CREAT. MATH. INFORM. **29** (2020), No. 2, 121 - 129
Online version at https://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

On L_3-affine surfaces

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ABSTRACT. A Riemannian manifold (M, g) is said to be an L_3-space if its Ricci tensor is cyclic parallel. This definition extends easily to the affine case. Here we investigate the torsion free affine surfaces (M, ∇) to be L_3-spaces and we study locally homogeneous L_3-affine surfaces.

1. INTRODUCTION

One of the most extensively studied objects in mathematics and physics are Einstein manifolds (see for example [3]), i.e. manifolds whose Ricci tensor is a constant multiple of the metric tensor. In his work [10] A. Gray defined a condition which generalize the concept of an Einstein manifold. Riemannian manifold (M, g) admitting a cyclic parallel Ricci tensor, that is

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0$$

where ∇ denotes the Levi-Civita connection of the metric g and X, Y, Z are arbitrary vector fields on M is called Einstein-like of class $\mathcal{A}$. It is noted that the above condition is equivalent to

$$(\nabla_X Ric)(X, X) = 0,$$

for any vector field $X \in \mathfrak{X}(M)$. If the Ricci tensor is a Codazzi tensor, i.e.,

$$(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z),$$

then (M, g) is called Einstein-like of class $\mathcal{B}$. Manifolds having a parallel Ricci tensor, i.e.,

$$(\nabla_X Ric)(Y, Z) = 0$$

are called Einstein-like of class $\mathcal{P}$. It is obvious that if the Ricci tensor of is parallel, then it is cyclic parallel. But, the converse statement is not true. Einstein-like manifolds admitting different curvature conditions are investigated by G. Calvaruso in [5]. Einstein-like manifolds of dimension 3 are studied in [2, 4] where as of dimension 4 are considered in [22]. An interesting study in [12] showed that Einstein-like Generalized Robertson-Walker space-times are perfect fluid space-times except one class of Grays decomposition.

The purpose of this paper is to investigate affine manifolds under cyclic parallel Ricci tensor condition. The paper is organized as follows. In Section 2, we recall some basic definitions and geometric concepts, namely, torsion, curvature and Ricci tensor on an affine manifold. In Section 3, we study the cyclic parallelism of the Ricci tensor for an affine connection on a two dimensional affine manifold also called L_3-affine surfaces. We establish geometric configurations of affine manifolds admitting a cyclic parallel Ricci tensor. In Section 4, characterization of locally homogeneous L_3-affine surfaces are given.

Received: 07.01.2020. In revised form: 17.05.2020. Accepted: 24.05.2020
2010 *Mathematics Subject Classification.* 53B05, 53C21.
Key words and phrases. *Affine connection, cyclic parallel Ricci tensor, Einstein-like manifolds, L_3-spaces.*
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2. Preliminaries

Let M be a n-dimensional smooth manifold and ∇ be an affine connection on M. Let us consider a system of coordinates $(x_1, x_2, \dots, x_n)$ in a neighborhood $\mathcal{U}$ of a point p in M. In $\mathcal{U}$, the affine connection is given by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k, \tag{2.1}$$

where $\{\partial_i = \frac{\partial}{\partial x_i}\}_{1\leq i \leq n}$ is a basis of the tangent space T_pM and the functions $\Gamma_{ij}^k(i,j,k = 1,2,3,\dots,n)$ are called the coefficients of the affine connection. The pair (M, ∇) shall be called affine manifold. We define a few tensors fields associated to a given affine connection ∇. The torsion tensor field T is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{2.2}$$

for any vector fields X and Y on M. The components of the torsion tensor T in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k. \tag{2.3}$$

If the torsion tensor of a given affine connection ∇ vanishes, we say that ∇ is torsion-free. The curvature tensor field $\mathcal{R}$ is defined by

$$\mathcal{R}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.4}$$

for any vector fields X, Y and Z on M. The components of the curvature tensor $\mathcal{R}$ in local coordinates are

$$\mathcal{R}(\partial_k, \partial_l)\partial_j = \sum_i R_{jkl}^i \partial_i. \tag{2.5}$$

