CREAT. MATH. INFORM. 28 (2019), No. 2, 215 - 221
Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Reduction approach to second order perturbed state-dependent sweeping process

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ABSTRACT. In this paper, we present a new approach to solving second order nonconvex perturbed sweeping process in finite dimensional setting. It consists in a reduction of the problem to a first order one without use of the standard methods of fixed point theory. The perturbation, that is the external force applied on the system is not necessary with bounded values.

1. INTRODUCTION

We deal, in this paper, with a class of evolution inclusions driven by second order time and state-dependent sweeping process. The so-called sweeping process is an evolution differential inclusion, governed by normal cones, that plays an important role in nonsmooth mechanics, elastoplasticity, quasistatics, convex optimization, planning procedures in mathematical economy, game theory, dynamics, $\cdots$. Such problems have been introduced and thoroughly studied in the 70's by Moreau in the setting where the sets are assumed to be convex (see [23]). Generalizations of the sweeping process have been the object of many studies, see e.g. [4, 6, 7, 8, 10, 11, 16, 17, 24, 26] and the references therein. It was also shown that quite similar formalisms apply to nonsmooth electrical networks as well as some problems of absolute stability [9, 18].

In general, the existence (and uniqueness) of solution for such problem is established by proving the convergence of the Moreau catching-up algorithm. This algorithm is constructed with a discretization of time and using of the projection property. The well-posedness of such algorithm in the convex case is clear. In the nonconvex case, that is for (uniformly) prox regular (or equivalently proximally smooth) sets, appropriate choices of the discretization make that the algorithm is still well-posed. Note that there are other approaches: the first is the regularization (Yoshida) method (like for maximal monotone operators) by reducing the problem to a differential equation (see for example [27]); and the second is based on the connection of the problem with an unconstrained differential inclusion governed by the subdifferential of the distance function (see [28]).

When the moving sets depend also on the state, one obtains a generalization of the classical sweeping process known as the state-dependent sweeping process. Such problems are motivated by parabolic quasi-variational inequalities arising e.g. in the evolution of sandpiles, and occur also in the treatment of 2-D or 3-D quasistatic evolution problems with friction, as well as in micro-mechanical damage models for iron materials with memory to describe the evolution of the plastic strain in presence of small damages. We refer to [19] for more details.

This problem has been studied for the first time for convex sets by Chraibi [14] in finite dimension, then by Kunze and Monteiro Marques [20] in Hilbert spaces under some compactness condition. Recently, Chemetov and Monteiro Marques [13] established the

Received: 08.06.2018. In revised form: 21.03.2019. Accepted: 28.03.2019
2010 *Mathematics Subject Classification.* 34A60, 49J52.
Key words and phrases. *state-dependent sweeping process, prox regular sets, unbounded perturbation, reduction.*

existence for prox regular sets with a Carathéodory perturbation by applying the Shauder fixed point theorem. By means of a generalized version of the Shauder theorem, Castaing, Ibrahim and Yarou [11] provided an other approach to prove the existence for prox regular and ball compact sets, and for the perturbed problem (even in presence of a delay). The approach is based on the Moreau's catching-up algorithm. To our knowledge, the first result concerning the second order time and state dependent sweeping process is due to [11]; for recent results about such problems, we refer to [1]-[5] and the references therein. A classical approach to resolve second order problems consists of a reduction to the first order and a use of the known results. Generally, this is made possible thanks to the use of fixed point theory. This was obtained in a recent paper [21] with strong conditions: the sets are contained in a strong compact, moreover only the particular case of a single-valued perturbation satisfying the linear growth condition has been considered.

In this paper, we present a new approach for solving second order sweeping process with set-valued perturbation in the finite dimensional setting: it consists in a reduction of the problem to a first order perturbed sweeping process and a use of the known results in this case without use of fixed point theory nor any compactness condition. Furthermore the perturbation is not necessary bounded or satisfying the linear growth condition. The paper is organized as follows. In Section 2, we recall notation and preliminaries needed throughout the paper. Section 3 is devoted to the reduction method for the existence of solutions to our problem in the finite dimensional setting. In Section 4, we give some applications to quasi-variational inequalities.

