f\|_{L^p_\nabla} := \begin{cases} \Big[\displaystyle\int_{J^*} |f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ inf\Big\{M \in \mathbb{R} \,/\, |f| \leq M \ \nabla - a.e., \ on \ J^*\Big\}, & p = \infty, \end{cases}$$

and $J^* = [a, b]_{\mathbb{T}}$. Moreover, if $f \in L^1_\nabla(J^*)$ and $g \in L^\infty_\nabla(J^*)$. Then $fg \in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla} \leq \|f\|_{L^1_\nabla}\|g\|_{L^\infty_\nabla}$.

For $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$, define the cone $P_\tau \subset X$ by

$$P_\tau = \Big\{(u_1, u_2) \in X : u_1(t) \geq 0,\ u_2(t) \geq 0 \text{ are symmetric, concave and } \min_{t \in [\tau, T-\tau]_{\mathbb{T}}} \big(u_1(t) + u_2(t)\big) \geq \frac{\gamma_\tau}{\gamma} \|\big(u_1(t), u_2(t)\big)\|_X \Big\},$$

where $\gamma_\tau = \min\big\{L_n(\tau)g_n(\tau), L_m(\tau)g_m(\tau)\big\}$ and $\gamma = \max\big\{\zeta_n^* g_n^*, \zeta_m^* g_m^*\big\}$.
For any $(u_1, u_2) \in P_\tau$, define an operator $F : P_\tau \to X$ by

$$F(u_1, u_2)(t) = \big(F_n(u_1, u_2), F_m(u_1, u_2)\big),$$

where

$$F_n(u_1, u_2) = \int_0^T K_n(t, s)\omega_1(s) f_1(u_1, u_2) \nabla s$$

and

$$F_m(u_1, u_2) = \int_0^T \mathcal{K}_m(t, s)\omega_2(s) f_2(u_1, u_2) \nabla s$$

Lemma 5.14. *Assume that (H1)-(H4) hold. Then $F(P_\tau) \subset P_\tau$ and $F : P_\tau \to P_\tau$ is completely continuous for each $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$.*

Proof. Fix $\tau \in (0, \frac{T}{2})_{\mathbb{T}}$. First note that $(u_1, u_2) \in P$ implies that $F_n(u_1, u_2)(t) \geq 0$ and $F_n(u_1, u_2)(t) \geq 0$ for all $t \in J$. On the other hand, by Lemma 3.6 and Lemma 3.12 we obtain

$$\begin{aligned}
&F_n(u_1, u_2)(t) + F_m(u_1, u_2)(t) \\
&= \int_0^T K_n(t, s)\omega_1(s) f_1(u_1, u_2) \nabla s + \int_0^T \mathcal{K}_m(t, s)\omega_2(s) f_2(u_1, u_2) \nabla s \\
&\leq \zeta_n^* g_n^* \int_0^T G_n(s, s)\omega_1(s) f_1(u_1, u_2) \nabla s + \zeta_m^* g_m^* \int_0^T \mathcal{G}_m(s, s)\omega_2(s) f_2(u_1, u_2) \nabla s \\
&\leq \gamma \Big(\int_0^T G_n(s, s)\omega_1(s) f_1(u_1, u_2) \nabla s + \int_0^T \mathcal{G}_m(s, s)\omega_2(s) f_2(u_1, u_2) \nabla s \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\min_{t \in [\tau, T-\tau]_{\mathbb{T}}} \big(F_n(u_1, u_2)(t) + F_m(u_1, u_2)(t)\big) \\
&= \min_{t \in [\tau, T-\tau]_{\mathbb{T}}} \Big(\int_0^T K_n(t, s)\omega_1(s) f_1(u_1, u_2) \nabla s + \int_0^T \mathcal{K}_m(t, s)\omega_2(s) f_2(u_1, u_2) \nabla s \Big) \\
&= L_n(\tau) g_n(\tau) \int_0^T G_n(s, s) w_1(s) f_1(u_1, u_2) \nabla s \\
&\qquad + L_m(\tau) g_m(\tau) \int_0^T \mathcal{G}_m(s, s) w_2(s) f_2(u_1, u_2) \nabla s \\
&\geq \gamma_\tau \Big(\int_0^T G_n(s, s) w_1(s) f_1(u_1, u_2) \nabla s + \int_0^T \mathcal{G}_m(s, s) w_2(s) f_2(u_1, u_2) \nabla s \Big) \\
&\geq \frac{\gamma_\tau}{\gamma} \|\big(F_n(u_1, u_2), F_m(u_1, u_2)\big)\|_X \\
&\geq \frac{\gamma_\tau}{\gamma} \|F\big(u_1, u_2)\|_X.
\end{aligned}$$

