

A relation between the Brocard and Miquel Angles

SÁNDOR NAGYDOBAI KISS

ABSTRACT. Let XYZ be an inscribed triangle in the reference triangle ABC ($X \in BC, Y \in CA, Z \in AB$) given by its absolute barycentric coordinates: $X = (0, 1 - \alpha, \alpha), Y = (\beta, 0, 1 - \beta), Z = (1 - \gamma, \gamma, 0)$, where α, β, γ are arbitrary real numbers. In this paper we deduce the following simple and beautiful formula between the Brocard and Miquel angles: $S_\omega = S_\theta + a^2\alpha + b^2\beta + c^2\gamma$, where ω is the Brocard and θ the Miquel angle of ABC .

1. PRELIMINARIES

In this paper we will use the barycentric coordinates [1] and the Conway triangle notations [2]. Given a reference triangle ABC whose sides are a, b and c and whose corresponding internal angles A, B and C , then the Conway triangle notations are

$$S = 2 \times \text{Area } ABC = bc \sin A = ca \sin B = ab \sin C \text{ and } S_\varphi = S \cot \varphi,$$

in particular $S_A = bc \cos A, S_B = ca \cos B, S_C = ab \cos C$. We will use also some important conditioned trigonometric identities:

$$\begin{aligned} b^2c^2 - S_A^2 &= c^2a^2 - S_B^2 = a^2b^2 - S_C^2 = S^2; \\ a^2S_A + S_{BC} &= b^2S_B + S_{CA} = c^2S_C + S_{AB} = S^2; \\ S_{BC} + S_{CA} + S_{AB} &= S^2; \\ a^2S_A + b^2S_B + c^2S_C &= 2S^2; \\ S_\omega &= \frac{a^2 + b^2 + c^2}{2} = S_A + S_B + S_C = a^2 + S_A = b^2 + S_B = c^2 + S_C; \\ b^2c^2 + c^2a^2 + a^2b^2 - S_\omega^2 &= S^2. \end{aligned}$$

We draw a circle through points B and C , tangent to side CA , another circle through points C and A , tangent to side AB and a circle through points A and B , tangent to side BC of the triangle ABC [3]. These three circles have a common point (Ω), the *first Brocard point of triangle ABC* (Figure 1). Moreover, $\angle\Omega AB = \angle\Omega BC = \angle\Omega CA = \omega$, and this angle ω is called the *Brocard angle of the triangle ABC* . The Brocard angle has the property that

$$\cot \omega = \cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{2S}.$$

Similarly we can construct the *second Brocard point* (Ω').

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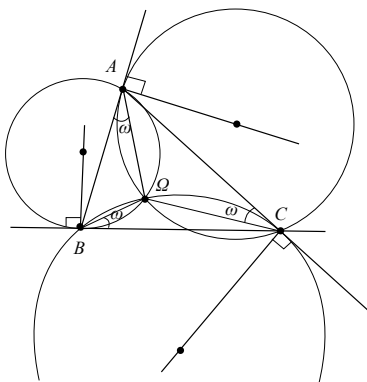


Figure 1.

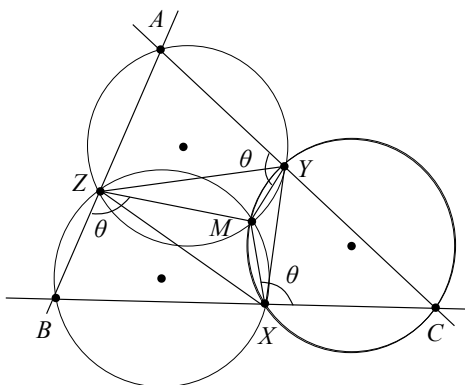


Figure 2.

The three circumcircles to triangles AYZ , BZX and CXY intersect in a single point M , called the *Miquel's point of the triangle ABC* [4]. In addition $\angle AYM = \angle BZM = \angle CXM = \theta$, and this angle θ is called the *Miquel angle of ABC* (Figure 2).

2. EQUATIONS OF CIRCUMCIRCLES OF THE RESIDUAL TRIANGLES

Firstly we will determine the barycentric coordinates of the circumcenters O_a, O_b, O_c of the residual triangles AYZ, BZX, CXY respectively. Denote with $L(K, PQ)$ the perpendicular from the point K to the line PQ . Consequently $O_a = L(Y_a, AC) \cap L(Z_a, AB)$, where Y_a and Z_a are the midpoints of segments AY and AZ . So $2Y_a = (1 + \beta, 0, 1 - \beta)$ and $2Z_a = (2 - \gamma, \gamma, 0)$ (Figure 3).