2. Preliminaries

Throughout this paper, H is a Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. The closed unit ball of H will be denoted by $\mathbf{B}$. If A is a subset of H, $\delta^*(x', A) = \sup_{y \in A} < x', y >$ stands for the support function of A at $x' \in H$. Let $T \geq 0$, the σ-algebra of Lebesgue measurable subsets of $[0,T]$ is denoted by $\mathcal{L}([0,T])$ and $\mathcal{B}(X)$ is the Borel tribe of any topological space X. $L^1_H([0,T], dt)$ (shortly $\mathrm{L}^1_H(0,T)$) is the Banach space of Lebesgue-Bochner integrable functions $f : [0,T] \to H$. A mapping $u : [0,T] \to H$ is *absolutely continuous* if there is a function $\dot{u} \in \mathrm{L}^1_H(0,T)$ such that $u(t) = u(0) + \int_0^t \dot{u}(s)\, ds, \ \forall t \in [0,T]$. If X is a topological space, $\mathcal{C}_X(H)$ is the space of continuous mappings $u : X \to H$ equipped with the norm of uniform convergence. A set-valued mapping $F : [0,T] \times H \rightharpoonup H$ is said to be upper semicontinuous if, for any open subset $\mathcal{V} \subset H$, the set $\{x \in H \ : \ F(x) \subset \mathcal{V}\}$ is open in H. F is said to be scalarly upper semicontinuous or upper hemicontinuous if, for any $y \in H$, the real-valued function $x \mapsto \delta^*(y, F(x))$ is upper semicontinuous. We refer to [12] for measurable set-valued mappings and convex analysis.

For a given $r \in]0, +\infty]$, a nonempty subset S of a Hilbert space H is r-prox-regular or equivalently r-proximally smooth ([27], [15]) if and only if every nonzero proximal normal to S can be realized by an r-ball. This is equivalent to say that for every $\bar{x} \in S$, and for every $v \neq 0, \ v \in N^p(S; \bar{x}) = N^p_S(\bar{x})$,

$$\left\langle \frac{v}{||v||}, x' - \bar{x} \right\rangle \leq \frac{1}{2r} ||x' - \bar{x}||^2$$

for all $x' \in S$ where $N^p_S(\bar{x})$ is the proximal normal cone of S at the point $\bar{x} \in S$ defined by

$$N^p_S(\bar{x}) := \{\xi \in H : \exists \rho > 0, \bar{x} \in \ \mathrm{Proj}_S(\bar{x} + \rho \xi)\}$$

where the projection on S is defined by $Proj_S(u) = \{y \in S : \quad d(u,S) = \|u - y\|\}$. As usual, we make the convention $\frac{1}{r} = 0$ for $r = +\infty$ and recall that for $r = +\infty$, the

r-proximal regularity of S is equivalent to the convexity of S. Let $f : H \to \mathbf{R} \cup \{+\infty\}$ a proper function and $\bar{x} \in dom f := \{x \in H \mid f(x) < +\infty\}$, the *proximal subdifferential* of f at $\bar{x}$ is the set $\partial^p f(\bar{x})$ of all elements $v \in H$ for which there exist $\delta > 0$ and $\rho > 0$ such that

$$f(y) \geq f(\bar{x}) + \langle v, y - \bar{x} \rangle - \rho ||y - \bar{x}||^2 \text{ for all } y \in \overline{B}_H(\bar{x}, \delta).$$

