CREAT. MATH. INFORM. **28** (2019), No. 2, 203 - 214

Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Sums and spectral norms of all almost balancing numbers

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ABSTRACT. In this work, we derive some algebraic relations on sums of all almost balancing numbers of first and second type. We also deduce some formulas on sums of Pell, Pell–Lucas and balancing numbers in terms of all almost balancing numbers of first and second type. Further, we formulate the eigenvalues and spectral norms of circulant matrices of all almost balancing numbers of first and second type.

1. INTRODUCTION

A positive integer n is called a balancing number (see [1] and [3]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1.1)

$$n^2=\frac{(n+r)(n+r+1)}{2} \text{ and } r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{1.2}$$

So from (1.2), n is a balancing number if and only if n^2 is a triangular number and $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. Behera and Panda ([1]), while accepting 1 as a balancing number (since it is the positive square root of the square triangular number 1), have set $B_0=1, B_1=6$ and so on, using the symbol B_n for the n^{th} balancing number. To standardize the notation at par with Fibonacci numbers, we relabel the balancing numbers by setting $B_0=0, B_1=1, B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 2$.

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (1.3)

$$n(n+1)=\frac{(n+r)(n+r+1)}{2} \text{ and } r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{1.4}$$

So from (1.4), n is a cobalancing number if and only if $n(n+1)$ is a triangular number and $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, we accept 0 as a cobalancing number, just like Behera and Panda ([1]) accepted 1 as a balancing number. Cobalancing number is denoted by b_n. Then it is easily seen that $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

Received: 01.03.2018. In revised form: 08.06.2018. Accepted: 15.03.2019
2010 *Mathematics Subject Classification.* 11B37, 11B39.
Key words and phrases. *balancing numbers, Pell numbers, Pell–Lucas numbers, almost balancing numbers, spectral norms.*

It is clear from (1.1) and (1.3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n = r_{n+1}$ and $R_n = b_n$ for $n \geq 1$, where R_n is the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n = b_n$, we get from (1.1) that

$$b_n = \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2} \text{ and } B_n = \frac{2b_n+1+\sqrt{8b_n^2+8b_n+1}}{2}. \tag{1.5}$$

Thus from (1.5), B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. So

$$C_n = \sqrt{8B_n^2+1} \text{ and } c_n = \sqrt{8b_n^2+8b_n+1}$$

are integers which are called the n^{th} Lucas–balancing number and n^{th} Lucas–cobalancing number (Here we notice that $C_0 = c_0 = 1$).

Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ be the roots of the characteristic equation for Pell and Pell–Lucas numbers (which are the numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$). Then Binet formulas for all balancing numbers are

$$B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n}+\beta^{2n}}{2} \text{ and } c_n = \frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$$

for $n \geq 1$ (for further details on balancing numbers see also [4, 8, 9, 11, 14]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [6, Theorem 4], Liptai proved that there is no Fibonacci balancing number except 1 and in [7, Theorem 4] he proved that there is no Lucas balancing number. In [16], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai and Olajos extended the concept of balancing numbers to the $(a, b)-$balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a+b)+\cdots+(a(n-1)+b) = (a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some positive integers n and r, then $an+b$ is an $(a, b)-$balancing number. The sequence of $(a, b)-$balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [8], the authors generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x with $x \leq y-2$ is called a $(k, l)-$power numerical center for y if

$$1^k+\cdots+(x-1)^k = (x+1)^l+\cdots+(y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for $(k, l)-$power numerical centers.

For positive integers k, x, let

$$\Pi_k(x) = x(x+1)\ldots(x+k-1).$$

Then it was proved in [5, Theorem 3 and Theorem 4] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [19, Theorem 1] Tengely, considered the case $k = 5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$.

Almost balancing numbers first defined by Panda and Panda in [13]. A natural number n is called an almost balancing number if the Diophantine equation

$$|[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]| = 1 \tag{1.6}$$

holds for some positive integer r which is called the almost balancer. From (1.6), if $nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = 1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 9}}{2}. \tag{1.7}$$

If $nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = -1$, then n is called an almost balancing number of second type and r is called an almost balancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 - 7}}{2}. \tag{1.8}$$

Let B_n^* denote the n^{th} almost balancing number of first type and let B_n^{**} denote the n^{th} almost balancing number of second type. Then from (1.7), B_n^* is an almost balancing number of first type if and only if $8(B_n^*)^2 + 9$ is a perfect square and from (1.8), B_n^{**} is an almost balancing number of second type if and only if $8(B_n^{**})^2 - 7$ is a perfect square. Thus

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} \text{ and } C_n^{**} = \sqrt{8(B_n^{**})^2 - 7}$$

are integers which are called the n^{th} almost Lucas–balancing number of first type and the n^{th} almost Lucas–balancing number of second type.