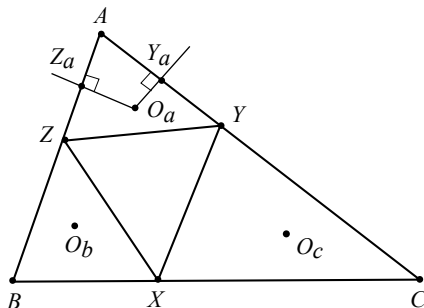


Figure 3.

The equations of the perpendiculars $L(Y_a, AC)$ and $L(Z_a, AB)$ are

$$\begin{vmatrix} x & y & z \\ 1 + \beta & 0 & 1 - \beta \\ S_C & -b^2 & S_A \end{vmatrix} = 0 \Leftrightarrow b^2(1 - \beta)x + (S_C - S_A - b^2\beta)y - b^2(1 + \beta)z = 0,$$

$$\begin{vmatrix} x & y & z \\ 2 - \gamma & \gamma & 0 \\ S_B & S_A & -c^2 \end{vmatrix} = 0 \Leftrightarrow -c^2\gamma x + c^2(2 - \gamma)y + (2S_A - c^2\gamma)z = 0.$$

With the Cramer rule from these equations we can obtain the coordinates of O_a :

$$\delta = \begin{vmatrix} b^2(1 - \beta) & S_C - S_A - b^2\beta \\ -c^2\gamma & c^2(2 - \gamma) \end{vmatrix} = b^2c^2(1 - \beta) - c^2S_A\gamma;$$

$$\delta_x = \begin{vmatrix} b^2(1 + \beta) & S_C - S_A - b^2\beta \\ -(2S_A - c^2\gamma) & c^2(2 - \gamma) \end{vmatrix} z = [2S^2 - b^2S_B(1 - \beta) - c^2S_C\gamma] z;$$

$$\delta_y = \begin{vmatrix} b^2(1 - \beta) & b^2(1 + \beta) \\ -c^2\gamma & -(2S_A - c^2\gamma) \end{vmatrix} z = [-b^2S_A(1 - \beta) + b^2c^2\gamma] z.$$

Consequently the coordinates of O_a are

$$O_a = (2S^2 - b^2S_B(1 - \beta) - c^2S_C\gamma : -b^2S_A(1 - \beta) + b^2c^2\gamma : b^2c^2(1 - \beta) - c^2S_A\gamma).$$

The sum of these coordinates is $2S^2$, so the absolute barycentric coordinates of O_a are

$$O_a = \left(1 - \frac{b^2S_B(1 - \beta) + c^2S_C\gamma}{2S^2}, \frac{-b^2S_A(1 - \beta) + b^2c^2\gamma}{2S^2}, \frac{b^2c^2(1 - \beta) - c^2S_A\gamma}{2S^2} \right).$$

Similarly we obtain the absolute barycentric coordinates of O_b and O_c :

$$O_b = \left(\frac{c^2a^2(1 - \gamma) - a^2S_B\alpha}{2S^2}, 1 - \frac{c^2S_C(1 - \gamma) + a^2S_A\alpha}{2S^2}, \frac{-c^2S_B(1 - \gamma) + c^2a^2\alpha}{2S^2} \right).$$

$$O_c = \left(\frac{-a^2S_C(1 - \alpha) + a^2b^2\beta}{2S^2}, \frac{a^2b^2(1 - \alpha) - b^2S_C\beta}{2S^2}, 1 - \frac{a^2S_A(1 - \alpha) + b^2S_B\beta}{2S^2} \right).$$

Denote with $O_a(R_a)$, $O_b(R_b)$, $O_c(R_c)$ the circumcircles of the residual triangles AYZ , BZX , CXY respectively, where R_a, R_b, R_c are the radii of these circles.