Given a nonempty closed set S and given a point $\bar{x} \in S$, the *Clarke normal cone* $N_S(\bar{x})$ to S at $\bar{x}$ defined by

$$N_S(\bar{x}) = \mathrm{cl}_\omega(\mathbf{R}_+ \partial\, d(\bar{x}, S))$$

where cl_ω denotes the closure with respect to the weak topology of H. With the definition of Clarke normal cones to nonempty closed sets in hand, the *Clarke subdifferential* $\partial f(\bar{x})$ of f at a point $\bar{x}$ (where f is finite) can be defined in terms of Clarke normal cone to the epigraph of the function by

$$\partial f(\bar{x}) := \{v \in H : (v, -1) \in N_{\mathrm{epi}(f)}((\bar{x}, f(\bar{x})))\},$$

where $\mathrm{epi}(f)$ denotes the epigraph of f, that is, $\mathrm{epi}(f) = \{(\bar{x}, \lambda) \in H \times \mathbf{R} : f(\bar{x}) \leq \lambda\}$. Further

$$\partial d(\bar{x}, S) \subset N_S(\bar{x}) \cap \mathbf{B} \quad \text{for all } \bar{x} \in S. \tag{2.1}$$

Let C, C' be two subsets of H, we denote by

$$\mathcal{H}(C, C') := \max\left\{ \sup_{u \in C} d(u, C'), \sup_{v \in C'} d(v, C) \right\}$$

the Hausdorff distance between C and C' and, for $r > 0$, $U_r(C)$ (respectively, $E_r(C)$) the open tube around the set C (respectively, the open enlargement of the set C), that is,

$$U_r(C) := \{v \in H : 0 < d(v, C) < r\},$$

respectively,

$$E_r(C) := \{v \in H : d(v, C) < r\}.$$

The following proposition provides some properties of the proximal and Clarke subdifferentials of the function distance $d(\cdot, C)$ when the set C is r-prox regular. It also summarizes some important consequences of the prox-regularity property which will be needed in the sequel of the paper. For the proof of these results we refer the reader to [6], [25].

Proposition 2.1. *Let C be a nonempty closed subset in the Hilbert space H and let $r > 0$. If the subset C is r-prox regular, then the following hold:*

a) *The projection mapping P_C is well defined and continuous over $E_r(C)$; hence, in particular, $\mathrm{P}_C(v)$ exists for all $v \in E_r(C)$;*
b) *For any $v \in U_r(C) \setminus C$ and $y = \mathrm{P}_C(v)$ one has $y \in \mathrm{Proj}_C\left(y + r\frac{v-y}{\|v-y\|}\right)$;*
c) *The Clarke and proximal subdifferentials of $d(\cdot, C)$ coincide at all points $v \in E_r(C)$;*
d) *The Clarke and proximal normal cone to C coincide at all points $u \in C$ and $\partial^p d(x, C) = N_C^p(x) \cap \mathbf{B}$;*
e) *Let $C : [0, T] \times H \rightharpoonup H$ be r-prox regular with closed values satisfying*

$$|d(x, C(t, u)) - d(y, C(s, v))| \leq ||x - y|| + V(t) - V(s) + L||u - v||$$

for all u, x, v, y in H and for all $s \leq t$ in $[0, T]$, where $V : [0, T] \to \mathbf{R}^+$ is a nondecreasing absolutely continuous function and L is a positive constant. Then the convex weakly compact valued mapping $(t, x, y) \to \partial^p d(y, C(t, x))$ satisfies the upper semicontinuity property: Let (t_n, x_n) be a sequence in $[0, T] \times H$ converging to some $(t, x) \in [0, T] \times H$, and (y_n) be a sequence in H with $y_n \in C(t_n, x_n)$ for all n, converging to $y \in C(t, x)$, then, for any $z \in H$,

$$\limsup_n \delta^*(z, \partial^p d(y_n, C(t_n, x_n))) \leq \delta^*(z, \partial^p d(y, C(t, x))).$$