Similarly in [10], Panda defined that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - (1 + 2 + \cdots + n)| = 1 \tag{1.9}$$

holds for some positive integer r which is called an almost cobalancer. From (1.9), if $nr + \frac{r(r+1)}{2} - \frac{n(n+1)}{2} = 1$, then n is called an almost cobalancing number of first type and r is called an almost cobalancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 9}}{2}. \tag{1.10}$$

If $nr + \frac{r(r+1)}{2} - \frac{n(n+1)}{2} = -1$, then n is called an almost cobalancing number of second type and r is called an almost cobalancer of second type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n - 7}}{2}. \tag{1.11}$$

Let b_n^* denote the n^{th} almost cobalancing number of first type and let b_n^{**} denote the n^{th} almost cobalancing number of second type. Then from (1.10), b_n^* is an almost cobalancing number of first type if and only if $8(b_n^*)^2 + 8b_n^* + 9$ is a perfect square and from (1.11), b_n^{**} is an almost cobalancing number of second type if and only if $8(b_n^{**})^2 + 8b_n^{**} - 7$ is a perfect square. Thus

$$c_n^* = \sqrt{8(b_n^*)^2 + 8b_n^* + 9} \text{ and } c_n^{**} = \sqrt{8(b_n^{**})^2 + 8b_n^{**} - 7}$$

are integers which are called the n^{th} almost Lucas–cobalancing number of first type and the n^{th} almost Lucas–cobalancing number of second type.

2. Sums of almost balancing numbers.

In [18], we considered the almost balancing and almost cobalancing numbers of first and second type and proved that the general terms of all almost balancing numbers of first type are

$$B_n^* = 3B_n, b_{2n}^* = 2b_{n+1} - b_n, b_{2n-1}^* = 4b_n - b_{n-1} + 1,$$
$$C_n^* = 3C_n, c_{2n}^* = c_{n+2} - 4c_{n+1}, c_{2n-1}^* = c_{n+1} - 2c_n$$

and the general terms of all almost balancing numbers of second type are

$$B^{**}_{2n-1} = B_{n-1} + C_{n-1}, B^{**}_{2n} = -B_n + C_n, b^{**}_n = 3b_n + 1,$$
$$C^{**}_{2n-1} = 8B_{n-1} + C_{n-1}, C^{**}_{2n} = 8B_n - C_n, c^{**}_n = 3c_n$$

for $n \geq 1$ (Here we note that $B^*_0 = 0, b^*_0 = 0, C^*_0 = 3, c^*_0 = 3, B^{**}_0 = 1, b^{**}_0 = 1, C^{**}_0 = -1$ and $c^{**}_0 = 3$).

In this section, we consider the sums of all almost balancing numbers of first and second type. We also deduce some formulas for the sums of Pell, Pell–Lucas and balancing numbers in terms of all almost balancing numbers of first and second type. Later we give the eigenvalues and spectral norms of circulant matrices of all almost balancing numbers of first and second type.

Theorem 2.1. *The sum of first $n-$terms of all almost balancing numbers of first type is given by the formulae*

$$\sum_{i=1}^{n} B^*_i = \frac{5B^*_n - B^*_{n-1} - 3}{4},\ n \geq 1$$

$$\sum_{i=1}^{n} b^*_i = \begin{cases} \frac{b^*_{n+2} - b^*_{n+1} + b^*_n - b^*_{n-1} - n - 3}{2} & n \geq 1 \text{ odd} \\ \frac{3b^*_{n+1} - 3b^*_n - n - 3}{2} & n \geq 2 \text{ even} \end{cases}$$

$$\sum_{i=1}^{n} C^*_i = \frac{7B^*_n - B^*_{n-1} - 3}{2},\ n \geq 1$$

$$\sum_{i=1}^{n} c^*_i = \begin{cases} \frac{6c^*_{n+2} - 6c^*_{n+1} - 12B^*_{\frac{n+1}{2}} + 4B^*_{\frac{n-1}{2}} - 18}{6} & n \geq 1 \text{ odd} \\ \frac{3c^*_{n+3} - 3c^*_{n+2} - 6c^*_{n+1} + 6c^*_n - 12B^*_{\frac{n}{2}} + 4B^*_{\frac{n-2}{2}} - 18}{6} & n \geq 2 \text{ even} \end{cases}$$