Proposition 2.1.

$$(a) \quad R_a = \frac{bc}{2S} \sqrt{b^2(1 - \beta)^2 - 2(1 - \beta)\gamma S_A + c^2\gamma^2},$$

$$(b) \quad R_b = \frac{ca}{2S} \sqrt{c^2(1 - \gamma)^2 - 2(1 - \gamma)\alpha S_B + a^2\alpha^2},$$

$$(c) \quad R_c = \frac{ab}{2S} \sqrt{a^2(1 - \alpha)^2 - 2(1 - \alpha)\beta S_C + b^2\beta^2}.$$

Proof. We have:

$$\begin{aligned} R_a^2 &= AO_a^2 = \left(\frac{b^2S_B(1 - \beta) + c^2S_C\gamma}{2S^2} \right)^2 S_A + \left(\frac{-b^2S_A(1 - \beta) + b^2c^2\gamma}{2S^2} \right)^2 S_B \\ &+ \left(\frac{b^2c^2(1 - \beta) - c^2S_A\gamma}{2S^2} \right)^2 S_C = \frac{1}{4S^4} \left[b^4(S_B^2S_A + S_A^2S_B + c^4S_C)(1 - \beta)^2 \right. \\ &\quad \left. + 2b^2c^2S_A(S_B S_C - b^2S_B - c^2S_C)(1 - \beta)\gamma + c^4(S_C^2S_A + S_A^2S_C + b^4S_B)\gamma^2 \right] \\ &= \frac{b^2c^2}{4S^2} [b^2(1 - \beta)^2 - 2(1 - \beta)\gamma S_A + c^2\gamma^2]. \end{aligned}$$

□

We will introduce the following notations:

$$\lambda = \lambda(\alpha, \beta, \gamma) = a^2\alpha + b^2\beta + c^2\gamma,$$

$$\lambda' = \lambda(1 - \alpha, 1 - \beta, 1 - \gamma) = a^2(1 - \alpha) + b^2(1 - \beta) + c^2(1 - \gamma),$$

$$E = E(\alpha, \beta, \gamma) = S_\omega - \lambda = \frac{1}{2} [a^2(1 - 2\alpha) + b^2(1 - 2\beta) + c^2(1 - 2\gamma)].$$

Consequently $\lambda' = 2S_\omega - \lambda$ i.e. $\lambda + \lambda' = 2S_\omega = a^2 + b^2 + c^2$ or $S_\omega - \lambda = \lambda' - S_\omega$.

Proposition 2.2.

$$(a) \quad O_b O_c = \frac{a}{2S} \sqrt{E^2 + S^2} = \frac{a}{2S} \sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2},$$

$$(b) \quad O_c O_a = \frac{b}{2S} \sqrt{E^2 + S^2} = \frac{b}{2S} \sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2},$$

$$(c) \quad O_a O_b = \frac{c}{2S} \sqrt{E^2 + S^2} = \frac{c}{2S} \sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2}.$$

Proof. We calculate the differences of coordinates of the points O_b and O_c :

$$\begin{aligned} x_{O_b} - x_{O_c} &= \frac{1}{2S^2} [c^2 a^2 (1 - \gamma) - a^2 S_B \alpha + a^2 S_C (1 - \alpha) - a^2 b^2 \beta] \\ &= \frac{a^2}{2S^2} (c^2 + S_C - a^2 \alpha - b^2 \beta - c^2 \gamma) = \frac{a^2 (S_\omega - \lambda)}{2S^2}, \end{aligned}$$

$$\begin{aligned} y_{O_b} - y_{O_c} &= \frac{1}{2S^2} [2S^2 - c^2 S_C (1 - \lambda) - a^2 S_A \alpha - a^2 b^2 (1 - \alpha) + b^2 S_C \beta] \\ &= \frac{1}{2S^2} [S_C (a^2 \alpha + b^2 \beta + c^2 \gamma - S_\omega) + S^2] = \frac{S_C (\lambda - S_\omega) + S^2}{2S^2}, \end{aligned}$$

$$\begin{aligned} z_{O_b} - z_{O_c} &= \frac{1}{2S^2} [-c^2 S_B (1 - \gamma) + c^2 a^2 \alpha - 2S^2 + a^2 S_A (1 - \alpha) + b^2 S_B \beta] \\ &= \frac{1}{2S^2} [S_B (a^2 \alpha + b^2 \beta + c^2 \gamma - S_\omega) - S^2] = \frac{S_B (\lambda - S_\omega) - S^2}{2S^2}. \end{aligned}$$