We shall assume that ∇ is torsion-free. If $\mathcal{R} = 0$ on M, we say that ∇ is flat affine connection. It is know that ∇ is flat if and only if around each point there exist a local coordinates system such that $\Gamma_{ij}^k = 0$, for all i, j and k. We define the Ricci tensor Ric by

$$Ric(X,Y) = \text{trace}\{Z \mapsto \mathcal{R}(Z,X)Y\}. \tag{2.6}$$

The components of the Ricci tensor in local coordinates are given by

$$Ric(\partial_j, \partial_k) = \sum_i R_{kij}^i. \tag{2.7}$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is, $Ric(Y,Z) = Ric(Z,Y)$. But this property is not true for an arbitrary affine connection with torsion-free. In fact, the property is closely related to the concept of parallel volume element (see [13] for more details).
In 2-dimensional manifold M, the curvature tensor $\mathcal{R}$ and the Ricci tensor Ric are related by

$$\mathcal{R}(X,Y)Z = Ric(Y,Z)X - Ric(X,Z)Y. \tag{2.8}$$

The covariant derivative of the curvature tensor $\mathcal{R}$ is given by

$$(\nabla_X\mathcal{R})(Y,Z)W = (\nabla_X Ric)(Z,W)Y - (\nabla_X Ric)(Y,W)Z, \tag{2.9}$$

where the covariant derivative of the Ricci tensor Ric is defined as

$$\begin{aligned}(\nabla_X Ric)(Z,W) &= X(Ric(Z,W)) - Ric(\nabla_X Z, W)\\ &\quad - Ric(Z, \nabla_X W).\end{aligned} \tag{2.10}$$

3. L_3-SPACES

Definition 3.1. [11] An affine manifold (M, ∇) is said to be an L_3-space if its Ricci tensor Ric is cyclic parallel, that is

$$(\nabla_X Ric)(X, X) = 0, \tag{3.11}$$

for any vector field X tangent to M or, equivalently, if

$$\mathfrak{S}_{X,Y,Z}(\nabla_X Ric)(Y, Z) = 0,$$

for any vector fields X, Y, Z tangent to M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y and Z.

For $X = \sum_i \alpha_i \partial_i$, it is easy to show that

$$(\nabla_X Ric)(X, X) = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k (\nabla_i Ric)_{jk}. \tag{3.12}$$

Lemma 3.1. *In particular for $n = 2$, the equations expressing the L_3-condition are:*

$$\begin{array}{rcl} (\nabla_{\partial_1} Ric)(\partial_1, \partial_1) = 0; \quad (\nabla_{\partial_2} Ric)(\partial_2, \partial_2) & = & 0 \\ (\nabla_{\partial_1} Ric)(\partial_1, \partial_2) + (\nabla_{\partial_1} Ric)(\partial_2, \partial_1) + (\nabla_{\partial_2} Ric)(\partial_1, \partial_1) & = & 0 \\ (\nabla_{\partial_1} Ric)(\partial_2, \partial_2) + (\nabla_{\partial_2} Ric)(\partial_2, \partial_1) + (\nabla_{\partial_2} Ric)(\partial_1, \partial_2) & = & 0. \end{array}$$

Let Σ be a smooth surface and ∇ be a torsion-free affine connection. By (2.1), we have