and sum of first $n-$terms of all almost balancing numbers of second type is given by the formulae

$$\sum_{i=1}^{n} B^{**}_i = \begin{cases} \frac{7B^{**}_n - 7B^{**}_{n-1} - B^{**}_{n-2} + B^{**}_{n-3}}{2} & n \geq 3 \text{ odd} \\ \frac{B^{**}_{n-1} - B^{**}_{n-2} + C^{**}_{n+1} - C^{**}_n}{2} & n \geq 2 \text{ even} \end{cases}$$

$$\sum_{i=1}^{n} b^{**}_i = \frac{3B^{**}_{2n+1} - 3B^{**}_{2n} - 2n}{4},\ n \geq 1$$

$$\sum_{i=1}^{n} C^{**}_i = \begin{cases} 10B^{**}_n - 10B^{**}_{n-1} - 2B^{**}_{n-2} + 2B^{**}_{n-3} - 3 & n \geq 3 \text{ odd} \\ \frac{68B^{**}_{n-1} - 68B^{**}_{n-2} - 12B^{**}_{n-3} + 12B^{**}_{n-4} - C^{**}_{n+1} + C^{**}_n - 6}{2} & n \geq 4 \text{ even} \end{cases}$$

$$\sum_{i=1}^{n} c^{**}_i = \frac{9B^{**}_{2n+1} - 9B^{**}_{2n} - 3B^{**}_{2n-1} + 3B^{**}_{2n-2} - 6}{4},\ n \geq 1.$$

Proof. For the almost balancing numbers B^*_n of first type, we have $B^*_n = 6B^*_{n-1} - B^*_{n-2}$ for $n \geq 2$. So we get $B^*_n + B^*_{n-2} = 6B^*_{n-1}$ and hence

$$B^*_2 + B^*_0 = 6B^*_1 \tag{2.12}$$
$$B^*_3 + B^*_1 = 6B^*_2$$
$$\cdots$$

$$B^*_{n-1} + B^*_{n-3} = 6B^*_{n-2}$$
$$B^*_n + B^*_{n-2} = 6B^*_{n-1}.$$

If we sum of both sides of (2.12), then we obtain $(B^*_2 + B^*_3 + \cdots + B^*_n) + (B^*_0 + B^*_1 + B^*_2 + \cdots + B^*_{n-2}) = 6(B^*_1 + B^*_2 + \cdots + B^*_{n-1})$ and hence $2(B^*_1 + B^*_2 + \cdots + B^*_n) - B^*_1 - B^*_{n-1} - B^*_n = 6(B^*_1 + B^*_2 + \cdots + B^*_n) - 6B^*_n$. So $-4(B^*_1 + B^*_2 + \cdots + B^*_n) = -5B^*_n + B^*_{n-1} + B^*_1$. Thus

$$B^*_1 + B^*_2 + \cdots + B^*_n = \frac{5B^*_n - B^*_{n-1} - 3}{4}$$

since $B^*_1 = 3$. The other cases can be proved similarly. □

In [15], Santana and Diaz–Barrero proved that the sum of first nonzero $4n + 1$ terms of Pell numbers is a perfect square, that is,

$$\sum_{i=1}^{4n+1} P_i = \left[\sum_{i=0}^{n} \binom{2n+1}{2i} 2^i\right]^2. \tag{2.13}$$

Later in [17], Tekcan and Tayat proved that the sum of first nonzero $2n + 1$ terms of Pell numbers is a perfect square if n is even or half of a perfect square if n is odd, that is, they proved that

$$\sum_{i=1}^{2n+1} P_i = \begin{cases} \left(\frac{\alpha^{n+1}+\beta^{n+1}}{2}\right)^2 & \text{for even } n \geq 2 \\ \frac{\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{2}}\right)^2}{2} & \text{for odd } n \geq 1, \end{cases}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. By considering this equality, they set two integer sequences

$$X_n = \frac{\alpha^{n+1} + \beta^{n+1}}{2} \quad \text{and} \quad Y_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{2}}$$

for $n \geq 0$ and proved that the right hand side of (2.13) is $[2X^2_n - 2X_nY_{n-1} + (-1)^{n+1}]^2$, that is,

$$\sum_{i=1}^{4n+1} P_i = [2X^2_n - 2X_nY_{n-1} + (-1)^{n+1}]^2.$$

Similarly we can determine the right hand side of (2.13) in terms of almost balancing and almost Lucas–balancing numbers of first and second type as follows:

