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On the existence of antiderivatives of some real functions

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ABSTRACT. An antiderivative of a real function $f(x)$ defined on an interval $I \subset \mathbb{R}$ is a function $F(x)$ whose derivative is equal to $f(x)$, that is, $F'(x) = f(x)$, for all $x \in I$. Antidifferentiation is the process of finding the set of all antiderivatives of a given function. If f and g are defined on the same interval I, then the set of antiderivatives of the sum of f and g equals the sum of the general antiderivatives of f and g. In general, the antiderivatives of the product of two functions f and g do not coincide to the product of the antiderivatives of f and g. Moreover, the fact that f and g have antiderivatives does not imply that the product $f \cdot g$ has antiderivatives. Our aim in this paper is to present some conditions which ensure that the product $f \cdot g$ and the composition $f \circ g$ of two functions f and g has antiderivatives.

1. INTRODUCTION

Let $I \subset \mathbb{R}$ be an interval. An antiderivative of a real function $f : I \to \mathbb{R}$ is a function $F : I \to \mathbb{R}$ whose derivative is equal to f, that is, $F'(x) = f(x)$, for all $x \in I$. In this context, for a given real function $f(x)$ defined on an interval $I \subset \mathbb{R}$, we are interested to know: a) does f possess antiderivatives ? and if YES (b) how to compute the antiderivatives of f ?

It is well known that any continuous function has antiderivatives and also that non-continuous functions can still have antiderivatives. A necessary but not sufficient condition for a function to have an antiderivative is to possess the intermediate value property, see [6], [13]. By another known result [13], we know that if f has an antiderivative, is bounded on closed finite subintervals of the domain and has the set of discontinuities of null Lebesgue measure, then its antiderivatives may be found by integration.

In this paper we establish two results which ensure the existence of antiderivatives for a product of two functions, respectively for the composition of two functions.

Example 1.1. The functions $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{2}, x = 0 \\ \sin^2 x - \sin \frac{1}{x}, x \neq 0 \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} \frac{1}{2}, x = 0 \\ \sin^2 x + \sin \frac{1}{x}, x \neq 0 \end{cases}$$

have antiderivatives. But the function $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = f(x) \cdot g(x) = \begin{cases} \frac{1}{4}, x = 0 \\ \sin^4 x - \sin^2 \frac{1}{x}, x \neq 0 \end{cases}$$

has no antiderivatives.

The example above shows that there exist situations when the product of two functions which have antiderivatives can provide a function with no antiderivatives.

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Thus, a result which establishes sufficient conditions to ensure the existence of antiderivatives for such a product of two functions is well motivated.

Example 1.2. The function $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0, x = 0 \\ x^2 \sin \frac{1}{x}, \ x \neq 0 \end{cases}$$

is an antiderivative for $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0, x = 0 \\ 2x \sin \frac{1}{x} - \cos \frac{1}{x}, x \neq 0. \end{cases}$$

If we consider $f = f_1 - f_2$, where

$$f_1 : \mathbb{R} \to \mathbb{R}, \ f_1(x) = f(x) = \begin{cases} 0, x = 0 \\ 2x \sin \frac{1}{x}, x \neq 0 \end{cases}$$

and

$$f_2 : \mathbb{R} \to \mathbb{R}, \ f_2(x) = \begin{cases} 0, x = 0 \\ \cos \frac{1}{x}, x \neq 0 \end{cases}$$

we can conclude that f_2 has antiderivatives, since f has antiderivatives and f_1 is continuous on $\mathbb{R}$ (and hence has antiderivatives, too).

2. THE EXISTENCE OF ANTIDERIVATIVES FOR THE PRODUCT OF TWO FUNCTIONS

The result from this section is similar to the one included in [5], but we give here another proof and present other illustrative examples.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which admits antiderivatives on $\mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be such that*

i) g is differentiable on $\mathbb{R}$,
ii) g' is continuous on $\mathbb{R}$.

Then the function $h : \mathbb{R} \to \mathbb{R}$, $h = f \cdot g$, admits antiderivatives on $\mathbb{R}$.

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be an antiderivative of f on $\mathbb{R}$. Then, $\varphi : \mathbb{R} \to \mathbb{R}, \varphi = F \cdot g$ is differentiable, since F and g are both differentiable functions. We have

$$\varphi'(x) = F'(x) \cdot g(x) + F(x) \cdot g'(x)$$

for all $x \in \mathbb{R}$. So,

$$f \cdot g = \varphi' - F \cdot g'$$

and since the function φ' has antiderivatives and $F \cdot g'$ is continuous, it follows that $f \cdot g$ has antiderivatives. □

Remark 2.1. It is easy to see that hypotheses (i) and (ii) also ensure that g has antiderivatives on $\mathbb{R}$.