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \text{ for } i, j, k = 1, 2, \tag{3.13}$$

where $\Gamma_{ij}^k = \Gamma_{ij}^k(x_1, x_2)$. The components of the curvature tensor $\mathcal{R}$ are given by

$$\mathcal{R}(\partial_1, \partial_2)\partial_1 = a\partial_1 + b\partial_2 \text{ and } \mathcal{R}(\partial_1, \partial_2)\partial_2 = c\partial_1 + d\partial_2,$$

where a, b, c and d are given by

$$\begin{array}{rcl} a & = & \partial_1 \Gamma_{12}^1 - \partial_2 \Gamma_{11}^1 + \Gamma_{12}^1 f_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^1, \\ b & = & \partial_1 \Gamma_{12}^2 - \partial_2 \Gamma_{11}^2 + \Gamma_{11}^2 \Gamma_{12}^1 + \Gamma_{12}^2 \Gamma_{12}^2 - \Gamma_{11}^1 \Gamma_{12}^2 - \Gamma_{11}^2 \Gamma_{22}^2, \\ c & = & \partial_1 \Gamma_{22}^1 - \partial_2 \Gamma_{12}^1 + \Gamma_{11}^1 \Gamma_{22}^1 + \Gamma_{12}^1 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{12}^1 - \Gamma_{12}^2 \Gamma_{22}^1, \\ d & = & \partial_1 \Gamma_{22}^2 - \partial_2 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^1 - \Gamma_{12}^1 \Gamma_{12}^2. \end{array}$$

From (2.8), the components of the Ricci tensor are given by

$$\begin{array}{rcl} Ric(\partial_1, \partial_1) & = & -b, \quad Ric(\partial_1, \partial_2) = -d, \\ Ric(\partial_2, \partial_1) & = & a, \quad Ric(\partial_2, \partial_2) = c. \end{array} \tag{3.14}$$

Proposition 3.1. *An affine surface (Σ, ∇) is L_3-space if and only if the coefficients of the torsion-free affine connection ∇ are the solutions of the following system of partial differential equations:*

$$\begin{array}{rcl} \partial_1 b - 2b\Gamma_{11}^1 + (d-a)\Gamma_{11}^2 = 0, \quad \partial_2 c - 2c\Gamma_{22}^2 + (d-a)\Gamma_{22}^1 & = & 0, \\ \partial_2 a - \partial_1 c - \partial_2 d + 2b\Gamma_{22}^1 - 4c\Gamma_{12}^2 + (d-a)(2\Gamma_{12}^1 + \Gamma_{12}^2) & = & 0, \\ \partial_1 a - \partial_2 b - \partial_1 d + 4b\Gamma_{12}^1 - 2c\Gamma_{11}^2 + (d-a)(\Gamma_{11}^1 + 2\Gamma_{12}^2) & = & 0. \end{array} \tag{3.15}$$

Proof. From a straightforward calculation using (2.10), the components of the covariant derivative of the Ricci tensor are given by

$$\begin{array}{rcl}
(\nabla_{\partial_1} Ric)(\partial_1, \partial_1) & = & -\partial_1 b + 2b\Gamma_{11}^1 + (d-a)\Gamma_{11}^2; \\
(\nabla_{\partial_1} Ric)(\partial_1, \partial_2) & = & -\partial_1 d + d(\Gamma_{11}^1 + \Gamma_{12}^2) + b\Gamma_{12}^1 - c\Gamma_{11}^2; \\
(\nabla_{\partial_1} Ric)(\partial_2, \partial_1) & = & \partial_1 a - a(\Gamma_{11}^1 + \Gamma_{12}^2) + b\Gamma_{12}^1 - c\Gamma_{11}^2; \\
(\nabla_{\partial_1} Ric)(\partial_2, \partial_2) & = & \partial_1 c + (d-a)\Gamma_{12}^1 - 2c\Gamma_{12}^2; \\
(\nabla_{\partial_2} Ric)(\partial_1, \partial_1) & = & -\partial_2 b + 2b\Gamma_{12}^1 + (d-a)\Gamma_{12}^2; \\
(\nabla_{\partial_2} Ric)(\partial_1, \partial_2) & = & -\partial_2 d + d(\Gamma_{12}^1 + \Gamma_{22}^2) + b\Gamma_{22}^1 - c\Gamma_{12}^2; \\
(\nabla_{\partial_2} Ric)(\partial_2, \partial_1) & = & \partial_2 a - a(\Gamma_{12}^1 + \Gamma_{22}^1) + b\Gamma_{22}^1 - c\Gamma_{12}^2; \\
(\nabla_{\partial_2} Ric)(\partial_2, \partial_2) & = & \partial_2 c + (d-a)\Gamma_{22}^1 - 2c\Gamma_{22}^2.
\end{array}$$

