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Warped product pointwise bi-slant submanifolds of trans-Sasakian manifold

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ABSTRACT. The purpose of this paper is to study pointwise bi-slant submanifolds of trans-Sasakian manifold. Firstly, we obtain a non-trivial example of a pointwise bi-slant submanifolds of an almost contact metric manifold. Next we provide some fundamental results, including a characterization for warped product pointwise bi-slant submanifolds in trans-Sasakian manifold. Then we establish that there does not exist warped product pointwise bi-slant submanifold of trans-Sasakian manifold $\tilde{M}$ under some certain considerations. Next, we consider that M is a proper pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distrbutions $\mathcal{D}_1 \oplus < \xi >$ and $\mathcal{D}_2$, then using Hiepko's Theorem, M becomes a locally warped product submanifold of the form $M_1 \times_f M_2$, where M_1 and M_2 are pointwise slant submanifolds with the slant angles θ_1 and θ_2 respectively. Later, we show that pointwise bi-slant submanifolds of trans-Sasakian manifold become Einstein manifolds admitting Ricci soliton and gradient Ricci soliton under some certain conditions..

1. INTRODUCTION

In [9], B. Y. Chen investigated the study of warped product slant submanifolds which are the generalization of both holomorphic and totally real submanifolds of a Kaehler manifold. In 1994, N. Papaghiuc initiated semi-slant submanifolds of a Kaehler manifold [20] as a natural generalization of slant submanifolds. In [14], the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold was established by A. Lota. In [1], P. Alegre studied and proved some important results of slant submanifolds of Lorentzian Sasakian and Para-Sasakian manifolds . In 2000, A. Carriazo introduced the notion of bi-slant submanifolds of an almost Hermitian manifold in [8], as a generalization of semi-slant submanifolds. Many authors have studied different types of submanifold of almost contact manifolds in [5], [12] etc.

In [9], Chen constructed the concept of warped product submanifolds. Later, many mathematicians extended the study of warped product submanifolds of almost Hermitian [3] as well as almost contact manifolds in [2], [4], [7], [13], [17], [23] etc.

The concept of warped product plays an important role in differential geometry as well as in physics, particularly in general theory of relativity [18]. The idea of warped product was first introduced by Bishop and O'Neil [6] to provide examples of Riemannian manifolds with negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds and $f > 0$ be a differential function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus, $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F is called the fiber. The function f is called the warping function of the warped product [18]. Now the following lemma is given in [18].

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Lemma 1.1. *Let $M = B \times_f F$ be a warped product, ∇, ∇^B, ∇^F be the Levi-Civita connection on M, B and F respectively. If $X, Y \in \chi(B), U, W \in \chi(F)$, then*
(i) $\nabla_X Y = \nabla^B_X Y$,
(ii) $\nabla_X U = \nabla_U X = (Xlnf)U$
(iii) $\nabla_U W = -\frac{g(U,W)}{f} grad_B f + \nabla^F_U W$,
for any $X, Y \in \Gamma(TB)$ and $U, W \in \Gamma(TF)$ where ∇ and ∇^F denote the Levi-Civita connections on M and F, respectively, and $gradf$ is the gradient of f.

The paper is organized as follows: In section 2, some basic definitions and preliminary formulas are stated which will be needful for this paper. In section 3, we observe some fundamental results of warped product pointwise bi-slant submanifolds of trans-Sasakian manifolds. In this section, we construct the necessary and sufficient condition for pointwise bi-slant submanifolds of trans-Sasakian manifolds to be locally warped product under some certain conditions.

2. Preliminaries

A $(2n+1)$ dimensional Riemannian manifold $(\tilde{M}, g)$ is called an almost contact metric manifold if there exists a (1,1) tensor field ϕ, a unit vector field ξ and a 1-form η on $\tilde{M}$ such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\phi X) = 0, \phi\xi = 0, \eta(X) = g(X, \xi), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.2}$$

for any vector fields X, Y on $\tilde{M}$. The notion of trans-Sasakian manifold was introduced by Oubina [19] in 1985. Then, J. C. Marrero [15] have studied the local structure of trans-Sasakian manifolds. An almost contact metric manifold $\tilde{M}$ is called a trans-Sasakian manifold if it satisfies the following condition

$$(\tilde{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.3}$$

for some smooth functions α, β on $\tilde{M}$ and we say that the trans-Sasakian structure is of type (α, β). For trans-Sasakian manifold, from (2.3) we have

