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A modified Krasnosellkii-Mann algorithms for computing fixed points of multivalued quasi-nonexpansive mappings in Banach spaces

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ABSTRACT. It is well known that Krasnoselskii-Mann iteration of nonexpansive mappings find application in many areas of mathematics and know to be weakly convergent in the infinite dimensional setting. In this paper, we introduce and study an explicit iterative scheme by a modified Krasnoselskii-Mann algorithm for approximating fixed points of multivalued quasi-nonexpansive mappings in Banach spaces. Strong convergence of the sequence generated by this algorithm is established. There is no compactness assumption. The results obtained in this paper are significant improvement on important recent results.

1. INTRODUCTION

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0,+\infty) \to [0,\infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, x \in E. \tag{1.1}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2 =\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), x \neq 0.$$

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E.

Following Browder [3], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is single-valued and is weak-to-$weak^*$ sequentially continous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see [4] for more details on duality maps). Finally recall that a Banach space E satisfies Opial property (see, e.g.,

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[12]) if $\limsup_{n\to+\infty}\|x_n - x\| < \limsup_{n\to+\infty}\|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property.

Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

For several years, the study of fixed point theory for multi-valued nonlinear mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [20], Kakutani [5], Nash [10, 11], Nadla [9], Sow [16], Xu et al. [20], Zuo et al. [23], Wu et al. [18], Panyanak [13]).

Interest in the study of fixed point theory for multi-valued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Non-Smooth Differential Equations, Optimization.

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [14]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in D\} = d(x, D),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The Pompeiu *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$ (see, Berinde [1]). A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L- Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \ \ \forall x, y \in D(T). \tag{1.2}$$

When $L \in (0, 1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

A multivalued map T is called quasi-nonexpansive if

$$H(Tx, Tp) \leq \|x - p\|$$

holds for all $x \in D(T)$ and $p \in F(T)$.

Remark 1.1. It is easy to see that the class of mulivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points.

Historically, one of the most investigated methods for approximating fixed points of single-valued nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [8]. Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \end{cases} \tag{1.3}$$

$\{\alpha_n\}$ is a sequence in $(0, 1)$. But Mann's iteration process has only weak convergence, even in Hilbert space setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators.

Recently, Sow et al. [17], motivated by the fact that Krasnoselskii-Mann algorithm method is remarkably useful for finding fixed points of single-valued nonexpansive mapping, proved the following theorem.

Theorem 1.1 (Sow et al. [17]). *Let E be a uniformly smooth real Banach space having a weakly continuous duality map and K a nonempty, closed and convex cone of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n. \tag{1.4}$$

Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$

(iii) $\lim_{n\to\infty} \lambda_n = 1$, $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$, *and* $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$

Then, the sequence $\{x_n\}$ generated by (1.4) *converges strongly to $x^* \in F(T)$,*

Motivated by Sow et al. [17], we construct an iterative algorithm for approximating fixed points of multivalued quasi-nonexpansive mappings which is also the solution of some variational inequality problems in real Banach spaces having a weakly continuous duality maps. Finally, our method of proof is of independent interest.

2. Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Definition 2.1. Let E be real Banach space and $T : D(T) \subset E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.

Lemma 2.1 (Demi-closedness Principle, [3]). *Let E be a uniformly convex Banach space satisfying the Opial condition, K be a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2.2 ([6]). *Assume that a Banach space E has a weakly continous duality mapping J_φ with jauge φ.*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle$$

for all x,y $\in E$.
In particular, for all x,y $\in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

Lemma 2.3. (Xu, [19], Zalinescu [22]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r := \{x \in E : ||x|| \leq r\}$ and $\beta_n \in [0,1]$. Then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+,\ g(0) = 0$$

such that for all $x, y \in B_r$

$$\|\beta_n x + (1-\beta_n)y\|^2 \leq \beta_n\|x\|^2 + (1-\beta_n)\|y\|^2 - (1-\beta_n)\beta_n g(\|x-y\|).$$

Lemma 2.4 (Xu, [21]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [7] *Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ \ t_n\} \leq t_{\tau(n)+1}.$$

Let C be a nonempty subsets of real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_Cx + t(x - Q_Cx)) = Q_Cx$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_Cx = x$ for each $x \in C$.