The length of segment $O_b O_c$ is

$$\begin{aligned} (O_b O_c)^2 &= \left(\frac{a^2 (S_\omega - \lambda)}{2S^2} \right)^2 S_A + \left(\frac{S_C (\lambda - S_\omega) + S^2}{2S^2} \right)^2 S_B + \left(\frac{S_B (\lambda - S_\omega) - S^2}{2S^2} \right)^2 S_C \\ &= \frac{1}{4S^4} [(a^4 S_A + S_B S_C^2 + S_B^2 S_C) (\lambda - S_\omega)^2 + (S_B + S_C) S^4] = \frac{a^2 ((\lambda - S_\omega)^2 + S^2)}{4S^2}. \end{aligned}$$

□

Proposition 2.3. *The equations of circumcircles of triangles AYZ , BZX , CXY are:*

$$O_a(R_a) : S_A x^2 + S_B y^2 + S_C z^2 - (x + y + z) [S_A x - (S_A - c^2 \gamma) y + (S_C - b^2 \beta) z] = 0,$$

$$O_b(R_b) : S_A x^2 + S_B y^2 + S_C z^2 - (x + y + z) [(S_A - c^2 \gamma) x + S_B y - (S_B - a^2 \alpha) z] = 0,$$

$$O_c(R_c) : S_A x^2 + S_B y^2 + S_C z^2 - (x + y + z) [-(S_C - b^2 \beta) x + (S_B - a^2 \alpha) y + S_C z] = 0.$$

Proof. The equation of the circle $O_a(R_a)$ is $PO_a^2 = R_a^2 \Leftrightarrow PO_a^2 = AO_a^2$, where $P = \left(\frac{x}{\mu}, \frac{y}{\mu}, \frac{z}{\mu}\right)$, $\mu = x + y + z \neq 0$. So

$$\begin{aligned} & \left(\frac{x}{\mu} - x_{O_a}\right)^2 S_A + \left(\frac{y}{\mu} - y_{O_a}\right)^2 S_B + \left(\frac{z}{\mu} - z_{O_a}\right)^2 S_C \\ &= (1 - x_{O_a})^2 S_A + y_{O_a}^2 S_B + z_{O_a}^2 S_C \Leftrightarrow S_A x^2 + S_B y^2 + S_C z^2 - \\ & \quad - 2\mu(S_A x_{O_a} x + S_B y_{O_a} y + S_C z_{O_a} z) - \mu^2(1 - 2x_{O_a})S_A = 0 \\ & S_A x^2 + S_B y^2 + S_C z^2 - \mu[S_A x + (S_A - 2S_A x_{O_a} + 2S_B y_{O_a})y + \\ & \quad + (S_A - 2S_A x_{O_a} + 2S_C z_{O_a})z] = 0 \\ & \Leftrightarrow S_A x^2 + S_B y^2 + S_C z^2 - (x + y + z)[S_A x - (S_A - c^2 \gamma)y + (S_C - b^2 \beta)z] = 0. \end{aligned}$$

□

3. BARYCENTRIC COORDINATES OF THE MIQUEL POINT

Denote with L_{bc}, L_{ca}, L_{ab} the radical axes of the pairs of circles $(O_b(R_b), O_c(R_c)), (O_c(R_c), O_a(R_a)), (O_a(R_a), O_b(R_b))$ respectively.

Proposition 3.4. *The equations of the radical axes L_{bc}, L_{ca}, L_{ab} are*

$$\begin{aligned} L_{bc} &: [b^2(1 - \beta) - c^2 \gamma]x + a^2 \alpha y - a^2(1 - \alpha)z = 0, \\ L_{ca} &: -b^2(1 - \beta)x + [c^2(1 - \gamma) - a^2 \alpha]y + b^2 \beta z = 0, \\ L_{ab} &: c^2 \gamma x - c^2(1 - \gamma)y + [a^2(1 - \alpha) - b^2 \beta]z = 0. \end{aligned}$$

Proof. We obtain the equation of the radical axis L_{bc} as the difference of the equations of circles $O_b(R_b)$ and $O_c(R_c)$, i.e.

$$\begin{aligned} (S_A - c^2 \gamma)x + S_B y - (S_B - a^2 \alpha)z - [- (S_C - b^2 \beta)x + (S_B - a^2 \alpha)y + S_C z] &= 0 \\ \Leftrightarrow [b^2(1 - \beta) - c^2 \gamma]x + a^2 \alpha y - a^2(1 - \alpha)z &= 0. \end{aligned}$$