The next examples show some functions which have antiderivatives but are discontinuous.

Example 2.3. Let $k : \mathbb{R} \to \mathbb{R}$ be a real function defined by

$$k(x) = \begin{cases} \cos x \cdot \cos \frac{1}{x}, x \neq 0 \\ 0, x = 0. \end{cases}$$

We have $k = k_1 \cdot k_2$, where

$$k_1 : \mathbb{R} \to \mathbb{R},\ k(x) = \begin{cases} 0, x = 0 \\ \cos \frac{1}{x}, x \neq 0 \end{cases}$$

and

$$k_2 : \mathbb{R} \to \mathbb{R},\ k_2(x) = \cos x.$$

Since k_1 has antiderivatives, see Example 1.2, and k_2 is differentiable with k_2' a continuous function, by Theorem 2.1 we obtain that k has antiderivatives on $\mathbb{R}$.

Remark 2.2.

$$l_1 : \mathbb{R} \to \mathbb{R},\ l_1(x) = \begin{cases} \sin x \cdot \cos \frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$$

and

$$l_2 : \mathbb{R} \to \mathbb{R},\ l_2(x) = \begin{cases} \sin x \cdot \sin \frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$$

are continuous on $\mathbb{R}$, so they both possess antiderivatives.

3. The existence of antiderivatives for the composition of two functions

The result from this section is similar to the one included in [5], but we give here another proof and illustrate it by means of other examples.

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two real functions such that*

i) f has antiderivatives on $\mathbb{R}$;
ii) g is differentiable on $\mathbb{R}$;
iii) g' is continuous on $\mathbb{R}$;
iv) $g(x) \neq 0$ for all $x \in \mathbb{R}$.

Then the function $h : \mathbb{R} \to \mathbb{R}$ with $h = f \circ H$ has antiderivatives on $\mathbb{R}$, where H is an antiderivative of the function $\dfrac{1}{g}$.

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be a antiderivatives of f on $\mathbb{R}$. Since the functions F, H and g are differentiable, we obtain the differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi = (F \circ H) \cdot g.$$

Now, we have

$$\varphi'(x) = (f \circ H)(x) + (F \circ H)(x) \cdot g'(x)$$

for all $x \in \mathbb{R}$. So,

$$f \circ H = \varphi' - (F \circ H) \cdot g'.$$

and this implies that $f \circ H$ is a linear combination between two functions which admit antiderivatives. □

Example 3.4. The function $u : \mathbb{R} \to \mathbb{R}$ defined by

$$u(x) = \begin{cases} 0, x = 0 \\ \cos \frac{1}{\ln(x+\sqrt{1+x^2})}, x \neq 0 \end{cases}$$

has antiderivatives on $\mathbb{R}$. To, prove that, consider the functions

$$u_1 : \mathbb{R} \to \mathbb{R},\ \ u_1(x) = \begin{cases} 0, x = 0 \\ \cos \frac{1}{x}, x \neq 0 \end{cases}$$

and

$$u_2 : \mathbb{R} \to \mathbb{R},\ u_2 = \ln(x + \sqrt{1 + x^2})$$

Now, we have $u = u_1 \circ u_2$ where u_1 has antiderivatives and u_2 is an antiderivative of $\frac{1}{g}$ with

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \sqrt{1 + x^2}.$$

The function g is differentiable and

$$g'(x) = \frac{x}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}.$$

Hence g' is continuous on $\mathbb{R}$ and $g(x) \neq 0$, for all $x \in \mathbb{R}$. So by Theorem 3.2 we obtain that $u = u_1 \circ u_2$ has antiderivatives.

4. Conclusions and final remarks

The study of the existence of antiderivatives is an important topic in many research domains. In Romanian literature, the interest for this topic is mainly due to the textbook [6], see also [1], [2], [4], [5], [7], [8], [14] etc. Sufficient conditions for existence of antiderivatives of real functions can be found in [11]. Other abstract results regarding definition and properties of generalized notions of antiderivatives for discrete functions where introduced in [9]. Applications of abstract antiderivatives are showed in [12], where the primitivable functions are used in convolution on L^p abstract spaces and in [3], where it is considered the family of convex antiderivatives. In [5] there are presented sufficient conditions for a product and the composition of the primitive functions to be primitive functions.

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