Lemma 2.6. [15] *Let C and D be nonempty subsets of a real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_Dz, j(y - Q_Dz)\rangle \leq 0$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 2.6 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 2.2. If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.

3. Main results

We now prove the following result.

Theorem 3.2. *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_φ and K be a nonempty, closed and convex cone of E. Let $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\} \ \forall p \in F(T)$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)v_n, \ \ v_n \in Tx_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n \end{cases} \tag{3.5}$$

$\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0, 1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; $\quad (ii)$ $\beta_n \in [a, b] \subset (0, 1)$.
(iii) $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.
Assume that $I - T$ is demiclosed at the origine.
Then, the sequence $\{x_n\}$ generated by (3.5) *converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(0)$ with $Q_{F(T)}$ the sunny nonexpansive retraction of K onto $F(T)$.*

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $p \in F(T)$.
Using (3.5), T is quasi-nonexpansive and $Tp = \{p\}$, we have

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1 - \beta_n)v_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|v_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)H(Tx_n, Tp) \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|x_n - p\|. \end{aligned}$$

Hence,

$$\|y_n - p\| \leq \|x_n - p\|. \tag{3.6}$$

Using (3.5) and inequality (3.6), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\
&\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\
&\leq [1-(1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\
&\leq \max\{\|x_n - p\|,\ \|p\|\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|,\ \|p\|\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded also are $\{y_n)\}$, and $\{Tx_n\}$.
Using Lemma 2.3 and (3.5), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n x_n + (1-\beta_n)v_n - p\|^2 \\
&\leq (1-\beta_n)\|v_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|v_n - x_n\|) \\
&\leq (1-\beta_n)H(Tx_n, Tp)^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|) \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|).
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|). \tag{3.7}$$

Therefore, by Lemma 2.2 and inequality (3.7), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\
&= \|\alpha_n\lambda_n\big(x_n - p\big) + (1-\alpha_n)\big(y_n - p\big) - (1-\lambda_n)\alpha_n p\|^2 \\
&\leq \|\alpha_n\big(\lambda_n x_n - \lambda_n p\big) + (1-\alpha_n)\big(y_n - p\big)\|^2 + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle \\
&\leq \alpha_n\lambda_n^2\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle \\
&\leq \alpha_n\lambda_n\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|)] \\
&\quad +2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle \\
&\leq [1-(1-\lambda_n)\alpha_n]\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - v_n\|) \\
&\quad +2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle.
\end{aligned}$$

Therefore,

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - v_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1-\lambda_n)\alpha_n\langle p, J(p - x_{n+1})\rangle. \tag{3.8}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$(1-\lambda_n)\langle p, J(p - x_{n+1})\rangle \leq B, \quad \text{for all}, \quad n \geq 0.$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - v_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n B. \tag{3.9}$$

Now we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

It then implies from (3.9) that

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - v_n\|) = 0. \tag{3.10}$$

Using the fact that $\beta_n \in [a,b] \subset (0,1)$ and property of g, we have

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.11}$$

Hence,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.12}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle = \lim_{k\to+\infty}\langle x^*, J_\varphi(x^* - x_{n_k})\rangle.$$