3. Set-valued Unbounded Perturbation

In this section, we study the second-order perturbed differential inclusion governed by the state-dependent sweeping process in finite dimensional setting: let $H = \mathbb{R}^d$, given two points a and b in $\mathbb{R}^d$, a moving set $C : [0,T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with *nonempty closed values* and a set-valued perturbation $G : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with *nonempty closed convex values*, consider the following problem

$$\begin{cases} -\dot{v}(t) \in N_{C(t,u(t))}(v(t)) + G(t, v(t), u(t)) & \text{a.e. } t \in [0,T], \\ v(t) \in C(t, u(t)) & \forall t \in [0,T], \\ u(t) = b + \int_0^t v(s)ds & \forall t \in [0,T], \\ v(t) = a + \int_0^t \dot{v}(s)ds & \forall t \in [0,T]. \end{cases} \qquad (\mathcal{P})$$

If $(u(\cdot); v(\cdot))$ is a solution of the above differential inclusion, then $u(\cdot)$ is a solution of the second order differential inclusion

$$\begin{cases} -\ddot{u}(t) \in N_{C(t,u(t))}(\dot{u}(t)) + G(t, \dot{u}(t), u(t)) & \text{a.e. } t \in [0,T], \\ \dot{u}(t) \in C(t, u(t)) & \forall t \in [0,T], \\ \dot{u}(0) = a \text{ and } u(0) = b. \end{cases}$$

Let assume the following assumptions

($\mathcal{A}_1$) There is some constant $r > 0$ such that, for each $t \in [0,T]$ and each $u \in \mathbb{R}^d$, the set $C(t,u)$ is r-prox regular.

($\mathcal{A}_2$) There are constants $k_1 > 0$, $k_2 \in]0,1[$ such that, for all $s, t \in [0,T]$ and $x, y, u, v \in \mathbb{R}^d$

$$\big| d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big) \big| \leq \|x - y\| + k_1|t - s| + k_2\|u - v\|.$$

In order to present the reduction approach for solving $(\mathcal{P})$ in the finite dimensional setting, let begin by the following weaker variant of existence result for first order sweeping process with unbounded perturbation proved in ([24], Theorem 3).

Theorem 3.1. *Assume that* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ *hold and let* $F : [0,T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ *be a set-valued mapping with nonempty closed convex values such that*

($\mathcal{A}_3$) *F is scalarly upper semicontinuous, such that for some real $\alpha > 0$,*

$$d(0, F(t,u)) \leq \frac{\alpha}{2}, \quad \textit{for all } t \in [0,T] \textit{ and } u \in \mathbb{R}^d \textit{ with } u \in C(t,u) \cap \frac{\alpha}{2}\mathbf{B}.$$

Then, for any $u_0 \in \mathbb{R}^d$ with $u_0 \in C(0, u_0)$, there exists a Lipschitz continuous mapping $u : [0,T] \to \mathbb{R}^d$ such that

$$\begin{cases} \dot{u}(t) \in -N_{C(t,u(t))}\big(u(t)\big) + F(t, u(t)) & \text{a.e.} \quad t \in [0,T], \\ u(t) \in C(t, u(t)) & \forall t \in [0,T], \\ u(t) = u_0 + \int_0^t \dot{u}(s)ds & \forall t \in [0,T] \end{cases}$$

that is, $u(\cdot)$ is solution to the differential inclusion with

$$\|\dot{u}(t)\| \leq \frac{k_1 + 2\alpha}{1 - k_2} \quad \textit{a.e. } t \in [0,T].$$

Proof. The proof is identical to that of Theorem 3 in [24], it is enough to see that the elements of the sequence of approximate solutions, constructed via the Moreau's catching-up algorithm, are bounded, therefore our condition $(\mathcal{A}_3)$ is more suitable (and weaker) than their condition on the perturbation, that is :

The set-valued mapping F is scalarly upper semicontinuous, such that for some real $\alpha > 0$,

$$d(0, F(t,u)) \leq \alpha, \quad \text{for all } t \in [0,T] \text{ and } u \in C(t,u).$$

In addition, in finite dimension, no compactness assumption is required. □

Now, we are able to establish the reduction of the second order differential inclusion to a first order.