Proposition 3.5. *The radical axes L_{bc}, L_{ca}, L_{ab} are concurrent.*

□

Proof. In the equations of the radical axes L_{bc}, L_{ca}, L_{ab} the sums of the coefficients of x, y respectively z are zero. □

Let M be the point of concurrence of the radical axes L_{bc}, L_{ca}, L_{ab} . This point is called the *radical center* of circles $O_a(R_a), O_b(R_b), O_c(R_c)$ or the *Miquel point determined by the inscribed triangle XYZ* (Figure 4):

$$M = L_{bc} \cap L_{ca} \cap L_{ab} = O_a(R_a) \cap O_b(R_b) \cap O_c(R_c).$$

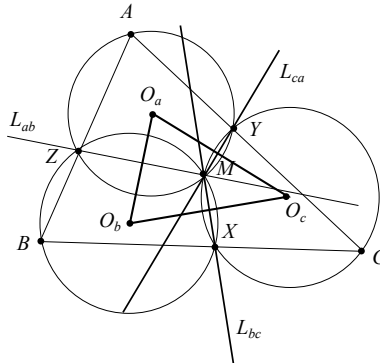


Figure 4.

From the equations of radical axis L_{bc} and L_{ca} , applied the Cramer rule we can determine the barycentric coordinates of Miquel point:

$$\begin{aligned}\delta &= \begin{vmatrix} b^2(1-\beta) - c^2\gamma & a^2\alpha \\ -b^2(1-\beta) & c^2(1-\gamma) - a^2\alpha \end{vmatrix} = \begin{vmatrix} b^2(1-\beta) - c^2\gamma & a^2\alpha \\ -c^2\gamma & c^2(1-\gamma) \end{vmatrix} \\ &= c^2[a^2\alpha + (1-\gamma)(b^2 - a^2\alpha - b^2\beta - c^2\gamma)] = c^2[(b^2 - \lambda)(1-\gamma) + a^2\alpha]; \\ \delta_x &= \begin{vmatrix} a^2(1-\alpha) & a^2\alpha \\ -b^2\beta & c^2(1-\gamma) - a^2\alpha \end{vmatrix} z = a^2[(c^2 - \lambda)(1-\alpha) + b^2\beta]z; \\ \delta_y &= \begin{vmatrix} b^2(1-\beta) - c^2\gamma & a^2(1-\alpha) \\ -b^2(1-\beta) & -b^2\beta \end{vmatrix} z = b^2[(a^2 - \lambda)(1-\beta) + c^2\gamma]z.\end{aligned}$$

Therefore

$$M = (a^2[(c^2 - \lambda)(1-\alpha) + b^2\beta] : b^2[(a^2 - \lambda)(1-\beta) + c^2\gamma] : c^2[(b^2 - \lambda)(1-\gamma) + a^2\alpha]).$$

We denote with μ_P the sum of coordinates of an arbitrary point P .

Therefore:

$$\begin{aligned}\mu_M &= b^2c^2 + c^2a^2 + a^2b^2 - \lambda[a^2(1-\alpha) + b^2(1-\beta) + c^2(1-\gamma)] \\ &= b^2c^2 + c^2a^2 + a^2b^2 - \lambda\lambda' = b^2c^2 + c^2a^2 + a^2b^2 + \lambda(\lambda - 2S_\omega) \\ &= b^2c^2 + c^2a^2 + a^2b^2 + (\lambda - S_\omega)^2 - S_\omega^2 = (\lambda - S_\omega)^2 + S^2 = (S_\omega - \lambda')^2 + S^2.\end{aligned}$$

Let x_M, y_M, z_M be the absolute barycentric coordinates of M :

$$\begin{aligned}x_M &= \frac{a^2[(c^2 - \lambda)(1-\alpha) + b^2\beta]}{(\lambda - S_\omega)^2 + S^2} = \frac{a^2[c^2(1-\gamma) + (b^2 - \lambda')\alpha]}{(\lambda - S_\omega)^2 + S^2}, \\ y_M &= \frac{b^2[(a^2 - \lambda)(1-\beta) + c^2\gamma]}{(\lambda - S_\omega)^2 + S^2} = \frac{b^2[a^2(1-\alpha) + (c^2 - \lambda')\beta]}{(\lambda - S_\omega)^2 + S^2}, \\ z_M &= \frac{c^2[(b^2 - \lambda)(1-\gamma) + a^2\alpha]}{(\lambda - S_\omega)^2 + S^2} = \frac{c^2[b^2(1-\beta) + (a^2 - \lambda')\gamma]}{(\lambda - S_\omega)^2 + S^2}.\end{aligned}$$

4. MAIN RESULTS

The lines MX, MY, MZ and the corresponding sides BC, CA, AB have the same angle named the *Miquel angle determined by the inscribed triangle XYZ* , which is a direct measured angle and change between 0 and π (Figure 2).

