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Some generalized definitions of uniform continuity for real valued functions

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ABSTRACT. The purpose of this study is to give some uniform continuity definitions for real valued functions.

1. INTRODUCTION AND BACKGROUND

The natural density of a set K of positive integers is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\big|\{k \leq n : k \in K\}\big|,$$

where $\big|\{k \leq n : k \in K\}\big|$ denotes the number of elements of K not exceeding n.

A sequence $x = \{x_k\}$ is said to be statistically convergent to the number u if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \leq n : |x_k - u| \geq \varepsilon\}\big| = 0.$$

In this case we write $st - \lim x_k = u$. If $u = 0$, then we say that $\{x_k\}$ is statistically null sequence.

The idea of statistical convergence was introduced by Fast [8] and studied by many authors. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \xi$, then $st - \lim x_k = \xi$. The converse does not hold in general.

A sequence $x = \{x_k\}$ is called statistically Cauchy if for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \leq n : |x_k - x_m| \geq \varepsilon\}\big| = 0.$$

If x is statistically Cauchy sequence, then

(1) There exists a subsequence $K = \{k_n\}$ of $\mathbb{N}$ with $\delta(K) = 1$ such that $|x_{k_m} - x_{k_n}| \to 0$ as $m, n \to \infty$,
(2) There exists $y = \{y_k\}$ and $z = \{z_k\}$ such that $x = y + z$, $|y_m - y_n| \to 0$ as $m, n \to \infty$ and $\lim\limits_{n\to\infty} \frac{1}{n}\big|\{k \leq n : z_k \neq 0\}\big| = 0$ ([9]).

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Statistical limit superior and inferior were defined in [10] as follows:

For a real number sequence $x=\{x_k\}$, let B_x denote the set

$$B_x := \big\{b\in\mathbb{R} : \delta\{k : x_k > b\} \neq 0\big\}$$

and similarly, let A_x denote the set

$$A_x := \big\{a\in\mathbb{R} : \delta\{k : x_k < a\} \neq 0\big\}.$$

If x is a real sequence, then the statistical limit superior of x is given by

$$st-\limsup x := \begin{cases} \sup B_x & , \quad B_x \neq \emptyset \\ -\infty & , \quad B_x = \emptyset. \end{cases}$$

Also, the statistical limit inferior of x is given by

$$st-\liminf x := \begin{cases} \sup A_x & , \quad A_x \neq \emptyset \\ \infty & , \quad A_x = \emptyset. \end{cases}$$

The following continuity definitions are known:

$f:\mathbb{R}\to\mathbb{R}$ is said to be statistically continuous at $u\in\mathbb{R}$ provided that whenever $st-\lim x_k = u$, then $st-\lim f(x_k) = f(u)$.

A sequence $x=\{x_n\}$ converges in the Cesàro sense if

$$C-\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{x_1+x_2+\ldots+x_n}{n}$$

exists and is finite.

$f:\mathbb{R}\to\mathbb{R}$ is said to be Cesàro continuous if

$$f\left(C-\lim_{n\to\infty} x_n\right) = C-\lim_{n\to\infty} f(x_n)$$

for every Cesàro convergent sequence $\{x_n\}$.

Definition 1.1. ([3]) A function $f:\mathbb{R}\to\mathbb{R}$ is said to be strongly sequentially continuous on $A\subseteq\mathbb{R}$ if for every $\{h_i\}\to 0$,

$$\lim_{n\to\infty}\inf_{i\leq n} f(x+h_i) = f(x)$$

uniformly for $x\in A$.

Strong sequential continuity can be viewed as an iterated limit property:

A function $f:\mathbb{R}\to\mathbb{R}$ is strongly sequentially continuous on $A\subseteq\mathbb{R}$ if and only if for each sequence $\{x_n\}\in A$ and each null sequence $\{h_i\}$,

$$\lim_{i\to\infty}\liminf_{n\to\infty}|f(x_n+h_i)-f(x)| = 0.$$

For a function $f:\mathbb{R}\to\mathbb{R}$, the following assertions are equivalent:

(1) f is not strongly sequentially continuous on $A\subseteq\mathbb{R}$,

(2) There exist a positive number ε, a sequence $\{x_n\}\subseteq A$ and a null sequence $\{h_i\}\subseteq\mathbb{R}$ such that

$$|f(x_n+h_i)-f(x_n)|\geq\varepsilon, \text{ whenever } n>i.$$

Definition 1.2. ([3]) A function $f:\mathbb{R}\to\mathbb{R}$ is said to be uniformly Cesàro continuous on $A\subseteq\mathbb{R}$ if for every null sequence $\{h_i\}\subseteq\mathbb{R}$ has a subsequence $\{h_i'\}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(x+h_i') = f(x)$$

uniformly for $x\in A$.