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.4}$$

$$(\tilde{\nabla}_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.5}$$

For 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned}
\tilde{R}(X,Y)Z &= [\tfrac{\tilde{r}}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)][g(Y,Z)X - g(X,Z)Y] \\
&- [\tfrac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\
&+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)][\phi\, grad\, \alpha - \, grad\, \beta] \\
&- [\tfrac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(Z)[\eta(Y)X - \eta(X)Y] \\
&- [Z\beta + (\phi Z)\alpha]\eta(Z)[\eta(Y)X - \eta(X)Y] \\
&- [X\beta + (\phi X)\alpha][g(Y,Z)\xi - \eta(Z)Y] \\
&- [Y\beta + (\phi Y)\alpha][g(X,Z)\xi - \eta(Z)X],
\end{aligned}$$

$$\begin{aligned}
\tilde{S}(X,Y) &= [\tfrac{\tilde{r}}{2} - (\alpha^2 - \beta^2 - \xi\beta)]g(X,Y) \\
&- [\tfrac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(X)\eta(Y) \\
&- [Y\beta + (\phi Y)\alpha]\eta(X) - [X\beta + (\phi X)\alpha]\eta(Y),
\end{aligned}$$

$\tilde{r}$ being scalar curvature on $\tilde{M}$.
When α and β are constants, the above equations give

$$\tilde{Q}X = (\frac{\tilde{r}}{2} - (\alpha^2 - \beta^2))X - (\frac{\tilde{r}}{2} - 3(\alpha^2 - \beta^2))\eta(X)\xi, \tag{2.6}$$

$$\tilde{R}(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y). \tag{2.7}$$

In general, trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$, $(0,\beta)$ are called cosymplectic, $\alpha-$Sasakian and $\beta-$Kenmotsu manifold, respectively.
Let M be a submanifold of an almost contact manifold $\tilde{M}$ with induced metric g. Let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and normal bundle $T^{\perp}M$ of M respectively. Let $\mathcal{F}$ denote the algebra of smooth functions on M and $\Gamma(TM)$ denotes the $\mathcal{F}$-module of smooth sections of TM over M. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.9}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\tilde{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.10}$$

where g denotes the Riemannian metric on $\tilde{M}$ as well as the one induced on M. The mean curvature H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i)$, where m is the dimension of M and $\{e_1, e_2......, e_m\}$ is a local orthonormal frame of vector fields on M.
A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be totally umbilical if the second fundamental form satisfies $h(X,Y) = g(X,Y)H$, for all $X,Y \in \Gamma(TM)$.
A submanifold M is said to be totally geodesic if $h(X,Y) = 0$, for all $X,Y \in \Gamma(TM)$ and minimal if $H = 0$.
A foliation L on a Riemannian manifold $\tilde{M}$ is called totally umbilical, if every leaf L is totally umbilical in $\tilde{M}$. If the mean curvature of every leaf is parallel in the normal bundle, then L is called a spheric foliation. If every leaf L is a totally geodesic, then L is called totally geodesic foliation, [10].
For any $X \in \Gamma(TM)$,

$$\phi X = PX + FX, \tag{2.11}$$

where PX is the tangential component and FX is the normal component of ϕX.

$$\phi N = BN + CN, \tag{2.12}$$

where BN is the tangential component and CN is the normal component of ϕN. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be invariant if F is identically zero, that is $\phi X \in \Gamma(TM)$ and anti-invariant if P is identically zero, that is $\phi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.
There is another class of submanifolds, called the slant submanifold. For each non-zero vector X tangent to M at x, such that X is not proportional to ξ_x. The angle $\theta(X)$ between ϕX and T_xM is constant for all nonzero $X \in T_xM - <\xi_x>$ and $x \in M$, then M is said to be a slant submanifold [7] and the angle θ is the slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \frac{\pi}{2}$, M is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.
We recall the following result which was obtained by Cabreizo et al. [7] for a slant submanifold of an almost contact metric manifold.