Theorem 2.2. *The sum of Pell numbers from* 1 *to* $4n + 1$ *is*

$$\sum_{i=1}^{4n+1} P_i = \begin{cases} \left(\frac{4B^*_n+C^*_n}{3}\right)^2 & \text{for } n \geq 1 \\ \left(\frac{4B^{**}_{2n+1}-4B^{**}_{2n}+C^{**}_{2n+1}-C^{**}_{2n}}{2}\right)^2 & \text{for } n \geq 1. \end{cases}$$

Proof. Note that $B^*_n = 3\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)$, $C^*_n = 3\left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right)$ and $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$. Since

$$P_1 + P_2 + \cdots + P_n = \frac{P_n + P_{n+1} - 1}{2},$$

we get

$$\begin{aligned}\sum_{i=1}^{4n+1} P_i &= \frac{P_{4n+1} + P_{4n+2} - 1}{2} \\ &= \frac{\frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}} + \frac{\alpha^{4n+2}-\beta^{4n+2}}{2\sqrt{2}} - 1}{2}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}\right)^2 \\
&= \left(\alpha^{2n}\Big(\frac{1}{\sqrt{2}}+\frac{1}{2}\Big)+\beta^{2n}\Big(\frac{-1}{\sqrt{2}}+\frac{1}{2}\Big)\right)^2 \\
&= \left(\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}}+\frac{\alpha^{2n}+\beta^{2n}}{2}\right)^2 \\
&= \left(\frac{12\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)+3\left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right)}{3}\right)^2 \\
&= \left(\frac{4B_n^*+C_n^*}{3}\right)^2
\end{aligned}
$$

as we wanted. The other case can be proved similarly. □

We can also rewrite Theorem 2.2, in terms of almost Lucas–cobalancing numbers of first and second type as follows:

Theorem 2.3. *The sum of Pell numbers from* 1 *to* $4n+1$ *is*

$$
\sum_{i=1}^{4n+1} P_i = \begin{cases} \left(\frac{c_{2n+1}^*-c_{2n}^*}{2}\right)^2 & \text{for } n \geq 1 \\ \left(\frac{c_{n+1}^{**}}{3}\right)^2 & \text{for } n \geq 1. \end{cases}
$$

As in Theorem 2.2, we can give the following theorem which can be proved similarly.

Theorem 2.4. *For the sums of Pell, Pell–Lucas and balancing numbers, we have*

(1) The sum of Pell numbers from 1 *to* $4n-1$ *adding* 1 *is a perfect square and is*

$$
1+\sum_{i=1}^{4n-1} P_i = \begin{cases} \left(\frac{C_n^*}{3}\right)^2 & \text{for } n \geq 1 \\ \left(\frac{C_{2n+1}^{**}-C_{2n}^{**}}{2}\right)^2 & \text{for } n \geq 1. \end{cases}
$$

Also

$$
\sum_{i=1}^{2n} P_i = \begin{cases} \frac{7B_n^*-B_{n-1}^*-3}{6} & \text{for } n \geq 1 \\ \frac{7B_{2n+1}^{**}-7B_{2n}^{**}-B_{2n-1}^{**}+B_{2n-2}^{**}-2}{4} & \text{for } n \geq 1 \end{cases}
$$

$$
\sum_{i=1}^{n} P_{2i} = \begin{cases} \frac{b_{2n+1}^*-b_{2n}^*-1}{2} & \text{for } n \geq 1 \\ \frac{b_{n+1}^{**}-1}{3} & \text{for } n \geq 1 \end{cases}
$$

$$\sum_{i=0}^{2n} P_{2i+1} = \begin{cases} \frac{(b^*_{2n+1}-b^*_{2n})(c^*_{2n+1}-c^*_{2n})}{2} & \text{for } n \geq 1 \\ \frac{c^{**}_{n+1}(2b^{**}_{n+1}+1)}{9} & \text{for } n \geq 1 \end{cases}$$

$$\sum_{i=1}^{2n} P_{2i-1} = \begin{cases} \frac{2B^*_n C^*_n}{9} & \text{for } n \geq 1 \\ \frac{(B^{**}_{2n+1}-B^{**}_{2n})(C^{**}_{2n+1}-C^{**}_{2n})}{2} & \text{for } n \geq 1. \end{cases}$$