2. Main results

In this section we give some definitions of uniform (sequentially) continuity for real valued functions.

The concept of continuity and any concept related to continuity play a very important role not only in pure mathematics but also in other branches of science involving mathematics especially in computer science, information theory, biological science, and dynamical systems. Similar studies on these concepts can be found in [4, 5, 11, 12, 13, 14]. Connor and Grosse-Erdmann [7] have investigated the impact of changing the definition of the convergence of sequences on the structure of sequential continuity of real functions as follows:

Let G be a method of sequential convergence and $f : A \subset \mathbb{R} \to \mathbb{R}$ a function. Then f is G-continuous at $u \in A$ provided that whenever an A-valued sequence $x = \{x_n\}$ is G-convergent to u, then the sequence $f(x) = (f(x_n))$ is G-convergent to $f(u)$. f is called G-continuous on A if it is G-continuous at every $u \in A$. The G-continuity of f (on its domain) can also be expressed briefly as follows:

If x is an A-valued G-convergent sequence, then $G(f(x)) = f(G(x))$.

Now we are ready to give new definitions.

Definition 2.3. A function $f : \mathbb{R} \to \mathbb{R}$ is uniform statistically continuous on $A \subseteq \mathbb{R}$ if for all sequences $\{x_n\}, \{y_n\} \in A$ such that $st - \lim |x_n - y_n| = 0$, one has

$$st - \lim_{n\to\infty} |f(x_n) - f(y_n)| = 0.$$

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ is uniform statistically Cesàro continuous on $A \subseteq \mathbb{R}$ if for every statistically null sequence $\{h_i\} \subseteq \mathbb{R}$ has a subsequence $\{h'_i\}$ such that

$$st - \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(x + h'_i) = f(x)$$

uniformly for $x \in A$.

Definition 2.5. A function $f : \mathbb{R} \to \mathbb{R}$ is strongly statistically continuous on $A \subseteq \mathbb{R}$ if for every statistically null sequence $\{h_i\}$

$$st - \lim_{n\to\infty} \inf_{i\leq n} |f(x + h_i) - f(x)| = 0$$

uniformly for $x \in A$.

A function $f : \mathbb{R} \to \mathbb{R}$ is strongly statistically continuous on $A \subseteq \mathbb{R}$ if and only if for each sequence $\{x_n\} \in A$ and each statistically null sequence $\{h_i\}$,

$$st - \lim_{i\to\infty} \Big(st - \liminf_{n\to\infty} |f(x_n + h_i) - f(x)| \Big) = 0.$$

Clearly, uniform statistically continuity on A implies strong statistically continuity on A.

Now we state a condition for uniform statistical continuity to fail.

Proposition 2.1. *A function $f : A \to \mathbb{R}$ is not uniform statistically continuous on A if and only if there exists $\varepsilon_1 > 0$ and sequences $\{x_n\}, \{y_n\}$ in A such that $st - \lim |x_n - y_n| = 0$ and*

$$\frac{1}{n} \big| \{ k \leq n : |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon \} \big| \geq \varepsilon_1$$

for all $n \in \mathbb{N}$ where $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$, respectively.

Theorem 2.1. *If $f : \mathbb{R} \to \mathbb{R}$ is uniform statistically Cesàro continuous on $A \in \mathbb{R}$, then f is strongly statistically continuous on A.*

Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ is not strongly statistically continuous on A. Choose $\varepsilon > 0$, statistically null sequence $\{h_i\}$ and $\{x_n\} \subseteq A$ such that

$$\lim_{n\to\infty} \frac{1}{p}\left|\left\{n \le p : \inf_{i\le n} |f(x_n+h_i)-f(x_n)| \ge \varepsilon\right\}\right| \ne 0.$$