So, $F(u_1, u_2) \in P_\tau$ and then $F(P_\tau) \subset P_\tau$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator T is completely continuous. The proof is complete. □

We consider three possible cases for $\omega_{1,2} \in L^p_\nabla(J) : p > 1, p = 1, p = \infty$. When $p > 1$ we have the following theorem.

Theorem 5.4. *Assume that* $(H1) - (H4)$ *hold, let* $\{\tau_k\}_{k=1}^\infty$ *be such that* $t_{k+1} < \tau_k < t_k$, $k = 1, 2, 3, \cdots$. *Let* $\{S_k\}_{k=1}^\infty$ *and* $\{r_k\}_{k=1}^\infty$ *be such that*

$$S_{k+1} < \frac{\gamma_{\tau_k}}{\gamma} r_k < C r_k < S_k,\ k \in \mathbb{N},$$

where

$$C = \max\left\{\frac{1}{L_n(\tau_1) g_n(\tau_1)\epsilon \int_{\tau_1}^{1-\tau_1} G_n(s,s)\nabla s},\ \frac{1}{L_m(\tau_1) g_m(\tau_1)\epsilon \int_{\tau_1}^{1-\tau_1} \mathcal{G}_n(s,s)\nabla s},\ 1\right\}.$$

Assume that f *satisfies*

(A1) $f_1(u_1, u_2) \le \frac{M_1 S_k}{2}$ *and* $f_2(u_1, u_2) \le \frac{M_1^{'} S_k}{2}$ *for all* $t \in J, 0 \le u_1 + u_2 \le S_k$, *where*

$$M_1 < \frac{1}{\zeta_n^* g_n^* \|G_n\|_{L^q_\nabla} \|\omega_1\|_{L^p_\nabla}} \text{ and } M_1^{'} < \frac{1}{\zeta_m^* g_m^* \|\mathcal{G}_m\|_{L^q_\nabla} \|\omega_2\|_{L^p_\nabla}}$$

(A2) $f_1(u_1, u_2) \ge C r_k$ *or* $f_2(u_1, u_2) \ge C r_k$ *for all* $t \in [\tau_k, T - \tau_k]_{\mathbb{T}}$, $\frac{\gamma_{\tau_k}}{\gamma} r_k \le u_1 + u_2 \le r_k$.

Then the system (1.1)-(1.2) *has countably infinitely many symmetric positive solutions* $\{(u_1^{[k]}, u_2^{[k]})\}_{k=1}^\infty$. *Furthermore,* $r_k \le \|(u_1^{[k]}, u_2^{[k]})\| \le S_k$ *for each* $k \in \mathbb{N}$.

Proof. Consider the sequences $\{\Omega_{1,k}\}_{k=1}^\infty$ and $\{\Omega_{2,k}\}_{k=1}^\infty$ of open subsets of X defined by

$$\Omega_{1,k} = \{(u_1, u_2) \in X : \|(u_1, u_2)\|_X < S_k\},$$
$$\Omega_{2,k} = \{(u_1, u_2) \in X : \|(u_1, u_2)\|_X < r_k\}.$$

Let $\{\tau_k\}_{k=1}^\infty$ be as in the hypothesis and note that $t^* < t_{k+1} < \tau_k < t_k < \frac{T}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone P_{τ_k} by

$$P_{\tau_k} = \Big\{(u_1, u_2) \in X : u_1(t) \ge 0,\ u_2(t) \ge 0 \text{ are symmetric, concave and } \min_{t \in [\tau_k,\, 1-\tau_k]_{\mathbb{T}}} \big(u_1(t) + u_2(t)\big) \ge \frac{\gamma_{\tau_k}}{\gamma} \|\big(u_1(t), u_2(t)\big)\|_X\Big\}.$$