Theorem 3.2. *Assume that C satisfies the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ and that G satisfies the following assertion:*

$(\mathcal{A}_3')$ *G is scalarly upper semicontinuous, such that for some real $\alpha > 0$,*

$$d(0, G(t,v,u)) \leq \frac{\alpha}{2}, \ \forall t \in [0,T] \ \text{and} \ \ (v,u) \in \frac{\alpha}{2}(\mathbf{B} \times \mathbf{B}) \ \text{with} \ v \in C(t,u).$$

Then, for any $b \in \mathbb{R}^d$ and for every $a \in C(0,b)$, there exists a mapping $u : [0,T] \to \mathbb{R}^d$ with Lipschitz continuous derivative $\dot{u}$, which is a solution of the problem $(\mathcal{P})$ and satisfies

$$\|\dot{u}(t)\| \leq \frac{k_1 + 2\alpha}{1 - k_2} \quad \text{a.e.} \ \ t \in [0,T]$$

and

$$\|\ddot{u}(t)\| \leq \frac{k_1 + 2\alpha}{1 - k_2} \quad \text{a.e.} \quad t \in [0,T].$$

Proof. . Define the set-valued mappings $Q : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d \times \mathbb{R}^d$ by

$$Q(t,x,y) = G(t,x,y) \times \{-x\},$$

and $S : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \rightharpoonup \mathbb{R}^d \times \mathbb{R}^d$ by

$$S(t,x,y) = C(t,y) \times \mathbb{R}^d.$$

Then $Q(\cdot,\cdot,\cdot)$ is convex-valued, scalarly upper semicontinuous, and the set $S(t,x,y)$ is r-prox regular.
The set-valued mapping Q satisfies the hypothesis $(\mathcal{A}_3)$. Indeed, for any $\eta = (x,y) \in \frac{\alpha}{2}(\mathbf{B} \times \mathbf{B})$ with $(x,y) \in S(t,x,y)$, hence $x \in C(t,y)$, then $\eta \in C(t,y) \times \mathbb{R}^d = S(t,\eta)$, by $(\mathcal{A}_3')$ one has

$$d(0, Q(t,\eta)) \leq d(0, G(t,x,y)) + d(0,-x) \leq \alpha.$$

For all $\xi = (u_1,u_2), \eta = (v_1,v_2), X = (x_1,x_2)$ and $Y = (y_1,y_2)$ in $\mathbb{R}^d \times \mathbb{R}^d$ and $t,s \in [0,T]$, we have

$$\begin{aligned} &|d(\xi, S(t,X)) - d(\eta, S(s,Y))| = \\ &\Big| d(u_1, C(t,x_2)) + d(u_2, \mathbb{R}^d) - \big(d(v_1, C(s,y_2)) + d(v_2,\mathbb{R}^d)\big) \Big| \\ &= |d(u_1, C(t,x_2)) - d(v_1, C(s,y_2))| \\ &\leq \|u_1 - v_1\| + k_1|t-s| + k_2\|x_2 - y_2\| \\ &\leq \|\xi - \eta\| + k_1|t-s| + k_2\|X - Y\|, \end{aligned}$$

so the set $S(t,x,y)$ moves in a Lipschitz-continuous way with respect to the Hausdorff distance, thus $(\mathcal{A}_2)$ is verified.

Further, for $\xi(t) = (v(t),u(t))$, one has $Q(t,\xi(t)) = G(t,v(t),u(t)) \times \{-v(t)\}$ and $S(t,\xi(t)) = C(t,u(t)) \times \mathbb{R}^d$. Thanks to Theorem 3.1, the following differential inclusion

$$\begin{cases} -\dot{\xi}(t) \in N_{S(t,\xi(t))}\big(\xi(t)\big) + Q(t,\xi(t)) & \text{a.e. } t \in [0,T], \\ \xi(0) = (a,b) \in S(0,\xi(0)). \end{cases}$$

has a Lipschitz continuous solution $\xi : [0,T] \to \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\xi(t) = \xi(0) + \int_0^t \dot{\xi}(s)ds = (a,b) + \int_0^t (\dot{v}(s), \dot{u}(s))ds = \Big(a + \int_0^t \dot{v}(s)ds \ , b + \int_0^t \dot{u}(s)ds\Big)$$

with $\|\dot{\xi}(t)\| \leq \frac{k_1 + 2\alpha}{1-k_2}$ a.e. $t \in [0,T]$. Thus, $(v(t),u(t))$ is solution of $(\mathcal{P})$, the proof is then complete. □