For each infinite set $I \subseteq \mathbb{N}$, let us define $I^{(2)}$ the set of all ordered pairs of I, that is, $I^{(2)} = \{(i,n) \in I \times I : i < n\}$. Now, we write

$$A_1 = \{(i,n) \in \mathbb{N}^{(2)}, n \le p : f(x_n+h_i)-f(x_n) \ge \varepsilon\},$$
$$A_2 = \{(i,n) \in \mathbb{N}^{(2)}, n \le p : f(x_n+h_i)-f(x_n) \le -\varepsilon\}.$$

By assumption, one has $A_1 \cup A_2 = \mathbb{N}^{(2)}$. Hence, by applying Ramsey's Theorem for pairs of integers, there is an infinite set $I \subseteq \mathbb{N}$ such that either $I^{(2)} \subseteq A_1$ or $I^{(2)} \subseteq A_2$. We may assume for instance that

$$\lim_{n\to\infty} \frac{1}{p}\left|\left\{n \le p : f(x_n+h_i)-f(x_n) \ge \varepsilon\right\}\right| \ne 0.$$

This implies that f is not uniform statistically Cesàro continuous on A. Indeed, if $\{h'_i\} = \{h_{k_i}\}$ is any subsequence of $\{h_i\}$, then for every $n \ge 1$ we have

$$\lim_{p\to\infty} \frac{1}{p}\left|\left\{n \le p : \frac{1}{n}\sum_{i=0}^{n-1}(f(x_{k_n}+h'_i)-f(x_{k_n})) \ge \varepsilon\right\}\right| \ne 0$$

and

$$\lim_{p\to\infty} \frac{1}{p}\left|\left\{n \le p : \frac{1}{n}\sum_{i=0}^{n-1}(f(x_{k_n}+h'_i)-f(x_{k_n})) \le -\varepsilon\right\}\right| \ne 0.$$

Hence

$$\lim_{p\to\infty} \frac{1}{p}\left|\left\{n \le p : \left|\frac{1}{n}\sum_{i=0}^{n-1} f(x_{k_n}+h'_i)-f(x_{k_n})\right| \ge \varepsilon\right\}\right| \ne 0$$

i.e., f is not uniform statistically Cesàro continuous on A. □

We know that under uniformly continuous function a Cauchy sequence goes to a Cauchy sequence. This is not true for continuous function in general. For example, take $f : (0,1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. $f(x)$ is continuous on $(0,1]$. The sequence $\{x_n\} = \{\frac{1}{n}\}$ is a Cauchy sequence in $(0,1]$ but its image under f, $(f(x_n)) = \{n\}$, is not Cauchy sequence in $\mathbb{R}$.

Theorem 2.2. .

(a) *If $f : \mathbb{R} \to \mathbb{R}$ is strongly statistically continuous on $A \in \mathbb{R}$, then f transforms statistically Cauchy sequences into statistically Cauchy sequences.*

(b) *In addition, If every sequence in A has a statistically Cauchy subsequence, then f is uniform statistically continuous on A.*

Proof. (a) Assume that there exists a statistically Cauchy sequence $\{x_j\}$ in A such that $(f(x_j))$ is not statistically Cauchy sequence. Then one can find a positive number ε and two subsequences $\{y_k\}, \{z_i\}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : |f(y_k)-f(z_i)| \ge \varepsilon\right\}\right| \ne 0$$

for all i. Now, f is uniform statistically continuous at all points z_i and $\{y_k\}$ is a statistically Cauchy sequence; hence, for each fixed i, there exists an integer k_i such that

$$\frac{1}{n}\left|\left\{k \le n : |f(y_k - y_{k_i} + z_i) - f(z_i)| \ge \varepsilon\right\}\right| < \varepsilon_1$$

whenever $n \geq k_i$. Moreover, we can assume that the sequence $\{k_i\}$ is increasing. Put $h_i = -y_{k_i} + z_i$. Since $\{x_j\}$ is a statistically Cauchy sequence, the sequence $\{h_i\}$ is statistically convergent to 0. Therefore by strong statistically continuity,

$$st - \liminf |f(x + h_i) - f(x)| = 0$$

uniformly on A.