Let $(u_1, u_2) \in P_{\tau_k} \cap \partial\Omega_{1,k}$. Then,

$$u_1(s) + u_2(s) \le S_k = \|(u_1, u_2)\|_X$$

for all $s \in J$. By $(A1)$,

$$\begin{aligned}
\|F_n(u_1, u_2)\| &= \max_{t \in J} \int_0^T K_n(t,s)\omega_1(s) f_1\big(u_1(s), u_2(s)\big)\nabla s \\
&\le \zeta_n^* g_n^* \int_0^T G_n(s,s)\omega_1(s) f_1(u_1, u_2)\nabla s \\
&\le \zeta_n^* g_n^* \|G_n\|_{L^q_\nabla} \|\omega_1\|_{L^p_\nabla} \frac{M_1 S_k}{2} \\
&\le \frac{S_k}{2} = \frac{\|(u_1, u_2)\|_X}{2}.
\end{aligned}$$

Thus we have $\|F_n(u_1,u_2)\| \le \frac{\|(u_1,u_2)\|_X}{2}$. Similarly we can see that

$$\|F_m(u_1,u_2)\| \le \frac{\|(u_1,u_2)\|_X}{2}.$$

Therefore, for $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{1,k}$, and $t \in J$ we get

$$\begin{aligned} \|F(u_1,u_2)\|_X &= \|\big(F_n(u_1,u_2), F_m(u_1,u_2)\big)\|_X \\ &= \|F_n(u_1,u_2)\| + \|F_m(u_1,u_2)\| \\ &\le \|(u_1,u_2)\|_X. \end{aligned} \tag{5.19}$$

Let $s \in [\tau_k, 1-\tau_k]_{\mathbb{T}}$. Then, for $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{2,k}$,

$$\begin{aligned} r_k = \|(u_1,u_2)\| &\ge u_1(s) + u_2(s) \\ &\ge \min_{s\in[\tau_k,\,1-\tau_k]_{\mathbb{T}}} \big(u_1(s)+u_2(s)\big) \\ &\ge \frac{\gamma_{\tau_k}}{\gamma}\|(u_1,u_2)\| \\ &\ge \frac{\gamma_{\tau_k}}{\gamma} r_k. \end{aligned}$$

By $(A2)$,

$$\begin{aligned} \|F(u_1,u_2)\| = \|F_n(u_1,u_2)\| + \|F_m(u_1,u_2)\| &\ge \|F_n(u_1,u_2)\| \\ &= \max_{t\in J} \int_0^T K_n(t,s)\omega_1(s) f_1\big(u_1(s),u_2(s)\big)\nabla s \\ &\ge \max_{t\in J} \int_{\tau_k}^{T-\tau_k} K_n(t,s)\omega_1(s) f_1\big(u_1(s),u_2(s)\big)\nabla s \\ &\ge \max_{t\in J} \int_{\tau_k}^{T-\tau_k} K_n(t,s)\omega_1(s)\nabla s C r_k \\ &\ge C r_k \epsilon \max_{t\in[\tau_1,\,1-\tau_1]_{\mathbb{T}}} \int_{\tau_1}^{T-\tau_1} K_n(t,s)\nabla s \\ &\ge C r_k L_n(\tau_1) g_n(\tau_1)\epsilon \int_{\tau_1}^{1-\tau_1} G_n(s,s)\nabla s \\ &\ge r_k = \|(u_1,u_2)\|_X. \end{aligned}$$

Thus, if $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{2,k}$, then

$$\|F(u_1,u_2)\| \ge \|(u_1,u_2)\|_X. \tag{5.20}$$

It is obvious that $0 \in \Omega_{2,k} \subset \bar{\Omega}_{2,k} \subset \Omega_{1,k}$. By (5.19),(5.20), it follows from Theorem 4.1 that the operator T has a fixed point $(u_1^{[k]}, u_2^{[k]}) \in P_{\tau_k} \cap \big(\bar{\Omega}_{1,k} \backslash \Omega_{2,k}\big)$ such that $r_k \le \|(u_1^{[k]}, u_2^{[k]})\| \le S_k$. Since $k \in \mathbb{N}$ was arbitrary, the proof is complete. □

Now we deal with the case $p = 1$.