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in TM$. Then, M is slant iff $\exists$ a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.13}$$

Again, if θ is slant angle of M, then $\lambda = \cos^2\theta$.
The following relations are straightforward consequences of (2.13):

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.14}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$.
For a pointwise slant submanifold of almost Hermitian manifold it is similarly derived in [16]

$$BFX = -X\sin^2\theta,\ CFX = -FPX, \tag{2.16}$$

for all $X \in \Gamma(TM)$.
Now, we explain the brief introduction of pointwise bi-slant submanifold of an almost contact metric manifold $\tilde{M}$.

Definition 2.1. [7, 8] A submanifold M of an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is said to be a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that:
(i) TM admits the orthogonal direct decomposition i.e. $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$, where $< \xi >$ is the one dimensional distribution spanned by the structure vector field ξ.
(ii) $\phi(\mathcal{D}_1) \perp \mathcal{D}_2$ and $\phi(\mathcal{D}_2) \perp \mathcal{D}_1$ that implies $P(\mathcal{D}_i) \subset \mathcal{D}_i$, $i = 1, 2$. (iii) The distribution $\mathcal{D}_1$ and $\mathcal{D}_2$ are pointwise slant with slant angles θ_1 and θ_2 respectively.

A pointwise bi-slant submanifold is called proper if its bi-slant angles satisfy $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ and θ_1, θ_2 are not constants on M.
For a pointwise bi-slant submanifold, we take

$$X = T_1X + T_2X,\ \forall X \in TM, \tag{2.17}$$

where T_i is the projection from TM onto D_i. So, T_iX are the components of X in D_i, $i = 1, 2$.
If we put $P_i = T_i \circ P$, then from the equation (2.17) we get

$$\phi X = P_1X + P_2X + FX,\ \forall X \in TM. \tag{2.18}$$

From Proposition we have

$$P^2 = \cos^2\theta_i(-I + \eta \otimes \xi),\ i = 1, 2. \tag{2.19}$$

Now, we provide the following non-trivial example of a pointwise bi-slant submanifolds of an almost contact metric manifold.

Example 2.1. Let M be a submanifold of R^7 with coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, z)$ Let us consider an isometric immersion x into R^7 as follows:

$$\psi(u, v, \alpha, \beta, z) = (u, -v, \sqrt{3}\sin\alpha, \cos\alpha, \sin\beta, \cos\beta, z).$$

We can easily to see that the tangent bundle TM is spanned by the tangent vectors $Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}$, $Z_2 = -\sqrt{3}\sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial y_1}$, $Z_3 = \sin\beta\frac{\partial}{\partial x_2} - \cos\beta\frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}$, $Z_4 = \sin\beta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_2} + \cos\alpha\frac{\partial}{\partial x_3} + \sqrt{3}\sin\alpha\frac{\partial}{\partial y_3}$, $Z_5 = \frac{\partial}{\partial z} = \xi$.
For any vector field $X = \gamma_i\frac{\partial}{\partial x_i} + \delta_j\frac{\partial}{\partial y_j} + v\frac{\partial}{\partial z} \in \Gamma(TR^7)$, then we have $g(X, X) = \gamma_i^2 + \delta_j^2 + v^2$, $g(\phi X, \phi X) = \gamma_i^2 + \delta_j^2$ and $\phi(X) = -\gamma_i\frac{\partial}{\partial x_i} - \delta_j\frac{\partial}{\partial y_j} = -X + \eta(X)\xi$, for any $i, j = 1, 2$. It is clear that $g(\phi X, \phi X) = g(X, X) - \eta(X)\eta(X)$. Thus (ϕ, ξ, η, g) is an almost contact metric

structure on R^7.
We define the almost contact structure ϕ of R^7, by

$$\phi(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i}, \phi(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j}, \frac{\partial}{\partial z} = 0,\ i,j \in \{1,2,3\}.$$

By direct calculations, we can infer that $D_1 = span\{Z_1, Z_2\}$ and $D_2 = span\{Z_3, Z_4\}$ are pointwise slant distributions with slant angles $\theta_1 = \cos^{-1}(\frac{\cos\alpha - \sqrt{3}\sin\alpha}{\sqrt{2}\sqrt{\cos^2\alpha + 2\sin^2\alpha}}), \theta_2 = \cos^{-1}(\frac{-\cos\alpha + \sqrt{3}\sin\alpha + \sin 2\beta}{\sqrt{3}\sqrt{\cos^2\alpha + 2\sin^2\alpha + 1}})$, respectively. Thus M is a pointwise bi-slant submanifold of R^7 such that $\xi = \frac{\partial}{\partial z}$ is tangent to M.