From Lemma 3.1 the proof is complete. □

Corollary 3.1. *Let ∇ be the torsion-free affine connection on $\mathbb{R}^2$ defined by $\nabla_{\partial_1}\partial_1 = \Gamma_{11}^1(x_1, x_2)\partial_1$, $\nabla_{\partial_1}\partial_2 = \Gamma_{12}^1(x_1, x_2)\partial_1$ and $\nabla_{\partial_2}\partial_2 = \Gamma_{22}^2(x_1, x_2)\partial_2$. Then $(\mathbb{R}^2, \nabla)$ is a L_3-space if and only if the functions $\Gamma_{11}^1, \Gamma_{12}^1$ and Γ_{22}^2 satisfy the following partial differential equations: $\partial_2 c - 2c\Gamma_{22}^2 = 0$, $\partial_1 a - \partial_1 d + (d-a)\Gamma_{11}^1 = 0$ and $\partial_2 a + \partial_1 c - \partial_2 d + (d-a)\Gamma_{22}^2 = 0$.*

To support this, we have the following example. Let us consider the torsion-free connection ∇ on $\mathbb{R}^2$ with the only non-zero coefficient functions given by :

$$\nabla_{\partial_1}\partial_1 = \alpha(x_1 + x_2)\partial_1, \quad \nabla_{\partial_1}\partial_2 = \beta x_1(x_1 + x_2 + 1)\partial_1, \quad \nabla_{\partial_2}\partial_2 = \alpha(x_1 + x_2)\partial_2$$

where $\alpha, \beta \in \mathbb{R}$. It is easy to check that $(\mathbb{R}^2, \nabla)$ is an L_3-space.

Corollary 3.2. *Let ∇ be the torsion-free affine connection on $\mathbb{R}^2$ defined by $\nabla_{\partial_1}\partial_1 = \Gamma_{11}^2\partial_2$ and $\nabla_{\partial_2}\partial_2 = \Gamma_{22}^1\partial_1$. Then $(\mathbb{R}^2, \nabla)$ is a L_3-space if and only if the functions $\Gamma_{11}^2 = \Gamma_{11}^2(x_1, x_2)$ and $\Gamma_{22}^1 = \Gamma_{22}^1(x_1, x_2)$ satisfy the following partial differential equations: $\partial_1 b - (d-a)\Gamma_{11}^2 = 0, \partial_2 c + (d-a)\Gamma_{22}^1 = 0$, $\partial_1 a - \partial_2 b - \partial_1 d - 2c\Gamma_{11}^2 = 0, \partial_2 a + \partial_1 c - \partial_2 d + 2b\Gamma_{22}^1 = 0$.*

For example, let us consider now on $\mathbb{R}^2$ the torsion-free affine connection with only the non-zero coefficient functions givent by: $\nabla_{\partial_1}\partial_1 = x_2\partial_2$ and $\nabla_{\partial_2}\partial_2 = x_1(1 + x_2)\partial_1$. It is easy to check that $(\mathbb{R}^2, \nabla)$ is an L_3-space.