Theorem 5.5. *Assume that $(H1)-(H4)$ hold, let $\{\tau_k\}_{k=1}^{\infty}$ be such that $t_{k+1} < \tau_k < t_k$, $k = 1,2,3,\cdots$. Let $\{S_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ be such that*

$$S_{k+1} < \frac{\gamma_{\tau_k}}{\gamma} r_k < C r_k < S_k, \ k \in \mathbb{N},$$

where C is defined in Theorem 5.4. Also assume that f satisfies
$(B1)$ $f_1(u_1,u_2) \le \frac{M_2 S_k}{2}$ *and* $f_2(u_1,u_2) \le \frac{M_2' S_k}{2}$ *for all* $t \in J$, $0 \le u_1+u_2 \le S_k$, *where*

$$M_2 < \min\left\{\frac{1}{\zeta_n^* g_n^* \|G_n\|_{L^\infty_\nabla}\|\omega_1\|_{L^1_\nabla}},\ C\right\},$$

$$M_2' < \min\left\{\frac{1}{\zeta_m^* g_m^* \|\mathcal{G}_m\|_{L^\infty_\nabla}\|\omega_2\|_{L^1_\nabla}},\ C\right\}$$

and $(A2)$. *Then the boundary value problem* (1.1)–(1.2) *has countably infinitely many symmetric positive solutions* $\{(u_1^{[k]},u_2^{[k]})\}_{k=1}^\infty$. *Furthermore, for each* $k \in \mathbb{N}$, $r_k \le \|(u_1^{[k]},u_2^{[k]})\| \le S_k$.

Proof. For a fixed k, let $\Omega_{1,k}$ be as in the proof of Theorem 5.4 and let (u_1,u_2) be an element of $P_{\tau_k} \cap \partial\Omega_{1,k}$. Then

$$u_1(s)+u_2(s) \le S_k = \|(u_1,u_2)\|_X,$$

for all $s \in J$. By $(B1)$ and Theorem 5.4,

$$\begin{aligned}
\|F(u_1,u_2)\| &= \|F_n(u_1,u_2)\| + \|F_m(u_1,u_2)\| \\
&\le \max_{t\in J}\int_0^T K_n(t,s)\omega_1(s) f_1\big(u_1(s),u_2(s)\big)\nabla s \\
&\qquad + \max_{t\in J}\int_0^T \mathcal{K}_m(t,s)\omega_2(s) f_2\big(u_1(s),u_2(s)\big)\nabla s \\
&\le \zeta_n^* g_n^* \|G_n\|_{L^\infty_\nabla}\|\omega_1\|_{L^1_\nabla}\frac{M_2 S_k}{2} + \zeta_m^* g_m^* \|\mathcal{G}_m\|_{L^\infty_\nabla}\|\omega_2\|_{L^1_\nabla}\frac{M_2' S_k}{2} \\
&\le S_k.
\end{aligned}$$

Thus,

$$\|F(u_1,u_2)\| \le \|(u_1,u_2)\|_X,$$

for $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{1,k}$. Now define $\Omega_{2,k} = \{(u_1,u_2) \in X : \|(u_1,u_2)\|_X < r_k\}$. Let $(u_1,u_2) \in P_{\tau_k} \cap \partial\Omega_{2,k}$ and let $s \in [\tau_k,\, 1-\tau_k]_{\mathbb{T}}$. Then, the argument leading to (5.20) carries over to the present case and completes the proof. □

Finally we consider the case of $p=\infty$.

Theorem 5.6. *Assume that* $(H1)-(H4)$ *hold. Let* $\{S_k\}_{k=1}^\infty$ *and* $\{r_k\}_{k=1}^\infty$ *be such that*

$$S_{k+1} < \frac{\gamma_\tau}{\gamma} r_k < C r_k < S_k,\ k \in \mathbb{N},$$

where C is defined in Theorem 5.4. Also assume that f satisfies
$(E1)$ $f_1(u_1,u_2) \le M_3 S_k$ *and* $f_1(u_1,u_2) \le M_3' S_k$ *for all* $t \in J$, $0 \le u_1+u_2 \le S_k$, *where*

$$M_3 < \min\left\{\frac{1}{\zeta_n^* g_n^* \|G_n\|_{L^1_\nabla}\|\omega_1\|_{L^\infty_\nabla}},\ C\right\},$$