(2) *The sum of $(2i-1)^{st}$ Pell–Lucas numbers from 1 to $2n$ is a perfect square and is*

$$\sum_{i=1}^{2n} Q_{2i-1} = \begin{cases} (\frac{4B^*_n}{3})^2 & \text{for } n \geq 1 \\ (2B^{**}_{2n+1} - 2B^{**}_{2n})^2 & \text{for } n \geq 1 \end{cases}$$

and the half of the sum of $(2i+1)^{st}$ Pell–Lucas numbers from 0 to $2n$ is a perfect square and is

$$\frac{\sum_{i=0}^{2n} Q_{2i+1}}{2} = \begin{cases} (\frac{c^*_{2n+1}-c^*_{2n}}{2})^2 & \text{for } n \geq 1 \\ (\frac{c^{**}_{n+1}}{3})^2 & \text{for } n \geq 1. \end{cases}$$

Also

$$\sum_{i=1}^{2n} Q_i = \begin{cases} 2b^*_{2n+1} - 2b^*_{2n} - 2 & \text{for } n \geq 1 \\ \frac{4b^{**}_{n+1}-4}{3} & \text{for } n \geq 1 \end{cases}$$

$$\sum_{i=0}^{2n-1} Q_i = \begin{cases} \frac{4B^*_n}{3} & \text{for } n \geq 1 \\ 2B^{**}_{2n+1} - 2B^{**}_{2n} & \text{for } n \geq 1 \end{cases}$$

$$\sum_{i=1}^{2n+1} Q_i = \begin{cases} \frac{4B^*_{n+1}-6}{3} & \text{for } n \geq 1 \\ 2B^{**}_{2n+3} - 2B^{**}_{2n+2} - 2 & \text{for } n \geq 1 \end{cases}$$

$$\sum_{i=1}^{2n} Q_{2i} = \begin{cases} \frac{8B^*_n(b^*_{2n+1}-b^*_{2n})}{3} & \text{for } n \geq 1 \\ \frac{(4B^{**}_{2n+1}-4B^{**}_{2n})(2b^{**}_{n+1}+1)}{3} & \text{for } n \geq 1. \end{cases}$$

(3) *The sum of $(2i-1)^{st}$ balancing numbers from 1 to n is a perfect square and is*

$$\sum_{i=1}^{n} B_{2i-1} = \begin{cases} (\frac{B^*_n}{3})^2 & \text{for } n \geq 1 \\ \left(\frac{B^{**}_{2n+1}-B^{**}_{2n}}{2}\right)^2 & \text{for } n \geq 1. \end{cases}$$

Also

$$\sum_{i=1}^{2n} B_i = \begin{cases} \frac{B_n^*(c_{2n+1}^*-c_{2n}^*)}{6} & \textit{for } n \geq 1 \\ \frac{c_{n+1}^{**}(B_{2n+1}^{**}-B_{2n}^{**})}{6} & \textit{for } n \geq 1 \end{cases}$$

$$\sum_{i=1}^{2n} B_{2i} = \begin{cases} \frac{B_n^* C_n^*(b_{2n+1}^*-b_{2n}^*)(c_{2n+1}^*-c_{2n}^*)}{9} & \textit{for } n \geq 1 \\ \frac{c_{n+1}^{**}(B_{2n+1}^{**}-B_{2n}^{**})(C_{2n+1}^{**}-C_{2n}^{**})(2b_{n+1}^{**}+1)}{18} & \textit{for } n \geq 1. \end{cases}$$

Now we can consider the eigenvalues and spectral norms of circulant matrices of all almost balancing numbers of first and second type. Recall that a circulant matrix (see [2]) is a matrix M defined as

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_n \\ m_n & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_n & m_1 & \cdots & m_{n-3} & m_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ m_3 & m_4 & m_5 & \cdots & m_1 & m_2 \\ m_2 & m_3 & m_4 & \cdots & m_n & m_1 \end{bmatrix},$$

where m_i are constant. In this case, the eigenvalues of M are

$$\lambda_j(M) = \sum_{u=0}^{n-1} m_u w^{-ju}, \tag{2.14}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. The spectral norm of $M = [m_{ij}]_{n\times n}$ is defined to be

$$||M||_{spec} = \max_{0\leq j\leq n-1}\{\sqrt{\lambda_j}\},$$

where λ_j are the eigenvalues of $M^H M$ and M^H denotes the conjugate transpose of M.