The cyclic parallelism of the Ricci tensor is sometimes called the "*First Ledger condition*" [17]. In [19], for instance, the author proved that a smooth Riemannian manifold satisfying the first Ledger condition is real analytic. Tod in [20] used the same condition to characterize the four-dimensional Kähler manifolds which are not Einstein. Also, Pedersen and Tod [17] showed if a smooth Riemannian three-manifold (M, g) is an L_3-space, then it is a locally homogeneous D'Atri space. The manifolds with cyclic parallel Ricci tensor, known as $\mathcal{A}$-manifolds, are well-developed in Riemannian geometry (see [23] and references therein). In affine setting, Diallo and Massamba proved that affine connections ∇ with cyclic parallel Ricci tensor are equivalently characterized both by being the so-called affine Szabó connections [6, 7, 8]

4. Locally homogeneous L_3-spaces in dimension two

Homogeneity is one of the fundamental notions in differential geometry. In this section we consider the homogeneity of manifolds with affine connections in dimension two. This homogeneity means that for each two points of a manifold there is an affine transformation which sends one point into another. In particular, we consider a local version of the homogeneity, that is, the affine transformations are given only locally, i.e., from a neighborhood onto a neighborhood. Note that the concept of homogeneity were first studied by Singer [18] on a Riemannian manifold (M, g). The first kind of homogeneity means that, for every smooth Riemannian manifold (M, g), its group of isometries $I(M)$ is acting transitively on M. Many years later, Opozda worked out an affine version of Singers theory in [14] and [15]. A smooth connection ∇ on M is locally homogeneous if and only if it admits, in neighborhoods of each point $p \in M$; at least two linearly independant affine Killing vectors fields. An affine Killing vector field X is characterized by the equation:

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0 \tag{4.16}$$

which has to be satisfied for arbitrary vectors fields Y, Z (see [16]). It is sufficient to satisfy (4.16) for the choices $(Y, Z) \in \{(\partial_1, \partial_1), (\partial_1, \partial_2), (\partial_2, \partial_1), (\partial_2, \partial_2)\}$. Moreover, we easily check from the basic identities for the torsion and the Lie brackets, that the choice $(Y, Z) = (\partial_1, \partial_2)$ gives the same conditions as the choice $(Y, Z) = (\partial_2, \partial_1)$.
In the sequel, let us express the vector field X in the form

$$X = F(x_1, x_2)\partial_1 + G(x_1, x_2)\partial_2.$$

Writing the formula (4.16) in local coordinates, we find that any affine Killing vector field X must satisfy six basic equations. We shall write these equations in the simplifed notation:

$$\begin{aligned}
\partial_{11}F + \Gamma_{11}^1\partial_1 F + \partial_1\Gamma_{11}^1 F - \Gamma_{11}^2\partial_2 F + \partial_2\Gamma_{11}^1 G + 2\Gamma_{12}^1\partial_1 G &= 0,\\
\partial_{11}G + 2\Gamma_{11}^2\partial_1 F + (2\Gamma_{12}^2 - \Gamma_{11}^1)\partial_1 G - \Gamma_{11}^2\partial_2 G + \partial_1\Gamma_{11}^2 F + \partial_2\Gamma_{11}^2 G &= 0,\\
\partial_{12}F + (\Gamma_{11}^1 - \Gamma_{12}^2)\partial_2 F + \Gamma_{22}^1\partial_1 G + \Gamma_{12}^1\partial_2 G + \partial_1\Gamma_{12}^1 F + \partial_2\Gamma_{12}^1 G &= 0,\\
\partial_{12}G + \Gamma_{12}^2\partial_1 F + \Gamma_{11}^2\partial_2 F + (\Gamma_{22}^2 - \Gamma_{11}^2)\partial_1 G + \partial_1\Gamma_{12}^2 F + \partial_2\Gamma_{12}^2 G &= 0,\\
\partial_{22}F - \Gamma_{22}^1\partial_1 F + (2\Gamma_{12}^1 - \Gamma_{22}^2)\partial_2 F + 2\Gamma_{22}^1\partial_2 G + \partial_1\Gamma_{22}^1 F + \partial_2\Gamma_{22}^1 G &= 0,\\
\partial_{22}G + 2\Gamma_{12}^2\partial_2 F - \Gamma_{22}^1\partial_1 G + \Gamma_{22}^2)\partial_2 G\partial_1\Gamma_{22}^2 F + \partial_2\Gamma_{22}^2 G &= 0.
\end{aligned}$$

The following result is the first classification of torsion free homogeneous connections on two dimensional manifolds.