$$M_3' < \min\left\{\frac{1}{\zeta_m^* g_m^* \|\mathcal{G}_m\|_{L^1_\nabla}\|\omega_2\|_{L^\infty_\nabla}},\ C\right\}$$

and $(A2)$. *Then the boundary value problem* (1.1)–(1.2) *has countably infinitely many symmetric positive solutions* $\{(u_1^{[k]},u_2^{[k]})\}_{k=1}^\infty$. *Furthermore, for each* $k \in \mathbb{N}$,
$r_k \le \|(u_1^{[k]},u_2^{[k]})\| \le S_k$.

Proof. By $(E1)$,

$$\begin{aligned}\|F(u_1,u_2)\| &= \|F_n(u_1,u_2)\| + \|F_m(u_1,u_2)\| \\ &\le \max_{t\in J}\int_0^T K_n(t,s)\omega_1(s)f_1\big(u_1(s),u_2(s)\big)\nabla s \\ &\qquad + \max_{t\in J}\int_0^T \mathcal{K}_m(t,s)\omega_2(s)f_2\big(u_1(s),u_2(s)\big)\nabla s \\ &\le \zeta_n^* g_n^* \|G_n\|_{L^1_\nabla}\|\omega_1\|_{L^\infty_\nabla}\frac{M_3S_k}{2} + \zeta_m^* g_m^*\|\mathcal{G}_m\|_{L^1_\nabla}\|\omega_2\|_{L^\infty_\nabla}\frac{M_3^{'}S_k}{2} \\ &\le S_k.\end{aligned}$$

This shows that if $(u_1,u_2)\in P_{\tau_k}\cap\partial\Omega_{1,k}$, where

$$\Omega_{1,k} = \{(u_1,u_2)\in X : \|(u_1,u_2)\| < S_k\}.$$

Then,

$$\|F(u_1,u_2)\| \le \|(u_1,u_2)\|.$$

Define $\Omega_{2,k} = \{(u_1,u_2)\in X : \|(u_1,u_2)\| < r_k\}$ and let $(u_1,u_2)\in P_{\tau_k}\cap\partial\Omega_{2,k}$. Then, the argument employed in the proof of Theorem 5.4 applies directly to yield $\|F(u_1,u_2)\| \ge \|(u_1,u_2)\|$. By the Theorem 4.1, completes the proof. □

6. Example

In this section, we provide an example of a family of functions $\omega(t)$ that satisfy conditions $(H1)$, $(H2)$ corresponding to the cases $p=1$ and $p=2$.

Let $\mathbb{T} = [0,\frac{1}{6}]\cup\{\frac{9}{50},\frac{1}{5},\frac{11}{50},\frac{6}{25}\}\cup[\frac{1}{4},\frac{3}{4}]\cup\{\frac{19}{25},\frac{39}{50},\frac{4}{5},\frac{41}{50}\}\cup[\frac{5}{6},1]$ be bounded symmetric time scale and consider the family of functions $\omega(t,\theta) : [0,1]_{\mathbb{T}}\to(0,+\infty]$ given by

$$\omega(t,\theta) = \begin{cases} \dfrac{1}{|t-\frac{1}{2}|^\theta} & if \quad 0\le t\le \dfrac{1}{4} \text{ or } \dfrac{3}{4}\le t\le 1, \\ \displaystyle\sum_{l=1}^{\infty}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{\big||t-\frac{1}{2}|+t_l-\frac{1}{2}\big|^\theta} & if \quad \dfrac{1}{4}<t<\dfrac{3}{4},\end{cases}$$

where

$$t_0 = \frac{5}{16},\ t_l = t_0 - \sum_{k=0}^{l-1}\frac{1}{(k+2)^4},\ l=1,2,3,\cdots,\ \text{and } \vartheta_0 = 1,\ \vartheta_l = \frac{1}{2}(t_l+t_{l+1}),\ l=1,2,3,\cdots.$$