Now, we consider the following lemma for later use.

Lemma 2.2. *Let M be a pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distributions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2$ with distinct slant angles θ_1 and θ_2 respectively. Then*

$$\begin{aligned}(\sin^2\theta_1 - \sin^2\theta_2) g(\nabla_X Y, Z) &= g(A_{FP_1Y}Z - A_{FY}P_2Z, X) \\ &+ g(A_{FP_2Z}Y - A_{FZ}P_1Y, X),\end{aligned} \tag{2.20}$$

where $X, Y \in \Gamma(\mathcal{D}_1 \oplus <\xi>)$ and $Z \in \Gamma(\mathcal{D}_2)$ and θ_1 and θ_2 are the slant angles of slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Proof. Proof is similar to [10]. □

3. Warped Product Pointwise Bi-slant Submanifold of trans-Sasakian Manifold:

In this section we assume that $M = M_1 \times_f M_2$ is a warped product pointwise bi-slant submanifold of trans-Sasakian manifold $\tilde{M}$ with certain condition on unit vector field ξ. Here, we establish that there do not exist warped product pointwise bi-slant submanifold of trans-Sasakian manifold $\tilde{M}$ under some certain considerations. Now we prove the following proposition.
First we prove the following proposition which will be helpful to prove later theorems.

Proposition 3.1. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with ξ is tangent to M_2. Then*

$$\begin{aligned}g(h(X,W), FP_2Z) - g(h(X,P_2Z), FW) &= X(\theta_2)\sin 2\theta_2 (g(Z,W) - \eta(Z)\eta(W)) \\ &- X(lnf)\eta(Z)\eta(W),\end{aligned} \tag{3.21}$$

where $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ and θ_1 and θ_2 are the slant angles of M_1 and M_2 respectively.

Proof. First we consider ξ is tangent to M_2. Then for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, we have

$$\begin{aligned}g(\tilde{\nabla}_X Z, W) &= g(\phi\tilde{\nabla}_X Z, \phi W) \\ &= g(\tilde{\nabla}_X \phi Z, \phi W) - g((\tilde{\nabla}_X \phi)Z, \phi W)\end{aligned}$$

From the equations (2.1)-(2.5), (2.11), (2.12), (2.14), (2.16) and Lemma 1.1. we obtain

$$\begin{aligned}g(\tilde{\nabla}_X Z, W) &= X(lnf)(\cos^2\theta_2 + \sin^2\theta_2)[g(Z,W) - \eta(Z)\eta(W)] + g(h(X,P_2Z), FW) \\ &+ \sin 2\theta_2 X(\theta_2)[g(Z,W) - \eta(Z)\eta(W)] - g(h(X,W), FP_2Z).\end{aligned} \tag{3.22}$$

On the other hand, we also have from Lemma 1.1.

$$g(\tilde{\nabla}_X Z, W) = g(\nabla_X Z, W) = X(lnf)g(Z, W). \tag{3.23}$$

From the equations (3.22) and (3.23) Proposition 3.1. is proved. □

Theorem 3.2. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively and also consider ξ is tangent to M_2. If M is mixed totally geodesic warped product submanifold and θ =constant, then M is a Riemannian product submanifold of M_1 and M_2.*

Proof. From Proposition 3.1. we can easily see that $Xlnf = 0$ that means f is constant on M. □

Proposition 3.2. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with ξ is tangent to M_1. Then*

$$\begin{aligned} a)\ g(h(X, Z), FW) + g(h(X, W), FZ) &= -2\alpha\eta(X)g(Z, W) + 2g(h(Z, W), FX) \\ &- 2P_1 X(lnf)g(Z, W), \end{aligned} \tag{3.24}$$

$$\begin{aligned} b)\ g(h(X, Z), FW) - g(h(X, W), FZ) &= -2\beta\eta(X)g(\phi Z, W) \\ &- 2X(lnf)g(Z, P_2 W), \end{aligned} \tag{3.25}$$

where $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ and θ_1 and θ_2 are the slant angles of M_1 and M_2 respectively.