Theorem 4.1. [16] *Let ∇ be a locally homogeneous torsion free affine connection on a two-dimensional manifold M. Then, in a neighborhood $\mathcal{U}(p)$ of each point $p \in M$, either ∇ is the Levi-Civita connection of the standard metric of the unit sphere or, there is a system (x_1, x_2) of local coordinates and constants a, b, c, d, e, f such that ∇ is expressed in $\mathcal{U}(p)$ by one of the following formulas:*

(1) Type A:

$$\nabla_{\partial_1}\partial_1 = a\partial_1 + b\partial_2, \quad \nabla_{\partial_1}\partial_2 = c\partial_1 + d\partial_2, \quad \nabla_{\partial_2}\partial_2 = e\partial_1 + f\partial_2.$$

(2) Type B:

$$\nabla_{\partial_1}\partial_1 = \frac{1}{x_1}(a\partial_1 + b\partial_2), \quad \nabla_{\partial_1}\partial_2 = \frac{1}{x_1}(c\partial_1 + d\partial_2), \quad \nabla_{\partial_2}\partial_2 = \frac{1}{x_1}(e\partial_1 + f\partial_2).$$

This result has been applied by many authors. Kowalski and Sekizawa [11] used it to examine Riemannian extensions of affine surfaces, Vanzurova [21] used it to study the metrizability of locally homogeneous affine surfaces, and Dusek [9] used it to study homogeneous geodesics. It plays a central role in the study of locally homogeneous connections with torsion of Arias-Marco and Kowalski [1].
Next, we characterize locally homogeneous torsion free affine connections which satisfy L_3-spaces condition on a two dimensional smooth manifold.

Theorem 4.2. *The affine locally homogeneous manifolds of Type A are L_3-spaces if and only the coefficients a, b, c, d, e and f satisfy the following:*

$$\begin{aligned} bc^2 + bde - acd - bcf &= 0 \\ bce - ade - cdf + d^2e &= 0 \\ abc + ad^2 - a^2d - abf + b^2e - bcd &= 0 \\ be^2 + c^2f - cf^2 - aef - cde + def &= 0. \end{aligned}$$

Proof. The components of the Ricci tensor are given by $Ric(\partial_1, \partial_1) = (ad - d^2 + bf - bc)$, $Ric(\partial_1, \partial_2) = (cd - be)$, $Ric(\partial_2, \partial_1) = (cd - be)$, $Ric(\partial_2, \partial_2) = (ae - de + cf - c^2)$. The Ricci tensor is symmetric. Then, the covariant derivatives of the Ricci tensor are given by

$$\begin{aligned} (\nabla_{\partial_1} Ric)(\partial_1, \partial_1) &= 2(abc + ad^2 - a^2d - abf + b^2e - bcd) \\ (\nabla_{\partial_1} Ric)(\partial_1, \partial_2) &= 2(bc^2 + bde - acd - bcf) \\ (\nabla_{\partial_1} Ric)(\partial_2, \partial_2) &= 2(bce - ade - cdf + d^2e) \\ (\nabla_{\partial_2} Ric)(\partial_1, \partial_1) &= 2(bc^2 + bde - acd - bcf) \\ (\nabla_{\partial_2} Ric)(\partial_1, \partial_2) &= 2(bce - ade - cdf + d^2e) \\ (\nabla_{\partial_2} Ric)(\partial_2, \partial_2) &= 2(be^2 + c^2f - cf^2 - aef - cde + def). \end{aligned}$$