From (3.12) and $I-T$ is demiclosed, we obtain $a \in F(T)$. On other hand, the assumption that the duality mapping J_φ is weakly continuous, the fact that $x^* = Q_{F(T)}(0)$ and Lemma 2.6, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_n)\rangle &= \lim_{k\to+\infty}\langle x^*, J_\varphi(x^* - x_{n_k})\rangle\\ &= \langle x^*, J_\varphi(x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma,\ \forall t \geq 0$, and φ is a gauge function, then for $1 \geq k \geq 0,\ \Phi(kt) \leq k\Phi(t)$. From (3.5) and Lemma 2.2, we get that

$$\begin{aligned}\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - x^*\|)\\ &\leq \Phi(\|\alpha_n\lambda_n(x_n - x^*) + (1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leq \Phi(\alpha_n\lambda_n\|x_n - x^*\| + \|(1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leq \Phi(\alpha_n\lambda_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leq \Phi((1-(1-\lambda_n)\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle\\ &\leq [1-(1-\lambda_n)\alpha_n]\Phi(\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - x_{n+1})\rangle.\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ B_k \leq B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From (3.9), we have

$$(1-\alpha_{\tau(n)})\beta_{\tau(n)}(1-\beta_{\tau(n)})g(\|x_{\tau(n)} - v_{\tau(n)}\|) \leq 2\alpha_{\tau(n)}B \to 0 \ \text{ as } \ n\to\infty.$$

Furthermore, we have

$$\|x_{\tau(n)} - v_{\tau(n)}\| \to 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d(x_{\tau(n)}, Tx_{\tau(n)}) = 0. \tag{3.13}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly in E and $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \Phi(\|x_{\tau(n)+1} - x^*\|) - \Phi(\|x_{\tau(n)} - x^*\|) \leq \Big(1-\lambda_{\tau(n)}\Big)\alpha_{\tau(n)}[-\Phi(\|x_{\tau(n)} - x^*\|) + \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle],$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty}\Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.5, we conclude that

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.2 for finding fixed points of multivalued nonexpansive mappings without demiclosedness assumption.

Theorem 3.3. *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_φ and K be a nonempty, closed and convex cone of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $F(T) \neq \emptyset$ and $Tp = \{p\}\ \forall p \in F(T)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)v_n, & v_n \in Tx_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n \end{cases} \tag{3.14}$$

$\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0,1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\beta_n \in [a,b] \subset (0,1)$.
(iii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by (3.14) *converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(0)$ with $Q_{F(T)}$ the unique sunny nonexpansive retraction of K onto $F(T)$.*

Proof. Since every multivalued nonexpansive mapping is multivalued quasi-nonexpansive, then the proof follows Lemma 2.1 and Theorem 3.2. □

Remark 3.3. In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, in the case where E is a real Hilbert space or $E = l_p,\ 1 < p < \infty$, our results can be used to approximated fixed ponts of multivalued quasi-nonexpansive mappings from the closed unit ball to itself.

Corollary 3.1. *Assume that $E = l_p,\ 1 < p < \infty$ or E is a real Hilbert space. Let B be the closed unit ball of E and $T : B \to CB(B)$ be a multivalued quasi-nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}\ \forall p \in F(T)$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in B$ by:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)v_n, & v_n \in Tx_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n \end{cases} \tag{3.15}$$

$\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0,1)$ satisfying:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\beta_n \in [a,b] \subset (0,1)$.
(iii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Assume that $I - T$ is demiclosed at the origine.
Then, the sequence $\{x_n\}$ generated by (3.15) *converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(0)$ with $Q_{F(T)}$ the unique sunny nonexpansive retraction of K onto $F(T)$.*

Now, we give some remarks on our results as follows:

(1) The proof methods of our result are very different from the ones of sow et al. [17] for single-valued version. Further, we remove the following conditions: $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n+1}|<\infty$, $\sum_{n=0}^{\infty}\alpha_n=\infty$, and $\sum_{n=0}^{\infty}|\lambda_n-\lambda_{n+1}|<\infty$ in Theorem 1.1 of [17] for single-valued version.

(2) Our results improve many recent results using Mann's method to approximate fixed points of multivalued nonexpansive mappings.

(3) Our results are applicable for finding minimum-norm fixed points of multivalued quasi-nonexpansive mappings in Hilbert spaces.

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