Proof. Let us assume that ξ be tangent to M_1. Then for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$,we have

$$\begin{aligned} g(h(X, Z), FW) &= g(\tilde{\nabla}_Z X, FW) \\ &= g((\tilde{\nabla}_Z \phi)X, W) - g(\tilde{\nabla}_Z \phi X, W) - g(\tilde{\nabla}_Z X, P_2 W). \end{aligned}$$

Taking the equations (2.3), (2.11) and Lemma 1.1 we can write

$$\begin{aligned} g(h(X, Z), FW) &= -\alpha\eta(X)g(Z, W) - \beta\eta(X)g(\phi Z, W) \\ &- P_1 X(lnf)g(Z, W) - X(lnf)g(Z, P_2 W) \\ &+ g(h(Z, W), FX). \end{aligned} \tag{3.26}$$

Now interchanging Z and W the above equation gives

$$\begin{aligned} g(h(X, W), FZ) &= -\alpha\eta(X)g(Z, W) - \beta\eta(X)g(\phi W, Z) \\ &- P_1 X(lnf)g(Z, W) - X(lnf)g(P_2 Z, W) \\ &+ g(h(Z, W), FX). \end{aligned} \tag{3.27}$$

Adding the equations (3.26) and (3.27) we get

$$\begin{aligned} g(h(X, Z), FW) + g(h(X, W), FZ) &= -2\alpha\eta(X)g(Z, W) + 2g(h(Z, W), FX) \\ &- 2P_1 X(lnf)g(Z, W). \end{aligned} \tag{3.28}$$

Substracting (3.27) from (3.26) we obtain

$$\begin{aligned} g(h(X, Z), FW) - g(h(X, W), FZ) &= -2\beta\eta(X)g(\phi Z, W) \\ &- 2X(lnf)g(Z, P_2 W). \end{aligned} \tag{3.29}$$

This completes the proof. □

Theorem 3.3. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively and also consider ξ is tangent to M_1. Then*
i) If $\alpha\eta(X)Z = \tilde{\nabla}_Z FX$, then M is a Riemannian product submanifold of M_1 and M_2.
ii) If M is mixed totally geodesic warped product submanifold and $\beta = 0$, then M is a Riemannian product submanifold of M_1 and M_2.

Proof. From Proposition 3.2 (a) and (b) we see that $Xlnf = 0$ which shows that f is constant on M. □

Proposition 3.3. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively and also consider ξ is tangent to M_1.Then*

$$[\sin 2\theta_2 X(\theta_2) + (2\beta\eta(X) - 2X(lnf))\cos^2\theta_2]g(Z,W) = 2\beta\eta(X)g(Z,FP_2W),$$

where $X \in \Gamma(TM_1)$. In particular, if $\beta = 0$, $Xlnf = \tan\theta_2 X(\theta_2)$.

Proof. For the equation (3.29) we have

$$\begin{aligned} g(h(X,Z),FW) - g(h(X,W),FZ) &= -2\beta\eta(X)g(\phi Z,W) \\ &- 2X(lnf)g(Z,P_2W), \end{aligned} \tag{3.30}$$

for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$.
Putting $W = P_2W$ in the equation (3.30) and then using the equations (2.11), (2.13) and (2.14) we derive

$$\begin{aligned} g(h(X,Z),FP_2W) - g(h(X,P_2W),FZ) &= 2\beta\eta(X)[-\cos^2\theta_2 g(Z,W) - g(Z,FP_2W)] \\ &+ 2X(lnf)\cos^2\theta_2 g(Z,W). \end{aligned} \tag{3.31}$$

On the other hand, we have

$$\begin{aligned} g(\tilde{\nabla}_X W, Z) &= g(\phi\tilde{\nabla}_X W, \phi Z) \\ &= g(\tilde{\nabla}_X \phi W, \phi Z) - g((\tilde{\nabla}_X\phi)W, \phi Z) \end{aligned} \tag{3.32}$$

Using the equations (2.11), (2.13), (2.14) and (2.16), the equation (3.32) reduces to

$$g(h(X,Z),FP_2W) - g(h(X,P_2W),FZ) = \sin(2\theta_2)X(\theta_2)g(Z,W). \tag{3.33}$$

From the equations (3.31) and (3.33) give

$$[\sin 2\theta_2 X(\theta_2) + (2\beta\eta(X) - 2X(lnf))\cos^2\theta_2]g(Z,W) = 2\beta\eta(X)g(Z,FP_2W).$$

In particular, if $\beta = 0$, $Xlnf = \tan\theta_2 X(\theta_2)$. □

Corollary 3.1. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively and also consider ξ is tangent to M_1. If $(\cos^2\theta)W = FP_2W$ and θ_2 =constant,then M is a Riemannian product submanifold of M_1 and M_2.*

Now, we prove the following lemmas for later use.