By direct calculation from Lemma 3.1, we get the following:

$$\begin{aligned} bc^2 + bde - acd - bcf = 0; \quad bce - ade - cdf + d^2e &= 0; \\ abc + ad^2 - a^2d - abf + b^2e - bcd &= 0; \\ be^2 + c^2f - cf^2 - aef - cde + def &= 0. \end{aligned}$$

The proof is complete. □

Theorem 4.3. *The affine locally homogeneous manifolds of Type B with symmetric Ricci tensor are L_3-spaces if and only if the coefficients a, b, c, d, e and f satisfy*

$$\begin{aligned} 2abc + 3bc - d - 2ad - a^2d - bcd + d^2 + ad^2 + b^2e &= 0 \\ -2c^3 + ace - 2cde + be^2 &= 0 \\ 2c + ac + 4bc^2 - 2cd - 3acd + 3be + 3bde + 2bce &= 0 \\ 3c^2 + 3c^2d + e - ae + 3bce + 2de - 3ade + 3d^2e &= 0. \end{aligned}$$

Proof. The components of the Ricci tensor are given by

$$\begin{aligned} Ric(\partial_1, \partial_1) &= \frac{1}{x_1^2}[d + d(a - d) + b(f - c)], \quad Ric(\partial_1, \partial_2) = \frac{1}{x_1^2}(f + cd - be), \\ Ric(\partial_2, \partial_1) &= \frac{1}{x_1^2}(-c + cd - be), \quad Ric(\partial_2, \partial_2) = \frac{1}{x_1^2}[-e + e(a - d) + c(f - c)] \end{aligned}$$

and it is symmetric if and only if $f = -c$ holds. So we set $f = -c$. Then, the covariant derivatives of the Ricci tensor are given by

$$\begin{aligned}
(\nabla_{\partial_1} Ric)(\partial_1, \partial_1) &= \frac{2}{x_1^3}(2abc + 3bc - d - 2ad - a^2 d - bcd + d^2 + ad^2 + b^2 e) \\
(\nabla_{\partial_1} Ric)(\partial_1, \partial_2) &= \frac{1}{x_1^3}(2c + ac + 4bc^2 - 2cd - 2acd + 3be + 2bde) \\
(\nabla_{\partial_1} Ric)(\partial_2, \partial_2) &= \frac{2}{x_1^3}(3c^2 + c^2 d + e - ae + bce + 2de - ade + d^2 e) \\
(\nabla_{\partial_2} Ric)(\partial_1, \partial_1) &= \frac{2}{x_1^3}(2bc^2 - acd + bde) \\
(\nabla_{\partial_2} Ric)(\partial_1, \partial_2) &= \frac{2}{x_1^3}(c^2 d + bce - ade + d^2 e) \\
(\nabla_{\partial_2} Ric)(\partial_2, \partial_2) &= \frac{2}{x_1^3}(-2c^3 + ace - 2cde + be^2).
\end{aligned}$$

By straightforward calculation from Lemma 3.1, we get the following:

$$\begin{aligned}
2abc + 3bc - d - 2ad - a^2 d - bcd + d^2 + ad^2 + b^2 e &= 0 \\
-2c^3 + ace - 2cde + be^2 &= 0 \\
2c + ac + 6bc^2 - 2cd - 3acd + 3be + 3bde + 2bce &= 0 \\
3c^2 + 3c^2 d + e - ae + 3bce + 2de - 3ade + 3d^2 e &= 0.
\end{aligned}$$

The proof is complete. □

Next we generalize the Theorem 4.3 when the Ricci tensor is not symmetric.