At first, it is easily seen that $\omega(t,\theta)\ge\omega(1,\theta) = \frac{1}{|1-\frac{1}{2}|^\theta} = 2^\theta,\ t_1 = \frac{1}{4} < \frac{1}{2},\ t_l - t_{l+1} = \frac{1}{(l+2)^4},\ l=1,2,3,\cdots$, and note that $\sum_{l=1}^{\infty}\frac{1}{l^4} = \frac{\pi^4}{90}$. So,

$$t^* = \lim_{l\to\infty} t_l = \frac{5}{16} - \sum_{k=0}^{\infty}\frac{1}{(k+2)^4} = \frac{5}{16} - \left(\frac{\pi^4}{90}-1\right) = \frac{21}{16} - \frac{\pi^4}{90} > \frac{1}{5}.$$

We claim that if $\theta=\frac{1}{2}$, then $\omega(t,\theta)\in L^1_\nabla[0,1]$. Note that $\sum_{l=1}^{\infty}\frac{1}{l^2}=\frac{\pi^2}{6}$, we have

$$\begin{aligned}\int_0^1\omega(t,\theta)\nabla t&=\int_0^{\frac{1}{6}}\omega(t,\theta)\nabla t+\int_{\frac{1}{4}}^{\frac{3}{4}}\omega(t,\theta)\nabla t+\int_{\frac{5}{6}}^{1}\omega(t,\theta)\nabla t\\&+\Bigg[\left(\frac{9}{50}-\frac{1}{6}\right)\omega\left(\frac{9}{50},\theta\right)+\left(\frac{1}{5}-\frac{9}{50}\right)\omega\left(\frac{1}{5},\theta\right)+\left(\frac{11}{50}-\frac{1}{5}\right)\omega\left(\frac{11}{50},\theta\right)\\&+\left(\frac{6}{25}-\frac{11}{50}\right)\omega\left(\frac{6}{25},\theta\right)+\left(\frac{1}{4}-\frac{6}{25}\right)\omega\left(\frac{1}{4},\theta\right)+\left(\frac{19}{25}-\frac{3}{4}\right)\omega\left(\frac{19}{25},\theta\right)\\&+\left(\frac{39}{50}-\frac{19}{25}\right)\omega\left(\frac{39}{50},\theta\right)+\left(\frac{4}{5}-\frac{39}{50}\right)\omega\left(\frac{4}{5},\theta\right)+\left(\frac{41}{50}-\frac{4}{5}\right)\omega\left(\frac{41}{50},\theta\right)\\&+\left(\frac{5}{6}-\frac{41}{50}\right)\omega\left(\frac{5}{6},\theta\right)\Bigg]\\&=\int_0^{\frac{1}{6}}\frac{1}{|t-\frac{1}{2}|^\theta}\nabla t+\int_{\frac{1}{4}}^{\frac{3}{4}}\sum_{l=1}^{\infty}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{\left||t-\frac{1}{2}|+t_l-\frac{1}{2}\right|^\theta}\nabla t+\int_{\frac{5}{6}}^{1}\frac{1}{|t-\frac{1}{2}|^\theta}\nabla t\\&+\frac{1}{50}\left[2\times\left(\frac{10}{3}\right)^\theta+2\times\left(\frac{25}{7}\right)^\theta+\left(\frac{50}{13}\right)^\theta+\left(\frac{25}{8}\right)^\theta\right]+\frac{1}{75}\left[\left(\frac{25}{8}\right)^\theta+3^\theta\right]\\&+\frac{1}{100}\left[4^\theta+\left(\frac{50}{13}\right)^\theta\right]\end{aligned}$$

Let

$$\begin{aligned}\Sigma=&\frac{1}{50}\left[2\times\left(\frac{10}{3}\right)^\theta+2\times\left(\frac{25}{7}\right)^\theta+\left(\frac{50}{13}\right)^\theta+\left(\frac{25}{8}\right)^\theta\right]+\frac{1}{75}\left[\left(\frac{25}{8}\right)^\theta+3^\theta\right]\\&+\frac{1}{100}\left[4^\theta+\left(\frac{50}{13}\right)^\theta\right]\end{aligned}$$