Lemma 3.3. *Let M be a pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distributions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2$ with distinct slant angles θ_1 and θ_2 respectively.*

Then

$$\begin{aligned}(\sin^2\theta_2-\sin^2\theta_1)g(\nabla_Z W,X) &= g(A_{FP_1X}W-A_{FX}P_2W,Z)\\ &+ g(A_{FP_2W}X-A_{FW}P_1X,Z)\\ &+ \beta\eta(X)\cos^2\theta_2 g(Z,W)-\alpha\eta(P_1X)g(Z,W)\\ &- \beta\eta(X)g(Z,FP_2W)+\alpha\eta(X)\sin^2\theta_1 g(Z,\phi W)\\ &+ \beta\eta(P_1X)g(Z,\phi W)+\beta\eta(X)\sin^2\theta_1 g(Z,W)\\ &+ \alpha\eta(X)g(Z,P_2W). \qquad (3.34)\end{aligned}$$

where $X\in\Gamma(\mathcal{D}_1\oplus<\xi>)$ and $Z,W\in\Gamma(\mathcal{D}_2)$ and θ_1 and θ_2 are the slant angles of slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Proof. For any $X\in\Gamma(\mathcal{D}_1\oplus<\xi>)$ and $Z,W\in\Gamma(\mathcal{D}_2)$,we have

$$\begin{aligned}g(\nabla_Z W,X)=g(\tilde\nabla_Z W,X) &= g(\phi\tilde\nabla_Z W,\phi X)\\ &= g(\tilde\nabla_X\phi Z,\phi W)-g((\tilde\nabla_Z\phi)W,\phi X) \qquad (3.35)\end{aligned}$$

Using the equations (2.1)-(2.5), (2.11)-(2.14), (2.16) we get (3.34). □

Lemma 3.4. *Let $M=M_1\times_f M_2$ be a warped product pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde M$ such that M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively and also consider ξ is tangent to M_1. Then*

$$(i)\ g(h(X,Y),FW)=g(h(X,W),FY). \qquad (3.36)$$

$$\begin{aligned}(ii)\ g(A_{FP_1X}W-A_{FX}P_2W,Z) &+ g(A_{FP_2W}X-A_{FW}P_1X,Z)=(\sin^2\theta_1\\ &- \sin^2\theta_2)X(lnf)g(Z,W)-\alpha\eta(P_1X)g(Z,W)\\ &- \eta(P_1X)g(\phi Z,W)+\alpha\sin^2\theta_1\eta(X)g(Z,\phi W)\\ &+ \beta\sin^2\theta_1\eta(X)g(Z,W)-\alpha g(Z,FW)\eta(X)\\ &+ \beta\eta(X)g(Z,\phi FW), \qquad (3.37)\end{aligned}$$

$X\in\Gamma(TM_1)$ and $Z,W\in\Gamma(TM_2)$.

Proof. Let us consider ξ be tangent to M_1. Then for any $X\in\Gamma(TM_1)$ and $Z,W\in\Gamma(TM_2)$,we have

$$\begin{aligned}g(h(X,Y),FW) &= g(\tilde\nabla_X Y,FW)\\ &= g(\tilde\nabla_X Y,\phi W)-g(\tilde\nabla_X Y,P_2W). \qquad (3.38)\end{aligned}$$

Using the equations (2.3), (2.8) and Lemma 1.1. we obtain

$$g(h(X,Y),FW)=g(h(X,W),FY).$$

Hence Lemma 3.4. (i) is proved.
Now, we have

$$\begin{aligned}g(\tilde\nabla_Z X,W) &= g(\phi\tilde\nabla_Z X,\phi W)\\ &= g(\tilde\nabla_Z\phi X,\phi W)-g((\tilde\nabla_Z\phi)X,\phi W) \qquad (3.39)\end{aligned}$$