Proposition 4.6.

$$\begin{aligned}(a) \quad AM &= \frac{bc\sqrt{b^2(1-\beta)^2 - 2(1-\beta)\gamma S_A + c^2\gamma^2}}{\sqrt{E^2 + S^2}} = \frac{aR_a}{O_bO_c}, \\ (b) \quad BM &= \frac{ca\sqrt{c^2(1-\gamma)^2 - 2(1-\gamma)\alpha S_B + a^2\alpha^2}}{\sqrt{E^2 + S^2}} = \frac{bR_b}{O_cO_a}, \\ (c) \quad CM &= \frac{ab\sqrt{a^2(1-\alpha)^2 - 2(1-\alpha)\beta S_C + b^2\beta^2}}{\sqrt{E^2 + S^2}} = \frac{cR_c}{O_aO_b}.\end{aligned}$$

Proof. We have:

$$\begin{aligned}AM^2 &= (1 - x_M)^2 S_A + y_M^2 S_B + z_M^2 S_C = (y_M + z_M)^2 S_A + y_M^2 S_B + z_M^2 S_C \\ &= c^2 y_M^2 + 2y_M z_M S_A + b^2 z_M^2 = \frac{b^2 c^2}{(E^2 + S^2)^2} \{b^2[(a^2 - \lambda)(1-\beta) + c^2\gamma]^2\end{aligned}$$

$$\begin{aligned}
 &+2[(a^2-\lambda)(1-\beta)+c^2\gamma][b^2(1-\beta)+(a^2-\lambda')\gamma]S_A+c^2[b^2(1-\beta)+(a^2-\lambda')\gamma]^2\} \\
 &= \frac{b^2c^2}{(E^2+S^2)^2}\{b^2[(S_\omega-\lambda-S_A)(1-\beta)+c^2\gamma]^2 \\
 &+2[(S_\omega-\lambda-S_A)(1-\beta)+c^2\gamma][b^2(1-\beta)+(S_\omega-\lambda'-S_A)\gamma]S_A \\
 &+c^2[b^2(1-\beta)+(S_\omega-\lambda'-S_A)\gamma]^2\} \\
 &= \frac{b^2c^2}{(E^2+S^2)^2}\{b^2[(E-S_A)(1-\beta)+c^2\gamma]^2 \\
 &+2[(E-S_A)(1-\beta)+c^2\gamma][b^2(1-\beta)-(E+S_A)\gamma]S_A \\
 &+c^2[b^2(1-\beta)-(E+S_A)\gamma]^2\} \\
 &= \frac{b^2c^2}{(E^2+S^2)^2}\{b^2[(E-S_A)^2+2(E-S_A)S_A+b^2c^2](1-\beta)^2 \\
 &+2[b^2c^2(E-S_A)-(E+S_A)(E-S_A)S_A+b^2c^2S_A-b^2c^2(E+S_A)](1-\beta)\gamma \\
 &+c^2[b^2c^2-2(E+S_A)S_A+(E+S_A)^2]\gamma^2\} = \\
 &= \frac{b^2c^2}{(E^2+S^2)^2}\{b^2(E^2+S^2)(1-\beta)^2-2(E^2+S^2)S_A(1-\beta)\gamma+c^2(E^2+S^2)\gamma^2\} \\
 &= \frac{b^2c^2}{E^2+S^2}[b^2(1-\beta)^2-2(1-\beta)\gamma S_A+c^2\gamma^2].
 \end{aligned}$$

□

From the law of sinus we have $2 \sin \theta = \frac{AM}{R_a} = \frac{BM}{R_b} = \frac{CM}{R_c}$ and from the Proposition

$$4.1 \quad \frac{AM}{R_a} = \frac{a}{O_bO_c'} \quad \frac{BM}{R_b} = \frac{b}{O_cO_a'} \quad \frac{CM}{R_c} = \frac{c}{O_aO_b}.$$

Therefore, the triangles ABC and $O_aO_bO_c$ are similar and its similarity ratio is $2 \sin \theta = \frac{a}{O_bO_c} = \frac{2S}{\sqrt{E^2+S^2}}$, from where $\sin \theta = \frac{S}{\sqrt{E^2+S^2}}$ (Figure 4).