with $\theta=\frac{1}{2}$. Then

$$\begin{aligned}\int_0^1\omega(t,\theta)\nabla t&=\sum_{l=1}^{\infty}\int_{\vartheta_l}^{\vartheta_{l-1}}\frac{1}{\left||t-\frac{1}{2}|+t_l-\frac{1}{2}\right|^\theta}\nabla t+\int_0^{\frac{1}{6}}\frac{1}{(\frac{1}{2}-t)^\theta}\nabla t\\&\qquad+\int_{\frac{5}{6}}^{1}\frac{1}{(t-\frac{1}{2})^\theta}\nabla t+\Sigma\\&=\sum_{l=1}^{\infty}\left[\int_{\vartheta_l}^{t_l}\frac{1}{(t_l-t)^\theta}\nabla t+\int_{t_l}^{\nu_{l-1}}\frac{1}{(t-t_l)^\theta}\nabla t\right]\\&\qquad+\frac{2}{1-\theta}\left[\frac{1}{2^{1-\theta}}-\frac{1}{3^{1-\theta}}\right]+\Sigma\\&=\sum_{l=1}^{\infty}\left[\int_{\frac{t_l+t_{l+1}}{2}}^{t_l}\frac{1}{(t_l-t)^\theta}\nabla t+\int_{t_l}^{\frac{t_{l-1}+t_l}{2}}\frac{1}{(t-t_l)^\theta}\nabla t\right]\\&\qquad+\frac{2}{1-\theta}\left[\frac{1}{2^{1-\theta}}-\frac{1}{3^{1-\theta}}\right]+\Sigma\end{aligned}$$

So that

$$\int_0^1 \omega(t,\theta)\nabla t = \frac{1}{1-\theta}\sum_{l=1}^{\infty}\left[\left(\frac{t_l - t_{l+1}}{2}\right)^{1-\theta} + \left(\frac{t_{l-1} - t_l}{2}\right)^{1-\theta}\right. \\ \left. + \frac{2}{1-\theta}\left[\frac{1}{2^{1-\theta}} - \frac{1}{3^{1-\theta}}\right] + \Sigma \right.$$

$$= \frac{1}{2^{1-\theta}(1-\theta)}\sum_{l=1}^{\infty}\left[\frac{1}{(l+2)^{4(1-\theta)}} + \frac{1}{(l+1)^{4(1-\theta)}}\right] + \frac{2}{1-\theta}\left[\frac{1}{2^{1-\theta}} - \frac{1}{3^{1-\theta}}\right] + \Sigma$$

$$= \sqrt{2}\sum_{l=1}^{\infty}\left[\frac{1}{(l+1)^2} + \frac{1}{(l+1)^2}\right] + 2\sqrt{2} - \frac{4}{3}\sqrt{3} + \Sigma$$

$$= \sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right) + 2\sqrt{2} - \frac{4}{3}\sqrt{3} + \Sigma,$$

This shows that $\omega(t,\theta) \in L^1_\nabla[0,1]$.

Next, we claim that if $\theta = \frac{1}{4}$, then $\omega(t,\theta) \in L^2_\nabla[0,1]$. In this case, we need the cauchy product,

$$\sum_{l=1}^{\infty} x_l \cdot \sum_{l=1}^{\infty} y_l = \sum_{l=1}^{\infty} z_l, \tag{6.21}$$

where

$$z_l = \sum_{n=1}^{l} x_n y_{l-n+1}. \tag{6.22}$$

Note that

$$\int_0^1 \omega^2(t,\theta)\nabla t = \int_0^{\frac{1}{4}} \omega^2(t,\theta)\nabla t + \int_{\frac{1}{4}}^{\frac{3}{4}}\left[\sum_{l=1}^{\infty}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{\left||t-\frac{1}{2}| + t_l - \frac{1}{2}\right|^{2\theta}}\right]^2\nabla t + \int_{\frac{3}{4}}^1 \omega^2(t,\theta)\nabla t, \tag{6.23}$$

we use (6.21) and (6.22) and the fact that, if $X \cap Y = \emptyset$, then $\chi[X]\cdot\chi[Y] = 0$ to simplify the integrand,

$$\left[\sum_{l=1}^{\infty}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{|t-t_l|^{\theta}}\right]^2 = \sum_{l=1}^{\infty}\sum_{n=1}^{l}\frac{\chi[\vartheta_n,\,\vartheta_{l-1}]}{|t-t_n|^{\theta}}\frac{\chi[\vartheta_{l-n+1},\,\vartheta_{l-n}]}{|t-t_{l-n+1}|^{\theta}} = \sum_{l=1}^{\infty}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{|t-t_l|^{2\theta}} \text{ a.e.},$$

and so (6.23) may be written as

$$\int_0^1 \omega^2(t,\theta)\nabla t = \sum_{l=1}^{\infty}\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{\chi[\vartheta_l,\,\vartheta_{l-1}]}{\left||t-\frac{1}{2}| + t_l - \frac{1}{2}\right|^{2\theta}}\nabla t + \int_0^{\frac{1}{4}}\omega^2(t,\theta)\nabla t + \int_{\frac{3}{4}}^1 \omega^2(t,\theta)\nabla t$$