Using the equations (2.3), (2.10)-(2.12), we obtain

$$\begin{aligned}g(\tilde\nabla_Z X,W) &= -\alpha\eta(P_1X)g(Z,W)-\beta\eta(P_1X)g(\phi Z,W)-g(\tilde\nabla_Z P_1^2X,W)\\ &- g(\tilde\nabla_Z FP_1X,W)-g(A_{FX}Z,P_2W)+g(\phi(\tilde\nabla_Z FW),X)\\ &+ g(\tilde\nabla_Z FW,P_1X)+\alpha\eta(X)g(Z,\phi W)\\ &+ \beta\eta(X)g(\phi Z,\phi W).\end{aligned}$$

Again, from the equations (2.1)-(2.5), (2.11)-(2.14), (2.16), Lemma 1.1. and the orthogonality of vector fields and symmetry of the shape operator it follows that

$$\begin{aligned} g(A_{FP_1X}W - A_{FX}P_2W, Z) &+ g(A_{FP_2W}X - A_{FW}P_1X, Z) = (\sin^2\theta_1 \\ &- \sin^2\theta_2)X(lnf)g(Z,W) - \alpha\eta(P_1X)g(Z,W) \\ &- \eta(P_1X)g(\phi Z, W) + \alpha\sin^2\theta_1\eta(X)g(Z,\phi W) \\ &+ \beta\sin^2\theta_1\eta(X)g(Z,W) - \alpha g(Z,FW)\eta(X) \\ &+ \beta\eta(X)g(Z,\phi FW). \end{aligned}$$

This completes the proof of the Lemma 3.4. (ii). □

Theorem 3.4. *Let M be a proper pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distrbutions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2$. If M is locally a warped product submanifold of the form $M_1 \times_f M_2$, where M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively with ξ tangent to M_1, then the shape operator A satisfies*

$$\begin{aligned} A_{FP_1X}W - A_{FX}P_2W &+ A_{FP_2W}X - A_{FW}P_1X = (\sin^2\theta_1 \\ &- \sin^2\theta_2)X(\mu)W - \alpha\eta(P_1X)W \\ &+ \beta\eta(P_1X)\phi W + \alpha\sin^2\theta_1\eta(X)\phi W \\ &+ \beta\sin^2\theta_1\eta(X)W - \alpha FW\eta(X) \\ &+ \beta\eta(X)\phi FW, \end{aligned} \tag{3.40}$$

with $\mu = lnf$.

Proof. Using Lemma 3.4. (i) and Lemma 3.4. (ii) we obtain the equation (3.40) with $\mu = lnf$. □

The next theorem shows a characterization result for pointwise bi-slant submanifold of a trans-Sasakian manifold. First we state the following theorem [11] according to S. Hiepko.

Hiepko's Theorem : Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distributions on a Riemannian manifold M. Suppose that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is a locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

Theorem 3.5. *Let M be a proper pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distrbutions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2$. If*

$$\begin{aligned} A_{FP_1X}W - A_{FX}P_2W &+ A_{FP_2W}X - A_{FW}P_1X = (\sin^2\theta_1 \\ &- \sin^2\theta_2)X(\mu)W - \alpha\eta(P_1X)W \\ &+ \beta\eta(P_1X)\phi W + \alpha\sin^2\theta_1\eta(X)\phi W \\ &+ \beta\sin^2\theta_1\eta(X)W - \alpha FW\eta(X) \\ &+ \beta\eta(X)\phi FW, \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} 2\alpha\eta(P_1X)W - 2\beta\eta(P_1X)W &+ \beta\eta(X)W - 2\alpha\sin^2\theta_1\eta(X)\phi W \\ &- 2\beta\sin^2\theta_1\eta(X)W - \alpha\phi W\eta(X) \\ &+ 2\alpha\eta(X)FW - 2\beta\eta(X)\phi FW. \end{aligned} \tag{3.42}$$

holds, then, M is locally a warped product submanifold of the form $M_1 \times_f M_2$, where M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively, where μ is a function on M satisfying $W\mu = 0$, for any $W \in \mathcal{D}_2$.