4. APPLICATIONS

The Reduction approach for second order perturbed state-dependent sweeping process makes it possible to study the antiplane frictional contact problem, the friction being modeled with Tresca's law, the classical model of the process is the following:

Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{array}{lll} div(\theta\nabla\dot{u} + \mu\nabla u) + f_0 = 0 & \text{in} & \Omega \times (0,T), \\ u = 0 & \text{on} & \Gamma_1 \times (0,T), \\ \theta\partial_\nu\dot{u} \ + \ \mu\partial_\nu u = f_2 & \text{on} & \Gamma_2 \times (0,T), \\ \left.\begin{array}{l} |\theta\partial_\nu\dot{u} \ + \ \mu\partial_\nu u| \leq g \\ \theta\partial_\nu\dot{u} \ + \ \mu\partial_\nu u = -g\frac{\dot{u}}{|\dot{u}|} \ if \ \dot{u} \neq 0, \end{array}\right\} & \text{on} & \Gamma_3 \times (0,T). \\ u(0) = u_0 & \text{in} & \Omega \end{array}$$

We refer to [22] for the physical interpretation of the problem. [22] obtained the following variational formulation

Find $u : I := [0,T] \to \mathbb{R}^d$ such that $\dot{u}(t) \in \Gamma$ a.e. $t \in I$ and $\forall v \in \Gamma$

$$a(u(t), v - \dot{u}(t)) + b(\dot{u}(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \geq < f(t,u(t)), v - \dot{u}(t) >$$

$$u(0) = u_0 \in \mathbb{R}^d,$$

where $a(\cdot,\cdot), b(\cdot,\cdot) : H \times H \to \mathbb{R}$ are two real continuous bilinear and symmetric forms. See also [2] for a similar problem. Following [2], one proves the equivalence between this variational inequality and the perturbed state-dependent sweeping process.

For evolution quasi-variational inequalities of second order type models, let consider the following variational formulation for quasistatic evolution problems with friction (also for problems arising in micro-mechanical damage models for iron materials with memory to describe the evolution of the plastic strain in presence of small damages): find two absolutely continuous mapping $u, \ v : I \to \mathbb{R}^d$ such that for all $t \in I, \ v(t) \in C(t, u(t))$ and $v(\cdot) = \dot{u}(\cdot)$ a.e. in I and

$$< \dot{v}(t) + A(u(t), v(t)), z - v(t) > \geq < f(t), z - v(t) >$$

for all $z \in C(t, u(t))$.
Under suitable assumptions on A and f, and taking $G(t,u,v) = \{A(u,v) - f(t)\}$ for all $(t,u,v) \in I \times \mathbb{R}^d \times \mathbb{R}^d$, it was shown in [3] that this problem is equivalent to our problem $(\mathcal{P})$. For other examples, we refer to [14], [19] and [22].

5. CONCLUSION

In this study, a new approach to solving second order perturbed state-dependent sweeping process is defined. This method consists in a reduction of the second order problem to a first order one, in order to obtain the existence of solution for the considered problem. This approach simplifies standard methods for solving problems governed by the sweeping process, that is based on the Moreau's catching-up algorithm (see [1] and the references therein). Furthermore, the reduction approach, in general, make use of fixed point theory as in [8] and [21]. In this study, the existence is obtained without use of fixed point theory nor any compactness condition, and the perturbation isn't necessary bounded or satisfying the linear growth condition.

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