Let B_n^*, b_n^*, C_n^* and c_n^* denote the circulant matrices of almost balancing numbers first type and let $B_n^{**}, b_n^{**}, C_n^{**}$ and c_n^{**} denote the circulant matrices of almost balancing numbers of second type. Then we can give the following theorem.

Theorem 2.5. *The eigenvalues of circulant matrices of all almost balancing numbers are*

$$\lambda_j(B^*_n) = \frac{(B^*_{n-1}+3)w^{-j} - B^*_n}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(b^*_{2n-1}) = \frac{(\frac{6B^*_{n-1}-C^*_{n-1}+27}{6})w^{-j} + \frac{-6B^*_n+C^*_n-3}{6}}{w^{-2j} - 6w^{-j} + 1} - \frac{n}{2}$$

$$\lambda_j(b^*_{2n}) = \frac{(\frac{6B^*_{n-1}+C^*_{n-1}+9}{6})w^{-j} + \frac{-6B^*_n-C^*_n+3}{6}}{w^{-2j} - 6w^{-j} + 1} - \frac{n}{2}$$

$$\lambda_j(C^*_n) = \frac{(C^*_{n-1}-9)w^{-j} - C^*_n + 3}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(c^*_{2n-1}) = \frac{(\frac{-4B^*_{n-1}+3C^*_{n-1}-39}{3})w^{-j} + \frac{4B^*_n-3C^*_n+9}{3}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(c^*_{2n}) = \frac{(\frac{4B^*_{n-1}+3C^*_{n-1}-15}{3})w^{-j} + \frac{-4B^*_n-3C^*_n+9}{3}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(B^{**}_{2n-1}) = \frac{\left\{\begin{array}{c} (\frac{-5B^{**}_{2n-1}+5B^{**}_{2n-2}+2C^{**}_{2n-1}-2C^{**}_{2n-2}-22}{2})w^{-j} \\ +\frac{5B^{**}_{2n+1}-5B^{**}_{2n}-2C^{**}_{2n+1}+2C^{**}_{2n}+4}{2} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(B^{**}_{2n}) = \frac{\left\{\begin{array}{c} (\frac{-B^{**}_{2n-1}+B^{**}_{2n-2}+C^{**}_{2n-1}-C^{**}_{2n-2}-8}{2})w^{-j} \\ +\frac{B^{**}_{2n+1}-B^{**}_{2n}-C^{**}_{2n+1}+C^{**}_{2n}+2}{2} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(b^{**}_n) = \frac{\left\{\begin{array}{c} (\frac{-6B^{**}_{2n-1}+6B^{**}_{2n-2}+3C^{**}_{2n-1}-3C^{**}_{2n-2}-30}{4})w^{-j} \\ +\frac{6B^{**}_{2n+1}-6B^{**}_{2n}-3C^{**}_{2n+1}+3C^{**}_{2n}+6}{4} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1} - \frac{n}{2}$$

$$\lambda_j(C^{**}_{2n-1}) = \frac{\left\{\begin{array}{c} (\frac{16B^{**}_{2n-1}-16B^{**}_{2n-2}-5C^{**}_{2n-1}+5C^{**}_{2n-2}+62}{2})w^{-j} \\ +\frac{-16B^{**}_{2n+1}+16B^{**}_{2n}+5C^{**}_{2n+1}-5C^{**}_{2n}-10}{2} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(C^{**}_{2n}) = \frac{\left\{\begin{array}{c} (\frac{8B^{**}_{2n-1}-8B^{**}_{2n-2}-C^{**}_{2n-1}+C^{**}_{2n-2}+22}{2})w^{-j} \\ +\frac{-8B^{**}_{2n+1}+8B^{**}_{2n}+C^{**}_{2n+1}-C^{**}_{2n}-2}{2} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1}$$

$$\lambda_j(c^{**}_n) = \frac{\left\{\begin{array}{c} (\frac{12B^{**}_{2n-1}-12B^{**}_{2n-2}-3C^{**}_{2n-1}+3C^{**}_{2n-2}+42}{2})w^{-j} \\ +\frac{-12B^{**}_{2n+1}+12B^{**}_{2n}+3C^{**}_{2n+1}-3C^{**}_{2n}-6}{2} \end{array}\right\}}{w^{-2j} - 6w^{-j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$ *and the spectral norms are*