Let

$$\Sigma = \frac{1}{75}\left[\left(\frac{25}{8}\right)^{2\theta} + 3^{2\theta}\right] + \frac{1}{100}\left[4^{2\theta} + \left(\frac{50}{13}\right)^{2\theta}\right] + \frac{1}{50}\left[2\times\left(\frac{10}{3}\right)^{2\theta} + 2\times\left(\frac{25}{7}\right)^{2\theta} + \left(\frac{50}{13}\right)^{2\theta} + \left(\frac{25}{8}\right)^{2\theta}\right]$$

with $\theta = \frac{1}{4}$. Then

$$\begin{aligned}
\int_0^1 \omega^2(t,\theta)\nabla t &= \sum_{l=1}^{\infty}\int_{\vartheta_l}^{\vartheta_{l-1}} \frac{1}{\left||t-\frac{1}{2}|+t_l-\frac{1}{2}\right|^{2\theta}}\nabla t + \int_0^{\frac{1}{6}} \frac{1}{(\frac{1}{2}-t)^{2\theta}}\nabla t \\
&\quad + \int_{\frac{5}{6}}^{1} \frac{1}{(t-\frac{1}{2})^{2\theta}}\nabla t + \Sigma \\
&= \sum_{l=1}^{\infty}\left[\int_{\vartheta_l}^{t_l} \frac{1}{(t_l-t)^{2\theta}}\nabla t + \int_{t_l}^{\vartheta_{l-1}} \frac{1}{(t-t_l)^{2\theta}}\nabla t\right] \\
&\quad + \frac{2}{1-2\theta}\left[\frac{1}{2^{1-2\theta}} - \frac{1}{3^{1-2\theta}}\right] + \Sigma \\
&= \sum_{l=1}^{\infty}\left[\int_{\frac{t_l+t_{l+1}}{2}}^{t_l} \frac{1}{(t_l-t)^{2\theta}}\nabla t + \int_{t_l}^{\frac{t_{l-1}+t_l}{2}} \frac{1}{(t-t_l)^{2\theta}}\nabla t\right] \\
&\quad + \frac{2}{1-2\theta}\left[\frac{1}{2^{1-2\theta}} - \frac{1}{3^{1-2\theta}}\right] + \Sigma \\
&= \frac{1}{1-2\theta}\sum_{l=1}^{\infty}\left[\left(\frac{t_l-t_{l+1}}{2}\right)^{1-2\theta} + \left(\frac{t_{l-1}-t_l}{2}\right)^{1-2\theta}\right] \\
&\quad + \frac{2}{1-2\theta}\left[\frac{1}{2^{1-2\theta}} - \frac{1}{3^{1-2\theta}}\right] + \Sigma \\
&= \frac{1}{2^{1-2\theta}(1-2\theta)}\sum_{l=1}^{\infty}\left[\frac{1}{(l+2)^{4(1-2\theta)}} + \frac{1}{(l+1)^{4(1-2\theta)}}\right] \\
&\quad + \frac{2}{1-2\theta}\left[\frac{1}{2^{1-2\theta}} - \frac{1}{3^{1-2\theta}}\right] + \Sigma \\
&= \sqrt{2}\sum_{l=1}^{\infty}\left[\frac{1}{(l+1)^2} + \frac{1}{(l+1)^2}\right] + 2\sqrt{2} - \frac{4}{3}\sqrt{3} + \Sigma \\
&= \sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right) + 2\sqrt{2} - \frac{4}{3}\sqrt{3} + \Sigma.
\end{aligned}$$

Which implies $\omega(t,\theta) \in L^2_{\nabla}[0,1]$.

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