In particular, one can find an integer i_0 such that for every k, there exists $i \in \{0, 1, 2, \ldots, i_0\}$ for which

$$\frac{1}{n}\left|\left\{k \leq n : |f(y_k) - f(y_k + h_i)| \geq \varepsilon\right\}\right| < \varepsilon_1.$$

Now, let $k = k(i_0)$ and choose an integer $i \leq i_0$ such that

$$\frac{1}{n}\left|\left\{k \leq n : |f(y_k) - f(y_k + h_i)| \geq \varepsilon\right\}\right| < \varepsilon_1$$

Since $i \leq i_0$, we also have

$$\frac{1}{n}\left|\left\{k \leq n : |f(y_k + h_i) - f(z_i)| \geq \varepsilon\right\}\right| < \varepsilon_1,$$

whence

$$\frac{1}{n}\left|\left\{k \leq n : |f(y_k) - f(z_i)| \geq \varepsilon\right\}\right| < 2\varepsilon_1$$

for all i which is a contradiction.

Part (b) is a straightforward consequence of (a). □

3. Statistical compactness

Definition 3.6. Let $A \subset \mathbb{R}$ and $u \in \mathbb{R}$. Then u is in the statistically closure of A if there is a sequence $x = \{x_n\}$ of points in A such that $st - \lim x_n = u$. We say that a set is statistically closed if it contains all of the points in its statistically closure.

Definition 3.7. A subset of A of $\mathbb{R}$ is called statistically compact if when $x = \{x_n\}$ is a sequence of points in A there is a subsequence $\{x_{n_k}\}$ of $x = \{x_n\}$ with $st - \lim x_{n_k} = u \in A$.

Theorem 3.3. *The statistically continuous image of any statistically compact subset of $\mathbb{R}$ is statistically compact.*

Proof. Let f be any statistically continuous function on $\mathbb{R}$ and A be any statistically compact subset of $\mathbb{R}$. $x = \{x_n\}$ be any sequence of points in A. Take any sequence $y = \{y_n\} = (f(x_n))$ of points in $f(A)$. Since A is statistically compact,there exists a subsequence $\{x_{n_k}\}$ of $x = \{x_n\}$ with $st - \lim x_{n_k} = u \in A$. Then the sequence $(f(x_{n_k}))$ is a subsequence of the sequence $y = \{y_n\}$. Since f is statistically continuous, we have

$$st - \lim f(x_{n_k}) = u \in f(A).$$

Thus $f(A)$ is statistically compact. □

The Heine-Cantor Theorem asserts that every continuous function on a compact set is uniformly continuous. In particular, if a function is continuous on a closed bounded interval of the real line, then it is uniformly continuous on that interval. We now show that the statistically continuous function on a statistically compact set is uniformly statistically continuous.

Theorem 3.4. *Every statistically continuous function on a statistically compact set is uniformly statistically continuous.*

Proof. Suppose that f is a statistically continuous function on the compact set A but not uniform statistically continuous. Then we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$st-\lim|x_n-y_n|=0 \text{ and } st-\lim|f(x_n)-f(y_n)|\neq 0.$$

Since A is statistically compact, there is a statistically convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $st-\lim x_{n_k}=x\in A$. Moreover since $st-\lim|x_{n_k}-y_{n_k}|=0$, it follows that

$$st-\lim y_{n_k}=st-\lim[x_{n_k}-(x_{n_k}-y_{n_k})]=st-\lim x_{n_k}-st-\lim(x_{n_k}-y_{n_k})=x$$

so $\{y_{n_k}\}$ also statistically convergent to x. Then since f is statistically continuous on A,

$$st-\lim|f(x_{n_k})-f(y_{n_k})|=|st-\lim f(x_{n_k})-st-\lim f(y_{n_k})|=|f(x)-f(x)|=0$$

but this contradicts the non-uniform statistical continuity condition

$$\frac{1}{n}\big|\{k\leq n:|f(x_{n_k})-f(y_{n_k})|\geq\varepsilon\}\big|\geq\varepsilon_1.$$

Therefore f is uniform statistically continuous. □

Conclusion

We gave definitions of uniform statistically continuity, uniform statistically Cesáro continuity and strongly statistically continuity for real valued functions. Our results include the following: If f is uniform statistically Cesàro continuous on a set A then f is strongly statistically continuous on A. If a function is statistically continuous on a closed bounded interval of the real line, then it is uniformly statistically continuous on that interval.

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