$$||B_n^*||_{spec} = \frac{5B_n^* - B_{n-1}^* - 3}{4}$$

$$||b_n^*||_{spec} = \begin{cases} \frac{b_{n+2}^* - b_{n+1}^* + b_n^* - b_{n-1}^* - n - 3}{2} & n \geq 1 \text{ odd} \\ \frac{3b_{n+1}^* - 3b_n^* - n - 3}{2} & n \geq 2 \text{ even} \end{cases}$$

$$||C_n^*||_{spec} = \frac{7B_n^* - B_{n-1}^* - 3}{2}, \ n \geq 1$$

$$||c_n^*||_{spec} = \begin{cases} \frac{6c_{n+2}^* - 6c_{n+1}^* - 12B_{\frac{n+1}{2}}^* + 4B_{\frac{n-1}{2}}^* - 18}{6} & n \geq 1 \text{ odd} \\ \frac{3c_{n+3}^* - 3c_{n+2}^* - 6c_{n+1}^* + 6c_n^* - 12B_{\frac{n}{2}}^* + 4B_{\frac{n-2}{2}}^* - 18}{6} & n \geq 2 \text{ even} \end{cases}$$

$$||B_n^{**}||_{spec} = \begin{cases} \frac{7B_n^{**} - 7B_{n-1}^{**} - B_{n-2}^{**} + B_{n-3}^{**}}{2} & n \geq 3 \text{ odd} \\ \frac{B_{n-1}^{**} - B_{n-2}^{**} + C_{n+1}^{**} - C_n^{**}}{2} & n \geq 2 \text{ even} \end{cases}$$

$$||b_n^{**}||_{spec} = \frac{3B_{2n+1}^{**} - 3B_{2n}^{**} - 2n}{4}, \ n \geq 1$$

$$||C_n^{**}||_{spec} = \begin{cases} 10B_n^{**} - 10B_{n-1}^{**} - 2B_{n-2}^{**} + 2B_{n-3}^{**} - 3 & n \geq 3 \text{ odd} \\ \frac{68B_{n-1}^{**} - 68B_{n-2}^{**} - 12B_{n-3}^{**} + 12B_{n-4}^{**} - C_{n+1}^{**} + C_n^{**} - 6}{2} & n \geq 4 \text{ even} \end{cases}$$

$$||c_n^{**}||_{spec} = \frac{9B_{2n+1}^{**} - 9B_{2n}^{**} - 3B_{2n-1}^{**} + 3B_{2n-2}^{**} - 6}{4}, \ n \geq 1.$$

Proof. Since $B_n^* = 3\left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}\right)$, we get from (2.14) that

$$\begin{aligned} \lambda_j(B_n^*) &= \sum_{u=0}^{n-1} B_u^* w^{-ju} \\ &= \frac{3}{4\sqrt{2}} \sum_{u=0}^{n-1} \left(\alpha^{2u} - \beta^{2u}\right) w^{-ju} \\ &= \frac{3}{4\sqrt{2}} \left(\sum_{u=0}^{n-1} (\alpha^2 w^{-j})^u - \sum_{u=0}^{n-1} (\beta^2 w^{-j})^u \right) \\ &= \frac{3}{4\sqrt{2}} \left(\frac{\alpha^{2n} - 1}{\alpha^2 w^{-j} - 1} - \frac{\beta^{2n} - 1}{\beta^2 w^{-j} - 1} \right) \\ &= \frac{3}{4\sqrt{2}} \left(\frac{w^{-j}(\alpha^{2n}\beta^2 - \beta^2 - \beta^{2n}\alpha^2 + \alpha^2) - \alpha^{2n} + \beta^{2n}}{w^{-2j} - 6w^{-j} + 1} \right) \end{aligned}$$

$$= \frac{w^{-j}}{w^{-2j} - 6w^{-j} + 1} \left[3 \left(\frac{\alpha^{2n-2} - \beta^{2n-2}}{4\sqrt{2}} \right) + 3 \right]$$
$$- \frac{1}{w^{-2j} - 6w^{-j} + 1} \left[3 \left(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \right) \right]$$
$$= \frac{(B^*_{n-1} + 3)w^{-j} - B^*_n}{w^{-2j} - 6w^{-j} + 1}.$$