Proof. Let M be a proper pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distrbutions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2$. From Lemma 2.2., we have

$$\begin{aligned}(\sin^2\theta_1 - \sin^2\theta_2) g(\nabla_Y X, Z) &= g(A_{FP_1X}Z - A_{FX}P_2Z, Y) \\ &+ g(A_{FP_2Z}X - A_{FZ}P_1X, Y),\end{aligned} \tag{3.43}$$

where $X, Y \in \Gamma(\mathcal{D}_1 \oplus <\xi>)$ and $Z \in \Gamma(\mathcal{D}_2)$. Now taking inner product with Y in the equation (3.40) and then using equation (3.42) we obtain

$$g(\nabla_Y X, Z) = X(\mu) g(Y, W) = 0.$$

Hence the leaves of the distributions are totally geodesic in M.
From Lemma 3.3., we have

$$\begin{aligned}(\sin^2\theta_2 - \sin^2\theta_1) g(\nabla_Z W, X) &= g(A_{FP_1X}W - A_{FX}P_2W, Z) \\ &+ g(A_{FP_2W}X - A_{FW}P_1X, Z) + \alpha\eta(X) g(Z, P_2W) \\ &+ \beta\eta(X)\cos^2\theta_2 g(Z, W) - \alpha\eta(P_1X) g(Z, W) \\ &- \beta\eta(X) g(Z, FP_2W) + \alpha\eta(X)\sin^2\theta_1 g(Z, \phi W) \\ &+ \beta\eta(P_1X) g(Z, \phi W) \\ &+ \beta\eta(X)\sin^2\theta_1 g(Z, W).\end{aligned} \tag{3.44}$$

Also, we have

$$g(\nabla_Z W, X) = -X(\mu) g(Z, W). \tag{3.45}$$

Interchanging W and Z and then using the definition of Lie bracket we obtain

$$g([Z, W], X) = 0. \tag{3.46}$$

Hence $\mathcal{D}_2$ is integrable.
Let us assume h_2 be the second fundamental form of a leaf M_2 of $\mathcal{D}_2$ in M. Then for any $X \in \Gamma(\mathcal{D}_1 \oplus <\xi>)$ and $Z, W \in \Gamma(\mathcal{D}_2)$ we have

$$g(h_2(Z, W), X) = g(\nabla_Z W, X). \tag{3.47}$$

From the equation (3.45) we can write

$$g(h_2(Z, W), X) = -X(\mu) g(Z, W). \tag{3.48}$$

From the definition of gradient it can be written as

$$g(h_2(Z, W), X) = -\nabla\mu g(Z, W). \tag{3.49}$$

It follows that the leaf M_2 is totally umbilical in M with mean curvature vector $H_2 = -\nabla\mu$. Then for any $Y \in \Gamma(\mathcal{D}_1 \oplus <\xi>)$ and $W \in \Gamma(\mathcal{D}_2)$ we have

$$\begin{aligned}g(\mathcal{D}_2^W H_2, Y) &= -g(\mathcal{D}_2^W \nabla\mu, Y), \\ &= -g(\nabla_W \nabla\mu, Y), \\ &= -W g(\nabla\mu, Y) + g(\nabla\mu, \nabla_W Y), \\ &= -W(Y\mu) - g(\nabla\mu, \nabla_Y W), \because [W, Y] = 0 \\ &= -Y(W\mu) + g(\nabla_Y \nabla\mu, W) = 0.\end{aligned} \tag{3.50}$$

Since $W\mu = 0$, for any $W \in \mathcal{D}_2$ and so $\nabla_Y \nabla\mu \in \Gamma(\mathcal{D}_1 \oplus <\xi>)$. This shows that the mean curvature of M_2 is parallel. Therefore, $\mathcal{D}_2$ is a spherical foliation. Hence by Hiepko's

Theorem, M is a locally warped product $M_1 \times_f M_2$, where M_1 and M_2 are pointwise slant submanifolds with the slant functions θ_1 and θ_2 respectively, where μ is a function on M. □

Conclusion We have established that there do not exist warped product pointwise bi-slant submanifold of trans-Sasakian manifold $\tilde{M}$ under some certain considerations. Next, we have proved that M is a proper pointwise bi-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ with pointwise slant distrbutions $\mathcal{D}_1 \oplus < \xi >$ and $\mathcal{D}_2$, then using Hiepko's Theorem, M is a locally warped product submanifold of the form $M_1 \times_f M_2$.

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