Proposition 4.7. *Between the Miquel and Brocard angles exist the following relation:*

$$S_\omega = S_\theta + a^2\alpha + b^2\beta + c^2\gamma.$$

Proof. We have: $\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{S^2}{E^2+S^2} = \frac{E^2}{E^2+S^2}$. So

$$\cos \theta = \frac{S_\omega - \lambda}{\sqrt{E^2+S^2}} \text{ and } S_\theta = S \cot \theta = S \frac{S_\omega - \lambda}{S} = S_\omega - (a^2\alpha + b^2\beta + c^2\gamma). \quad \square$$

Remark 4.1. The Miquel angle is right if and only if the triangle XYZ is pedal triangle.

5. ISOTOMIC INSCRIBED TRIANGLES

If $X'Y'Z'$ and XYZ are isotomic inscribed triangles then $X' = (0, \alpha, 1 - \alpha)$, $Y' = (1 - \beta, 0, \beta)$, $Z' = (\gamma, 1 - \gamma, 0)$. Consequently all results referred to the triangle $X'Y'Z'$ can be obtained from above results replacing α, β, γ by $1 - \alpha, 1 - \beta, 1 - \gamma$ respectively. For example

$$O'_bO'_c = \frac{a}{2S} \sqrt{(\lambda' - S_\omega)^2 + S^2} = \frac{a}{2S} \sqrt{(S_\omega - \lambda)^2 + S^2} = O_bO_c,$$

$$O'_c O'_a = \frac{b}{2S} \sqrt{(\lambda' - S_\omega)^2 + S^2} = \frac{b}{2S} \sqrt{(S_\omega - \lambda)^2 + S^2} = O_c O_a,$$

$$O'_a O'_b = \frac{c}{2S} \sqrt{(\lambda' - S_\omega)^2 + S^2} = \frac{c}{2S} \sqrt{(S_\omega - \lambda)^2 + S^2} = O_a O_b.$$

It follows that the triangles $O_a O_b O_c$ and $O'_a O'_b O'_c$ are congruent.

Proposition 5.8. *If θ' is the Miquel angle of the isotomic inscribed triangle $X'Y'Z'$ then*

- (a) $S_\omega = -S_{\theta'} + a^2\alpha + b^2\beta + c^2\gamma$
- (b) $\theta + \theta' = \pi$.

Proof. We have:

- (a) $S_{\theta'} = S_\omega - [a^2(1-\alpha) + b^2(1-\beta) + c^2(1-\gamma)] = -S_\omega + a^2\alpha + b^2\beta + c^2\gamma$
- (b) $S_\theta + S'_{\theta} = S \cot \theta + S \cot \theta' = 0$. Therefore $\frac{\sin(\theta + \theta')}{\sin \theta \cdot \sin \theta'} = 0$,

i.e. $\theta + \theta' = \pi$ [5]. □

Remark 5.2. For $\alpha = \beta = \gamma = 0$ the Miquel point coincide with the 1st Brocard point $\Omega = (c^2a^2 : a^2b^2 : b^2c^2)$ (Figure 1) and for $\alpha = \beta = \gamma = 1$ we obtain the 2nd Brocard point $\Omega' = (a^2b^2 : b^2c^2 : c^2a^2)$ [6].

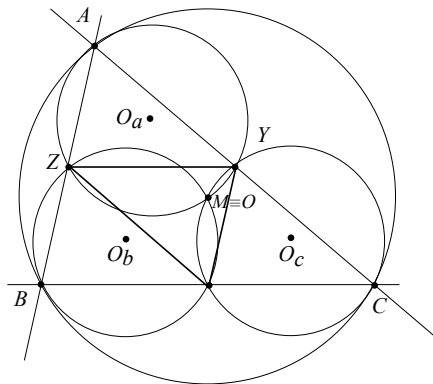


Figure 5.

If $\alpha = \beta = \gamma = \frac{1}{2}$ the Miquel point coincide with the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$ of the reference triangle ABC (Figure 5) [6].

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440095 SATU MARE,
 AVRAM IANCU STR., NO 58, SC. A, AP. 8
 ROMANIA
 E-mail address: d.sandor.kiss@gmail.com