CREAT. MATH. INFORM. **29** (2020), No. 2, 161 - 164
Online version at https://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Some regularity properties of the functions obtained from some algebraic properties

DAN ŞTEFAN MARINESCU and MIHAI MONEA

ABSTRACT. The main result of this note is stated as follows. Let $a, b \in \mathbb{R}$, $a < b$, and a function $f : (a, b) \to \mathbb{R}$. We consider $u, v : (a, b) \to \mathbb{R}$ such that $u(x) > 0$ and $v(x) < 0$, for any $x \in (a, b)$, and define $g, h : (a, b) \to \mathbb{R}$ by $g(x) = u(x) f(x)$ and $h(x) = v(x) f(x)$, for any $x \in (a, b)$. The main result we establish is stated as follows:

Theorem. *Let n a positive integer, $n \geq 3$. If u, v are $(n-1)$-times differentiable on (a, b) and g, h are n-convex functions then f is $(n-1)$-times differentiable on (a, b).*

1. INTRODUCTION

The following contest problem, stated here as a Proposition 1.1, has been proposed to the 2018's district round of the Romanian National Mathematical Olympiad [7].

Proposition 1.1. *Let $a, b \in \mathbb{R}$, $a < b$, and the function $f : (a, b) \to \mathbb{R}$, such that the functions $g, h : (a, b) \to \mathbb{R}$ defined by $g(x) = (x - a) f(x)$ and $h(x) = (x - b) f(x)$, for any $x \in (a, b)$, are nondecreasing. Then f is continuous on (a, b).*

We remark that the reverse implication of Proposition 1.1 fails. For example, the function $f : (-1, 1) \to \mathbb{R}$, $f(x) = |x|$ is continuous but the functions $g, h : (-1, 1) \to \mathbb{R}$, defined by $g(x) = (x + 1) f(x)$ and $h(x) = (x - 1) f(x)$, for any $x \in (-1, 1)$, are not monotone.

A similar result has been published later by the authors of this note, see [4].

Proposition 1.2. *Let $a, b \in \mathbb{R}$, $a < b$, and the function $f : (a, b) \to \mathbb{R}$ such that the functions $g, h : (a, b) \to \mathbb{R}$ defined by $g(x) = (x - a) f(x)$ and $h(x) = (x - b) f(x)$, for any $x \in (a, b)$, are convex. Then f is differentiable on (a, b).*

Starting from these interesting results, the aim of this note is to present some similar but more general results.

2. ABOUT THE n-CONVEX FUNCTIONS

The concept of n-convex function, where n is a positive integer, is due to Hopf [1] and Popoviciu [6]. We remind that a function f, defined on the real interval I, is called *n-convex* if

$$\frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_n)} + \ldots$$
$$+ \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_{n-1})} \geq 0,$$

Received: 12.07.2019. In revised form: 18.02.2020. Accepted: 25.02.2020
2010 *Mathematics Subject Classification.* 26B25.
Key words and phrases. *continuous functions, differentiable functions, convex functions, n-convex functions.*
Corresponding author: Mihai Monea; mihaimonea@yahoo.com

for any distinct $x_0, x_1, ..., x_n$ from the interval I.

It can be easily seen that a 1-convex function is a nondecreasing function while a 2-convex function is convex in the usual sense.

Some properties of n-convex functions are similar to that of classical convex functions. Let n a positive integer, $n \geq 3$ and I an open interval. If $f : I \to \mathbb{R}$ is a n-convex function then f is $(n-2)$-times differentiable on I and the derivative $f^{(n-2)}$ is convex. Then, for any $x \in I$, there exist the finite one-sided $(n-1)$-th derivatives $f_-^{(n-1)}(x), f_+^{(n-1)}(x)$ and $f_-^{(n-1)}(x) \leq f_+^{(n-1)}(x)$.

More on this kind of results - which are not trivial - can be found in [2] and [3].

Now, we are in position to state a result that generalize the results from the previous section. First, let n be a positive integer, $n \geq 3$ and $a, b \in \mathbb{R}, a < b$.

For any function $f : (a,b) \to \mathbb{R}$, we consider $g, h : (a,b) \to \mathbb{R}$ defined by $g(x) = (x-a) f(x)$ and $h(x) = (x-b) f(x)$, for any $x \in (a,b)$.

Proposition 2.1. *If g, h are n-convex functions, then f is $(n-1)$-times differentiable on (a,b).*

In the next section we will present the proof of this proposition as a consequence of a more general result.