Theorem 4.4. *The affine locally homogeneous manifolds of Type B with not symmetric Ricci tensor are L_3-spaces if and only if the coefficients a, b, c, d, e and f satisfy*

$$\begin{aligned}
2d - 3bc - 2d^2 + 4ad + 3bf - 2abc - 2ad^2 + 2a^2 d + 2abf + 2bcd - 2b^2 c = 0 \\
ce + ef + 2be^2 - 2cde - 2aef - 2cf^2 + 2c^2 f + 2def = 0 \\
-2f + 6be - 5cd - af - 6acd - 2bcf + 10bc^2 + 6bde - fd + 2c + ac = 0 \\
6bce - 6ade + 6d^2 e - f^2 - 6cdf + 4c^2 + 2e - 2ae - 3cf + 4de = 0
\end{aligned}$$

Proof. The components of the Ricci tensor are given by

$$\begin{aligned}
Ric(\partial_1, \partial_1) &= \frac{1}{x_1^2}[d + d(a-d) + b(f-c)], \quad Ric(\partial_1, \partial_2) = \frac{1}{x_1^2}(f + cd - be), \\
Ric(\partial_2, \partial_1) &= \frac{1}{x_1^2}(-c + cd - be), \quad Ric(\partial_2, \partial_2) = \frac{1}{x_1^2}[-e + e(a-d) + c(f-c)].
\end{aligned}$$

Here, the Ricci tensor is not symmetric. Then, the covariant derivatives of the Ricci tensor are given by

$$
\begin{aligned}
(\nabla_{\partial_1} Ric)(\partial_1,\partial_1) &= \frac{1}{x_1^3}\Big(-2d+3bc+2d^2-4ad-3bf+2abc \\
&\quad +2ad^2-2a^2d-2abf-2bcd+2b^2e\Big)\\
(\nabla_{\partial_1} Ric)(\partial_1,\partial_2) &= \frac{1}{x_1^3}\Big(-2f+3be-3cd-af-2acd-2bcf+2bc^2+2bde-fd\Big)\\
(\nabla_{\partial_1} Ric)(\partial_2,\partial_1) &= \frac{1}{x_1^3}\Big(2c-2cd+3be+2bc^2-2acd-2bcf+2bde+ac\Big)\\
(\nabla_{\partial_2} Ric)(\partial_1,\partial_1) &= \frac{1}{x_1^3}\Big(4bc^2-2acd+2bde\Big)\\
(\nabla_{\partial_2} Ric)(\partial_1,\partial_2) &= \frac{1}{x_1^3}\Big(-cf+2bce-2ade+2d^2e-f^2-2cdf\Big)\\
(\nabla_{\partial_1} Ric)(\partial_2,\partial_2) &= \frac{1}{x_1^3}\Big(2e-2ae-3cf+3c^2+4de-2ade-2cdf+2d^2e+bce\Big)\\
(\nabla_{\partial_2} Ric)(\partial_2,\partial_1) &= \frac{1}{x_1^3}\Big(2bce+2d^2e-2ade+cf-2cdf+c^2\Big)\\
(\nabla_{\partial_2} Ric)(\partial_2,\partial_2) &= \frac{1}{x_1^3}\Big(ef+2be^2-2cde-2aef-2cf^2+2c^2f+2def+ce\Big)
\end{aligned}
$$

By straightforward calculation from Lemma 3.1, we get the following:

$$
\begin{aligned}
2d-3bc-2d^2+4ad+3bf-2abc-2ad^2+2a^2d+2abf+2bcd-2b^2c=0\\
ce+ef+2be^2-2cde-2aef-2cf^2+2c^2f+2def=0\\
-2f+6be-5cd-af-6acd-2bcf+10bc^2+6bde-fd+2c+ac=0\\
6bce-6ade+6d^2e-f^2-6cdf+4c^2+2e-2ae-3cf+4de=0
\end{aligned}
$$

□

The locally homogeneous surfaces of Type A and Type B have quite different geometric properties. For instance, the Ricci tensor of any Type A surface is symmetric while this property can fail for a Type B surface. Thus the geometry of a Type B surface is not as rigid as that of a Type A surface. This difference in terms of geometric properties is also remarkable when those surfaces satisfy the L_3 condition (Theorems 4.2, 4.3 and 4.4).

Acknowledgments. The authors would like to thank the referee for his/her valuable suggestions and comments that helped them improve the paper.

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