For the circulant matrix

$$B^*_n = \begin{bmatrix} B^*_1 & B^*_2 & B^*_3 & \cdots & B^*_{n-1} & B^*_n \\ B^*_n & B^*_1 & B^*_2 & \cdots & B^*_{n-2} & B^*_{n-1} \\ B^*_{n-1} & B^*_n & B^*_1 & \cdots & B^*_{n-3} & B^*_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ B^*_3 & B^*_4 & B^*_5 & \cdots & B^*_1 & B^*_2 \\ B^*_2 & B^*_3 & B^*_4 & \cdots & B^*_n & B^*_1 \end{bmatrix},$$

for almost balancing numbers B^*_n of first type, we have

$$(B^*_n)^H B_n = \begin{bmatrix} B^*_{11} & B^*_{12} & \cdots & B^*_{1(n-1)} & B^*_{1n} \\ B^*_{21} & B^*_{22} & \cdots & B^*_{2(n-1)} & B^*_{2n} \\ . & . & & . & . \\ . & . & \cdots & . & . \\ . & . & & . & . \\ B^*_{(n-1)1} & B^*_{(n-1)2} & \cdots & B^*_{(n-1)(n-1)} & B^*_{(n-1)n} \\ B^*_{n1} & B^*_{n2} & \cdots & B^*_{n(n-1)} & B^*_{nn} \end{bmatrix},$$

where

$$\begin{aligned}
B^*_{11} &= (B^*_1)^2 + (B^*_n)^2 + \cdots + (B^*_3)^2 + (B^*_2)^2 \\
B^*_{12} &= B^*_1 B^*_2 + B^*_n B^*_1 + \cdots + B^*_3 B^*_4 + B^*_2 B^*_3 \\
&\cdots \\
B^*_{1(n-1)} &= B^*_1 B^*_{n-1} + B^*_n B^*_{n-2} + \cdots + B^*_3 B^*_1 + B^*_2 B^*_n \\
B^*_{1n} &= B^*_1 B^*_n + B^*_n B^*_{n-2} + \cdots + B^*_3 B^*_1 + B^*_2 B^*_n \\
B^*_{21} &= B^*_2 B^*_1 + B^*_1 B^*_n + \cdots + B^*_4 B^*_3 + B^*_3 B^*_2 \\
B^*_{22} &= (B^*_2)^2 + (B^*_1)^2 + \cdots + (B^*_4)^2 + (B^*_3)^2 \\
&\cdots \\
B^*_{2(n-1)} &= B^*_2 B^*_{n-1} + B^*_1 B^*_{n-2} + \cdots + B^*_4 B^*_1 + B^*_3 B^*_4 \\
B^*_{2n} &= B^*_2 B^*_n + B^*_1 B^*_{n-1} + \cdots + B^*_4 B^*_2 + B^*_3 B^*_1 \\
&\cdots \\
B^*_{n1} &= B^*_n B^*_1 + B^*_{n-1} B^*_n + \cdots + B^*_2 B^*_3 + B^*_1 B^*_2 \\
B^*_{n2} &= B^*_n B^*_2 + B^*_{n-1} B^*_1 + \cdots + B^*_2 B^*_4 + B^*_1 B^*_3 \\
&\cdots \\
B^*_{n(n-1)} &= B^*_n B^*_{n-1} + B^*_{n-1} B^*_{n-2} + \cdots + B^*_2 B^*_1 + B^*_1 B^*_n \\
B^*_{nn} &= (B^*_n)^2 + (B^*_{n-1})^2 + \cdots + (B^*_2)^2 + (B^*_1)^2.
\end{aligned}$$

The eigenvalues of $(B_n^*)^H B_n^*$ are $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$. Here λ_0 is maximum and is

$$\begin{aligned}\lambda_0 &= (B_1^*)^2 + (B_2^*)^2 + \cdots + (B_{n-1}^*)^2 + (B_n^*)^2 \\ &+ 2\left[\begin{array}{c} B_1^*(B_2^* + B_3^* + \cdots + B_{n-1}^* + B_n^*) \\ + B_2^*(B_3^* + \cdots + B_{n-1}^* + B_n^*) \\ + \cdots \\ + B_{n-1}^* B_n^* \end{array}\right] \\ &= (B_1^* + B_2^* + \cdots + B_n^*)^2.\end{aligned}$$

Thus the spectral norm of B_n^* is $||B_n^*||_{spec} = \sqrt{\lambda_0} = B_1^* + B_2^* + \cdots + B_n^*$. Since

$$B_1^* + B_2^* + \cdots + B_n^* = \frac{5B_n^* - B_{n-1}^* - 3}{4}$$

by Theorem 2.1, we conclude that

$$||B_n^*||_{spec} = \frac{5B_n^* - B_{n-1}^* - 3}{4}.$$

The others can be proved similarly. □

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