3. Main results

Let $a, b \in \mathbb{R}$, $a < b$, and $f : (a.b) \to \mathbb{R}$ a function. We consider $u, v : (a,b) \to \mathbb{R}$ such that $u(x) > 0$ and $v(x) < 0$, for all $x \in (a,b)$ and define the functions $g, h : (a,b) \to \mathbb{R}$ by $g(x) = u(x) f(x)$ and $h(x) = v(x) f(x)$, for any $x \in (a,b)$.

Proposition 3.1. *If u, v are continuous on (a,b) and g, h are nondecreasing on (a,b), then f is continuous on (a,b).*

Proof. Let $c \in (a,b)$. We will prove that $\lim_{x \nearrow c} f(x) = \lim_{x \searrow c} f(x) = f(c)$. First, we observe that $\lim_{x \nearrow c} f(x)$ and $\lim_{x \searrow c} f(x)$ exist due to the assumptions in the hyphotesis.

Now, for any $x \in (a,c)$, we have $g(x) \leq g(c)$, also $u(x) f(x) \leq u(c) f(c)$. We obtain $f(x) \leq \frac{u(c)}{u(x)} f(c)$. Hence $\lim_{x \nearrow c} u(x) = u(c) \neq 0$, then $\lim_{x \nearrow c} f(x) \leq f(c)$. On the other side, we have $h(x) \leq h(c)$, also $v(x) f(x) \leq v(c) f(c)$. We obtain $f(x) \geq \frac{v(c)}{v(x)} f(c)$. Hence, $\lim_{x \nearrow c} v(x) = v(c) \neq 0$, then $\lim_{x \nearrow c} f(x) \geq f(c)$. As a consequence, we find $\lim_{x \nearrow c} f(x) = f(c)$.

In the same way, by starting from $x \in (c,b)$, we obtain $\lim_{x \searrow c} f(x) = f(c)$. □

The proof of the following proposition will use some properties of the convex functions, that can be found, for example, in [5]. We recall that if $f : (a,b) \to \mathbb{R}$ is a convex function, then f is continuous on (a,b) and there exist $f'_-(x)$ and $f'_+(x)$, for any $x \in (a,b)$. Moreover, they are finite and $f'_-(x) \leq f'_+(x)$, for any $x \in (a,b)$.

Proposition 3.2. *If u, v are differentiale on (a,b) and g, h are convex on (a,b) then f is differentiable on (a,b).*

Proof. Let $c \in (a,b)$. We will prove that $f'_-(c) = f'_+(c) \in \mathbb{R}$. From $g(x) = u(x) f(x)$, we obtain $f(x) = \frac{g(x)}{u(x)}$. Then there exist $f'_-(x)$ and $f'_+(x)$. Moreover, for any $x \in (a,b)$, we have

$$f'_-(x) = \frac{g'_-(x) u(x) - g(x) u'(x)}{u^2(x)} = \frac{g'_-(x)}{u(x)} - \frac{g(x) u'(x)}{u^2(x)}$$

and

$$f'_+(x) = \frac{g'_+(x)}{u(x)} - \frac{g(x)\,u'(x)}{u^2(x)}.$$

Hence $g'_-(x) \le g'_+(x)$ and $u(x) \ge 0$, we obtain $f'_-(x) \le f'_+(x)$.

In the same way, the equality $h(x) = v(x)\,f(x)$ lead us to

$$f'_-(x) = \frac{h'_-(x)}{v(x)} - \frac{h(x)\,v'(x)}{v^2(x)}$$

and

$$f'_+(x) = \frac{h'_+(x)}{v(x)} - \frac{h(x)\,v'(x)}{v^2(x)},$$

for any $x \in (a,b)$. Hence, $h'_-(x) \le h'_+(x)$ and $v(x) \le 0$, we obtain $f'_-(x) \ge f'_+(x)$.

We conclude that $f'_-(x) = f'_+(x)$. □

The main result of this paper is given by the following theorem.

Theorem 3.3. *Let n be a positive integer, $n \ge 3$. If u, v are $(n-1)$-times differentiable on (a,b) and g, h are n-convex function, then f is $(n-1)$-times differentiable on (a,b).*

Proof. We will prove that $f_+^{(n-1)}(x) = f_-^{(n-1)}(x)$, for any $x \in (a,b)$.

From $f(x) = \frac{g(x)}{u(x)}$, we obtain that f is $(n-1)$-times differentiable on (a,b). Using Leibniz's formula for the derivatives of a product, we obtain

$$\begin{aligned} g^{(n-2)}(x) &= (u(x)\,f(x))^{(n-2)} \\ &= \sum_{k=0}^{n-3} \binom{n-2}{k} u^{(n-2-k)}(x)\,f^{(k)}(x) + u(x)\,f^{(n-2)}(x), \end{aligned}$$

for any $x \in (a,b)$. Hence, $u^{(n-2)}$ is differentiable and $g^{(n-2)}$ is convex, then $f^{(n-2)}$ has one-sided derivatives, for any $x \in (a,b)$. Moreover, for any $x \in (a,b)$, we have

$$\begin{aligned} g_-^{(n-1)}(x) &= \sum_{k=0}^{n-3} \binom{n-2}{k} \left(u^{(n-1-k)}(x)\,f^{(k)}(x) + u^{(n-2-k)}(x)\,f^{(k+1)}\right) \\ &\quad + u'(x)\,f^{(n-2)}(x) + u(x)\,f_-^{(n-1)}(x). \end{aligned}$$

In the same way, we obtain

$$\begin{aligned} g_+^{(n-1)}(x) &= \sum_{k=0}^{n-3} \binom{n-2}{k} \left(u^{(n-1-k)}(x)\,f^{(k)}(x) + u^{(n-2-k)}(x)\,f^{(k+1)}\right) \\ &\quad + u'(x)\,f^{(n-2)}(x) + u(x)\,f_+^{(n-1)}(x). \end{aligned}$$

Then

$$g_+^{(n-1)}(x) - g_-^{(n-1)}(x) = u(x)\left(f_+^{(n-1)}(x) - f_-^{(n-1)}(x)\right).$$

Hence, $g^{(n-2)}$ is convex, then $g_+^{(n-1)}(x) \ge g_-^{(n-1)}(x)$. We find

$$f_+^{(n-1)}(x) \ge f_-^{(n-1)}(x),$$

for any $x \in (a,b)$.

Starting from equality $h(x) = v(x)\,f(x)$, we obtain

$$h_+^{(n-1)}(x) - h_-^{(n-1)}(x) = v(x)\left(f_+^{(n-1)}(x) - f_-^{(n-1)}(x)\right).$$

Hence $h_{+}^{(n-1)}(x) \geq h_{-}^{(n-1)}(x)$ and since $v(x) < 0$, we find

$$f_{+}^{(n-1)}(x) \leq f_{-}^{(n-1)}(x),$$

for any $x \in (a, b)$. Then

$$f_{+}^{(n-1)}(x) = f_{-}^{(n-1)}(x),$$

for any $x \in (a, b)$ and the proof is complete. □

Now, if we choose $u(x) = x - a$ and $v(x) = x - b$, we obtain the proof of the Proposition 2.1.

Finally, since a 1-convex function means that it is nondecreasing and a 2-convex function means that it is convex, with the convention that 0-time differentiable means continuity, we can state the following unitary result.

Theorem 3.4. *If the functions u, v are $(n-1)$-times differentiable and g, h are n-convex, then f is $(n-1)$-times differentiable, for any positive integer n.*

References

[1] Hopf, E., *Über die Zusammenhänge zwischen gewissen höheren Differenzenquotienten reeller Funktionen einer reellen Variablen und deren Differenzierbarkeitseigenschaften*, PhD dissertation, Friedrich-Wilhelms-Universität, Berlin, 1926

[2] Karlin, S. and Studden, W. J., *Tchebycheff Systems: With Applications in Analysis and Statistics*, Pure Appl. Math., vol. XV, Interscience Publishers John Wiley & Sons, New York, London, Sydney, 1966

[3] Kuczma, M., *An Introduction to the Theory of Functional Equations and Inequalities*, Birkhäuser Verlag AG, 2009

[4] Marinescu, D. Ş. and Monea, M., *Problem 12096*, Amer. Math. Monthly, **126** (2019), pag. 181

[5] Niculescu, C. P. and Persson, L.-E., *Convex Functions and Their Applications*, Springer, 2018

[6] Popoviciu, T., *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*, Mathematica, **8** (1934), 1–85

[7] ***, *Romanian Mathematical Competitions*, RMC 2018, Bucharest, 2018

Colegiul Naţional "Iancu de Hunedoara" Hunedoara, Romania
E-mail address: marinescuds@gmail.com

Colegiul Naţional "decebal" Deva, Romania
E-mail address: mihaimonea@yahoo.com